Volume Conjecture Seminar 3

Hyperbolic volumes of knot complements II

Craig Hodgson

15 August 2011

Abstract. We continue with last week's introduction to hyperbolic geometry and hyperbolic structures on three-manifolds. In particular, we explain how ideal triangulations can be used to compute hyperbolic structures and hyperbolic volumes for knot complements, illustrating this method for the figure-eight knot complement.

1 Computing hyperbolic structures

Given a knot K in the three-sphere $S^3 = \mathbb{R}^3 \cup \{\infty\}$, we saw last week that asides from a few cases, its *knot complement* $S^3 \setminus K$ may be endowed with a unique complete finite-volumed hyperbolic structure. This hyperbolic structure and its corresponding volume are knot invariants for K and may be obtained in the following step:

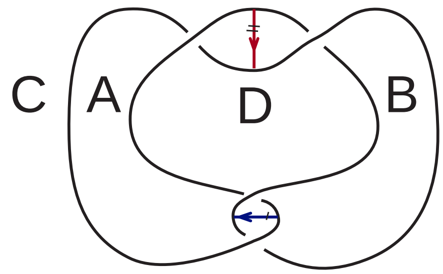
1. Find a topological decomposition of $M = S^3 \setminus K$ into ideal tetrahedra – topological tetrahedra without their vertices.
2. Appropriately specify the geometry for these ideal tetrahedra in $\mathbb{H}^3$ so that they assemble correctly to give the unique complete hyperbolic structure on M.
3. Compute and sum the volume of these ideal tetrahedra.

We now explain in detail various points of this procedure, demonstrating it via the example of the figure eight knot complement.

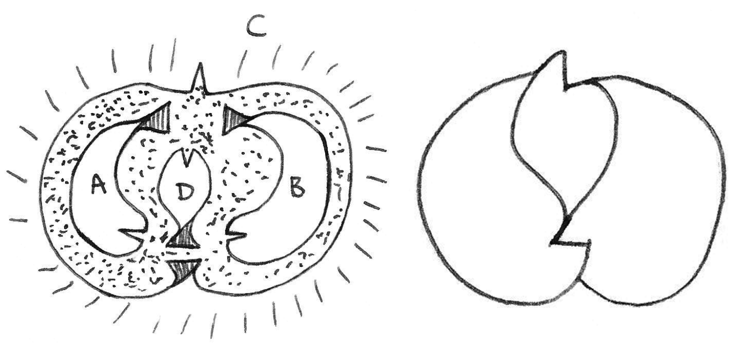
2 Topological decomposition

For our purposes, we will defer the description of the triangulation procedure to section 3 of Jeffrey Weeks' chapter in the *Handbook of knot theory*. His algorithm implemented in SnapPea and results in the decomposition of the figure eight knot described below. See Thurston's Princeton Notes or Book for other nice examples, e.g. Whitehead link, Borromean rings.

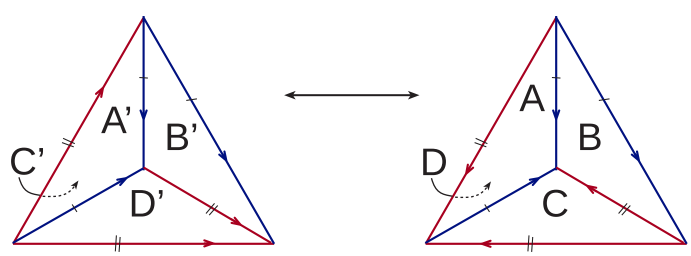
We begin by adding a red and a blue edge joining sections of the figure eight knot K in $S^3 = \mathbb{R}^3 \cup \{\infty\}$, we obtain the following 1-dimensional CW-complex L:



Now take four additional 2-cells labelled A, B, C and D, and glue them along their boundaries so that these boundaries lie on L. We describe this in the left figure below, taking care to specify that region C also includes a point at infinity. In particular, this point at infinity compactifies the resulting 2-complex so that it is homotopy equivalent to a 2-sphere, thereby cutting S^3 into two open 3-balls[1]. Moreover, the figure on the right illustrates what can be seen from one of these 3-balls.

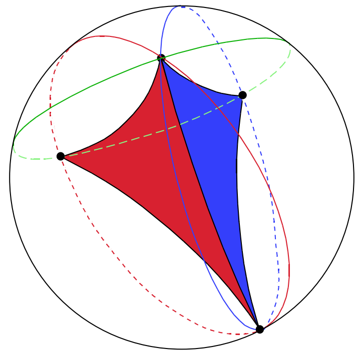


When we endow M with the necessary geometric structure to give it a complete finite volume hyperbolic structure, the knot K disappears into the ideal points at the cusps of M. Therefore, by looking carefully at the pattern of faces on the boundary of the two aforementioned 3-balls and removing the edges along the knot, we obtain the following triangulation for M:



The edges of these two tetrahedra fit together in two groups of 6 partitioned by colour. Consider a regular *ideal* tetrahedron in $\mathbb{H}^3$, with all its vertices on the sphere at infinity. This has all dihedral angles 60 degrees.

[1]This is analogous to how a great circle S^1 cuts up S^2 into two open 2-balls.



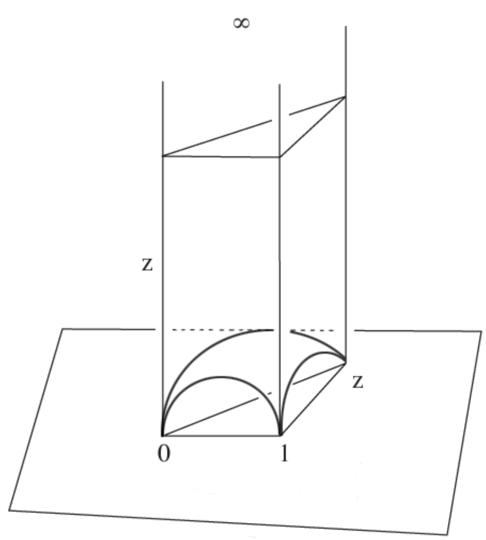
Taking two regular ideal tetrahedra and gluing their faces together by isometries then gives a hyperbolic structure on X. We now show that this is precisely the unique complete finite volumed hyperbolic structure on M.

3 Parametrizing ideal tetrahedra in $\mathbb{H}^3$

An hyperbolic ideal tetrahedron is the hyperbolic convex hull of four points $\{v_1, v_2, v_3, v_4\}$ in $\partial\mathbb{H}^3 = S^2 = \mathbb{C} \cup \{\infty\}$; since there exists a unique[2] Möbius transformation $g \in \mathrm{PSL}_2(\mathbb{C}) = \mathrm{Aut}(\mathbb{H}^3)$ taking

$$v_1 \mapsto \infty,\ v_2 \mapsto 0,\ v_3 \mapsto 1,$$

the image of the fourth point, $g(v_4) \in S^2 \setminus \{0, 1, \infty\} = \mathbb{C} \setminus \{0, 1\}$, uniquely specifies a hyperbolic ideal tetrahedron up to orientation preserving isometry. Viewed in the upper half space model, a hyperbolic ideal tetrahedron resembles:



The intersection of any sufficiently small horosphere centred at ∞ with the above tetrahedron yields similar triangles and this in turn motivates an alternate way to specify an (hyperbolic) ideal tetrahedron:

[2]The existence and uniqueness of this function is left as an exercise.

by measuring the angles of the corners of such a horocyclic triangle. These angles α, β, γ are the *dihedral angles* to the corresponding edges of our ideal tetrahedron and we can use the fact that $\alpha + \beta + \gamma = \pi$ to establish the following:

Exercise 1. Show that opposite edges in an ideal tetrahedron have the same dihedral angles.

Going back to our first method of specifying the geometry of our ideal tetrahedron, we associate $z = g(v_4)$ to the edge from 0 to ∞ and note that this is the *cross ratio* of v_1, v_2, v_3 and v_4. We then respectively associate to the edges over z and 1 the complex parameters z' and z'' in a similar fashion. Observe that the affine Möbius transformation rotating anti-clockwise about 0 and resizing the edge between 0 and 1 to that between 0 and z is given by:

$$g_1 : x \mapsto z \cdot x.$$

Then, the affine map taking the edge between 0 and z and rotating it anti-clockwise about z and resizing the edge to that between 1 and z is given by:

$$g_2 : x \mapsto z' \cdot x - zz' + z.$$

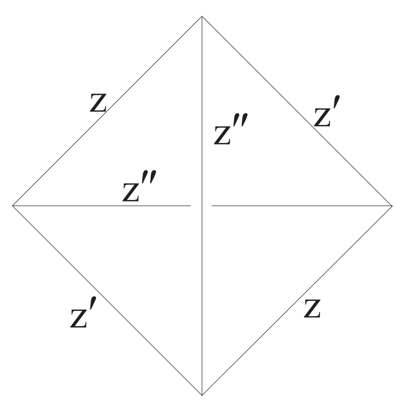
Since $0 \mapsto 1 = -zz' + z$ under this map, we see that $z' = \frac{z-1}{z}$.

Similarly, the affine map taking the edge between 1 and z and rotating it anti-clockwise about 1 and resizing the edge to that between 0 and 1 is given by:

$$g_3 : x \mapsto z'' \cdot x - z'' + 1.$$

Like before, this tells us that $z'' = \frac{1}{1-z}$.

Observe that these parameters satisfy $zz'z'' = -1$. This should not be surprising, since we know that $g_3 \circ g_2 \circ g_1$ takes the form: $x \mapsto zz'z''x + c$ and that the composition gives an affine transformation reversing the direction of the edge between 0 and 1. Furthermore, by noticing that the dihedral angle of an edge is the argument of its complex parameter, we can use Exercise 1 to show that the complex parameter associated to opposite edges are the same:

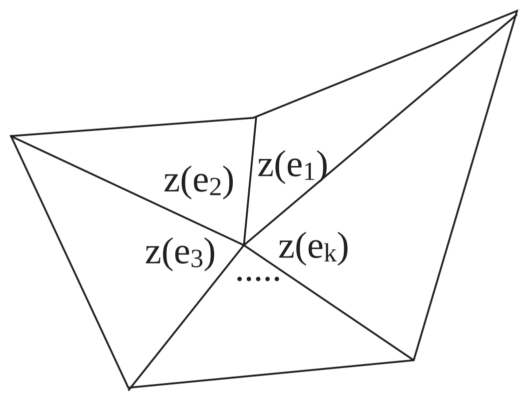


The above parameterisations tell us that there is a complex 1-dimensional parameter space for ideal tetrahedra, and one immediate consequence is that there is essentially no freedom in how we glue together two hyperbolic ideal tetrahedra if we want edges to match up correctly. We now phrase last week's gluing conditions for obtaining a hyperbolic structure from a topological ideal triangulation in terms of this complex parametrisation.

4 Gluing conditions to obtain a hyperbolic structure

4.1 Edge gluing condition(s)

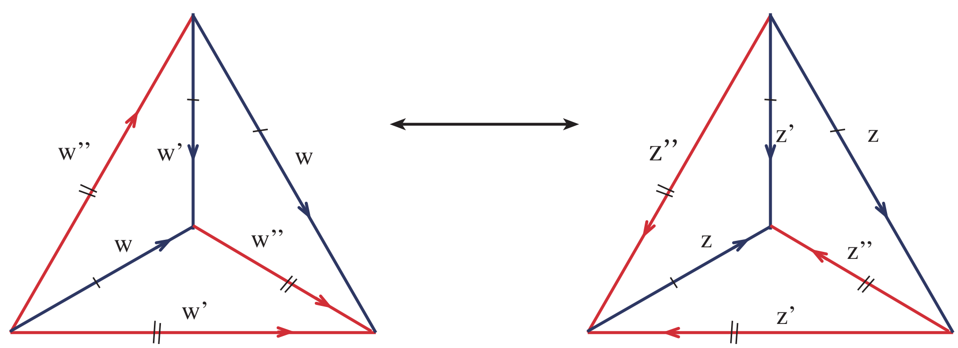
Given n ideal tetrahedra with complex edge parameters $z(e)$, the first gluing condition says that we want the dihedral angles around each edge to sum to 2π and that when glued together, there should be no translation along the edge as one travels around it. This is equivalent to asking that the Euclidean horospherical triangles around each edge fit together:



Again, by thinking of $z(e_i)$ in terms of an affine rotate and resize map, we see that

$$z(e_1)z(e_2)\cdots z(e_k) = 1 \text{ and } \arg z(e_1) + \arg z(e_2) + \ldots + \arg z(e_k) = 2\pi$$

We stress that the angle sum is 2π, not $4\pi, 6\pi, \ldots$ and presently consider the example of the figure eight knot complement to illustrate this condtion.



For the figure eight knot complement as drawn above, the respective edge equations for the blue and red edges are:

$$z^2 z' w^2 w' = 1, \quad z'(z'')^2 w'(w'')^2 = 1,$$

expressing z', z'', w' and w'' in terms of z and w yields:

$$zw(1-z)(1-w) = 1$$
$$z^{-1}w^{-1}(1-z)^{-1}(1-w)^{-1} = 1.$$

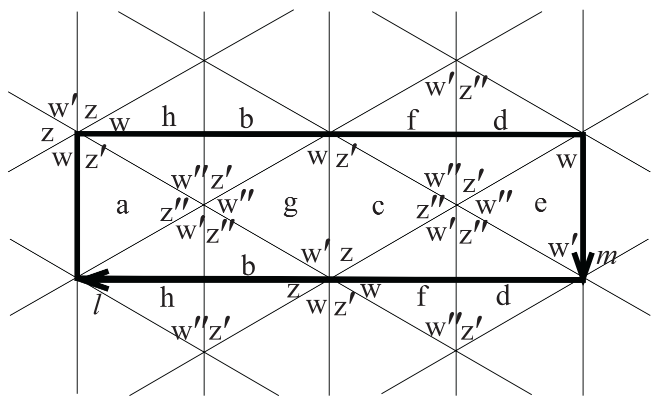
These equations are equivalent, so there is a complex 1-dimensional solution space for this polynomial in $\mathbb{C}^2$. The Prosad-Mostow rigidity theorem tells us that precisely one solution in this family will give a complete finite volume hyperbolic metric on our figure eight knot complement. The other solutions may also glue together correctly, but yield incomplete hyperbolic structures - which leads us to the second gluing condition:

4.2 Completeness condition

Recall now that the completeness condition tells us that the horospherical triangles of our ideal triangulation must glue together to give an Euclidean torus T that bounds a tubular neighbourhood of our knot K. Consider the developing map from the universal cover of the torus to T

$$\pi : \tilde{T} = \mathbb{R}^2 \to T,$$

since the horospherical triangulation on $T \subset M$ decomposes it into Euclidean triangles, we may lift this triangulation information to obtain a tiling of $\tilde{T}$.



Given a simple closed loop γ on T, consider a lift $\tilde{\gamma}$ of γ starting at an intersection point between γ and an edge e of the horospherical triangulation of T. Let $\{\tilde{e}_i\}_{i=0,\ldots,n}$ be the set of edges we intersect as we traverse along $\tilde{\gamma}$. We know from construction that $\tilde{e}_0$ and $\tilde{e}_n$ are both lifts of e and let the following sequence of affine transformations

$$\{x \mapsto z_i \cdot x + c_i\}_{i=1,\ldots,n}$$

respectively rotate and resize the edge $\tilde{e}_{i-1}$ to $\tilde{e}_i$. Since $\tilde{e}_0$ and $\tilde{e}_n$ are parallel, the compositions of these transformations must take the form $x \mapsto x + c$, thereby telling us that

$$\prod_{i=1}^{n} z_i = 1.$$

Now, since each of these z_i is either a complex edge parameter associated to an edge or its reciprocal, we obtain two relations for our edge parameters from the longitude and meridian of T. The relation obtained from any two loops representing the same homotopy class are the same[3], and since the

[3] Since we're concerned with the case that T is flat, holonomy is the same as monodromy. However, should one wish to consider this picture for a non-Euclidean cuspoidal surface - such as in the incomplete case, holonomy should be employed.

fundamental group of T is $\mathbb{Z} \oplus \mathbb{Z}$ generated by the longitude and meridian, these are the only (potentially) new relations that we can obtain via this method. By applying this argument to the figure eight knot complement, we get that:

$$h(l) = z^2(1-z)^2 = 1, \text{ and } h(m) = w(1-z) = 1,$$

where $h(l)$ and $h(m)$ denote the respective *holonomies* of the longitude and the meridian. Combining this with our previous relations yields the unique solution (with $\mathrm{Im}(z), \mathrm{Im}(w) > 0$) of

$$z = w = \exp(\frac{i\pi}{3}) = \frac{1+\sqrt{-3}}{2}$$

as being the geometric parameters we need to obtain a complete hyperbolic structure on M. (Note that we expect this uniqueness due to Mostow-Prasad Rigidity!)

4.3 Computing hyperbolic volume

We finish off by describing three ways of representing the volume of an ideal tetrahedron given its geometry. Let $\Delta(z)$ be an ideal tetrahedron in $\mathbb{H}^3$ with complex parameter $z \in \mathbb{C}$ and dihedral angles

$$\alpha = \arg(z), \ \beta = \arg(\frac{z-1}{z}) \text{ and } \gamma = \arg(\frac{1}{1-z}).$$

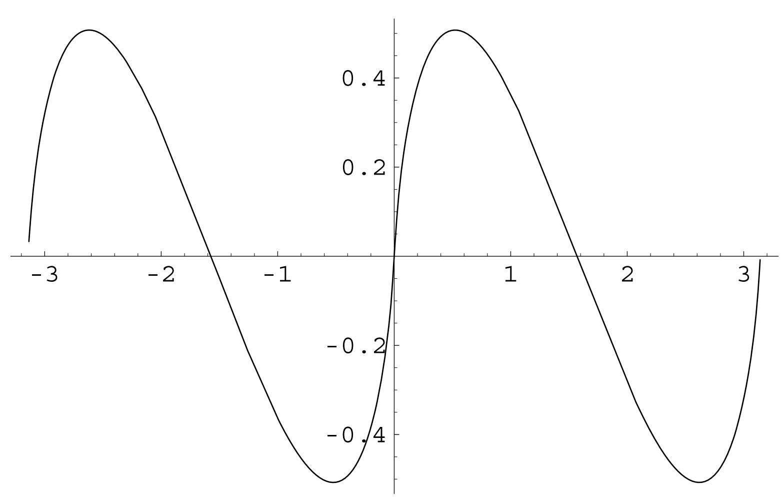
For our first method, by explicitly integrating the hyperbolic volume element, we obtain that:

$$\mathrm{Vol}(\Delta(z)) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma),$$

where

$$\Lambda(\theta) = -\int_0^\theta |\log(2\sin t)|\, dt$$

is the *Lobachevsky function* (See [Milnor]). Here's its graph:



Graph of Lobachevsky function for $-\pi \leq \theta \leq \pi$

Using this, we numerically compute the volume of the complete hyperbolic structure on the figure eight knot complement to be $6\Lambda(\pi/3) \approx 2.0298$.

Exercise 2. Prove that the regular ideal tetrahedron is the unique tetrahedra in $\mathbb{H}^3$ of maximal volume ($\approx$ 1.0149...). [Hint: Maximize $\Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma)$ subject to the constraint $\alpha + \beta + \gamma = \pi$.]

For our second method, notice that this volume can also be expresse nicely in terms of the *dilogarithm function*

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = -\int_0^z \frac{\log(1-w)}{w}\, dw, \quad \text{for } |z| \leq 1.$$

Making a change of variable $w = e^{2i\theta}$ in the last integral, then taking the imaginary part gives

$$\Lambda(\theta) = \frac{1}{2}\mathrm{Im}\ \mathrm{Li}_2(e^{2i\theta}) = \frac{1}{2}\sum \frac{\sin(2n\theta)}{n^2} \quad \text{for all } \theta.$$

Lastly, the *Bloch-Wigner dilogarithm* function

$$D(z) = \mathrm{Im}\ \mathrm{Li}_2(z) + \log|z| \arg(1-z)$$

turns out to be a well-defined real analytic function on $\mathbb{C} - \{0, 1\}$. It follows from an identity of Kummer implies that

$$D(z) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma).$$

Thus,

$$\mathrm{Vol}(\Delta(z)) = D(z).$$

5 Some references

J. Weeks, *Computation of hyperbolic structures in knot theory*, (I don't know how to cite Arxiv...well, either Arxiv or a proper book...?)
Available online at http://arxiv.org/abs/math/0309407.

W. Thurston, *The Geometry and Topology of Three-manifolds*, Princeton University lecture notes, 1980.
Available online at http://library.msri.org/books/gt3m.

W. Thurston, *Three-Dimensional Geometry and Topology*, Princeton University Press, 1997.

J. Milnor, *Hyperbolic geometry: the first 150 years*, Bulletin of the AMS 6 (1982), 9–24.

W. Neumann and D. Zagier, *Volumes of hyperbolic three-manifolds*, Topology 24 (1985), 307–332.