Volume Conjecture Seminar 2

Hyperbolic volumes of knot complements I

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Abstract. We will begin with a brief introduction to hyperbolic geometry and hyperbolic structures on three-manifolds. We will then explain how ideal triangulations can be used to compute hyperbolic structures and hyperbolic volumes for knot complements, illustrating this method for the figure-eight knot complement.

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0 Motivation

Throughout this talk, we will take K to be a knot in the three-sphere $S^3 = \mathbb{R}^3 \cup \{\infty\}$. We consider the non-compact three-manifold $S^3 \setminus K$ and refer to it as the *knot complement*. It is known from the work of Gordon and Luecke that the topology of the knot complement completely determines the knot up to a homeomorphism of S^3. Therefore, any topological invariant of the knot complement is an invariant of the corresponding knot. In particular, a consequence of the following theorem is the fact that the volume of the complement of a hyperbolic knot is a knot invariant. Recall that a *hyperbolic knot* is one whose complement admits a *hyperbolic structure* — that is, a complete metric of constant curvature -1.

Theorem (Thurston, late 1970s)**.** *Most knots — more precisely, those which are not torus knots or satellite knots — are hyperbolic.*[1]

A manifold which possesses a metric of constant curvature -1 can be locally modelled on hyperbolic space $\mathbb{H}^n$. For our purposes, we only need to consider the cases $n = 2$ and $n = 3$.

Theorem (Mostow–Prasad Rigidity)**.** *A hyperbolic knot complement has a unique hyperbolic structure up to isometry.*

More precisely, the above result is a corollary of Mostow–Prasad Rigidity, which states that the geometry of a finite-volume hyperbolic manifold of dimension greater than two is uniquely determined by its fundamental group. The key point here is that geometric properties derived from this unique hyperbolic structure — such as the volume — are topological invariants of $S^3 \setminus K$ and hence, topological invariants of K.

1 Some hyperbolic geometry

We first introduce the *upper half-space model* for the hyperbolic space $\mathbb{H}^3$. The underlying set for the model is $\{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$, endowed with the Riemannian metric

$$ds = \frac{\sqrt{dx^2 + dy^2 + dz^2}}{z}.$$

[1] Recall that a *torus knot* is one which lies on the surface of an unknotted torus. A *satellite knot* is one which is obtained by taking a knot lying non-trivially inside a solid torus and then knotting the solid torus.

We calculate the length $\ell(\gamma)$ of a path $\gamma : [a, b] \to \mathbb{H}^3$ with $\gamma(t) = (x(t), y(t), z(t))$ in this metric using the integral

$$\ell(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}}{z(t)} \, \mathrm{d}t.$$

We will often consider the boundary $\partial\mathbb{H}^3$ of $\mathbb{H}^3$ topologically as a sphere in the following way.

$$\{(x, y, z) \mid z = 0\} \cup \{\infty\} = \mathbb{R}^2 \cup \{\infty\} = \mathbb{C} \cup \{\infty\} = S^2$$

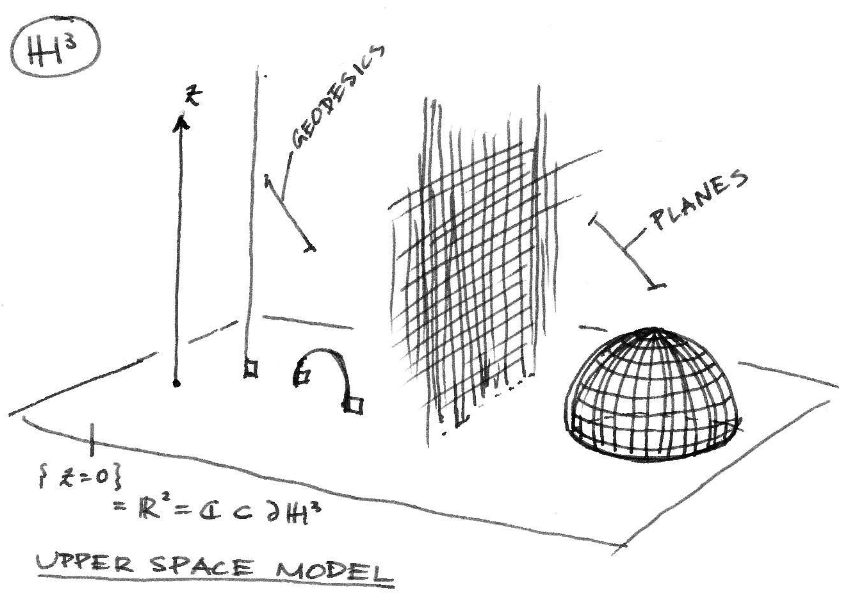
This is also known as the *sphere at infinity*, since the points of $\partial\mathbb{H}^3$ are at infinite distance from any point of $\mathbb{H}^3$.

In the upper half-space model, we have the following facts.

- *Hyperbolic angles are equal to Euclidean angles*
  The angles measured using the given Riemannian metric on the upper half-space are equal to the angles measured using the standard Euclidean metric on the upper half-space.
- *Hyperbolic lines are Euclidean rays and semicircles perpendicular to the boundary.*
  Note that a hyperbolic line is a complete geodesic in $\mathbb{H}^3$. Euclidean rays perpendicular to the boundary may be thought of as semicircles perpendicular to the boundary with infinite radius and one endpoint at ∞.
- *Hyperbolic planes are Euclidean planes and hemispheres perpendicular to the boundary.*
  Note that a hyperbolic plane is a complete, totally geodesic two-dimensional subspace in $\mathbb{H}^3$.

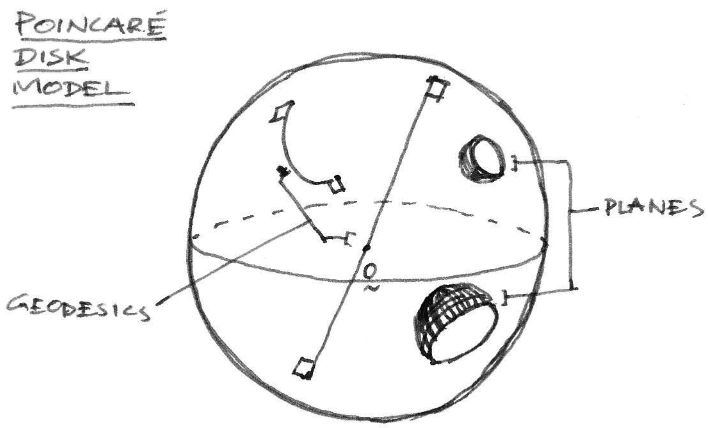
The volume form induced by the Riemannian metric on $\mathbb{H}^3$ is given by

$$\mathrm{d}V = \frac{\mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z}{z^3}.$$



2 The Poincaré disk model

It is also sometimes useful to know about the Poincaré disk model for the hyperbolic space $\mathbb{H}^3$.



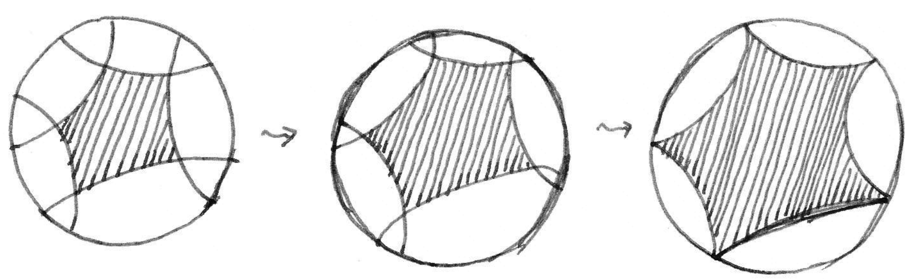
The underlying set for the model is the open unit ball in $\mathbb{R}^3$ with Riemannian metric

$$ds = \frac{2}{1 - x^2 - y^2 - z^2} \sqrt{dx^2 + dy^2 + dz^2},$$

and the boundary $\partial\mathbb{H}^3$ corresponds to the boundary of the ball. We also introduce

- *polygons*, which are two-dimensional regions bounded by hyperbolic lines lying in a hyperbolic plane; and
- *polyhedra*, which are three-dimensional regions bounded by hyperbolic planes.

We also have a notion of *ideal polygons* and *ideal polyhedra*, which are polygons and polyhedra with their vertices lying on the sphere at infinity.



Ideal polygons and polyhedra are not compact, but have finite hyperbolic area and volume, respectively. The following exercise may be used to prove this fact for polyhedra, and an analogous exercise suffices for the case of polygons.

Exercise. For $S \subseteq \mathbb{R}^2$, show that the hyperbolic volume of the set

$$R = \{(x, y, z) \in \mathbb{H}^3 \mid (x, y) \in S \text{ and } z \geq 1\}$$

in the upper half-space model is equal to half the Euclidean area of S.

3 Isometries of hyperbolic space

Every isometry of $\mathbb{H}^3$ extends continuously to the sphere at infinity, and in fact, gives a conformal diffeomorphism of $\partial\mathbb{H}^3$. This yields a homomorphism ϕ from the group of orientation preserving isometries of $\mathbb{H}^3$ to the group of Möbius transformations. This group is equal to $PSL_2(\mathbb{C}) = SL_2(\mathbb{C})/\langle \pm I \rangle$ via the isomorphism

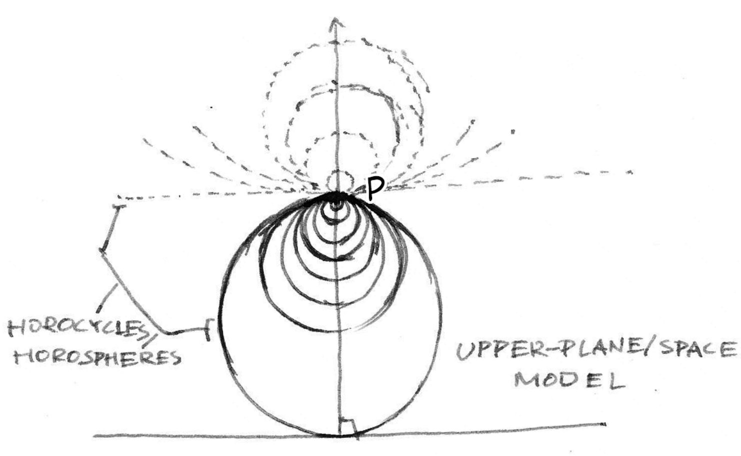
$$f(z) = \frac{az + b}{cz + d} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We can use the fact that any point in $\mathbb{H}^3$ is the intersection of three hyperbolic planes to prove that the map ϕ is injective. Moreover, by writing down explicit formulas for how an arbitrary element of $PSL_2(\mathbb{C})$ acts on $\mathbb{H}^3$ — for example, by using quaternions — we can show that the map ϕ is surjective. Therefore, the group of orientation preserving isometries of $\mathbb{H}^3$ is naturally isomorphic to the group of Möbius transformations.

$$\mathrm{Isom}^+(\mathbb{H}^3) \cong PSL_2(\mathbb{C})$$

4 Horospheres

Fix a point $P \in \mathbb{H}^3$ and consider a family of Euclidean spheres through P with varying radius. These are actually hyperbolic spheres with constant positive intrinsic curvature and intrinsic spherical geometry. In the limit as the radius of the hyperbolic sphere approaches infinity, we obtain a *horosphere*. Horospheres have zero intrinsic curvature and intrinsic Euclidean geometry. In the upper half-space model, they are represented by Euclidean spheres which are tangent to the plane $\{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$ or by Euclidean planes given by the equation $z = \textit{constant}$.



5 Computing hyperbolic structures

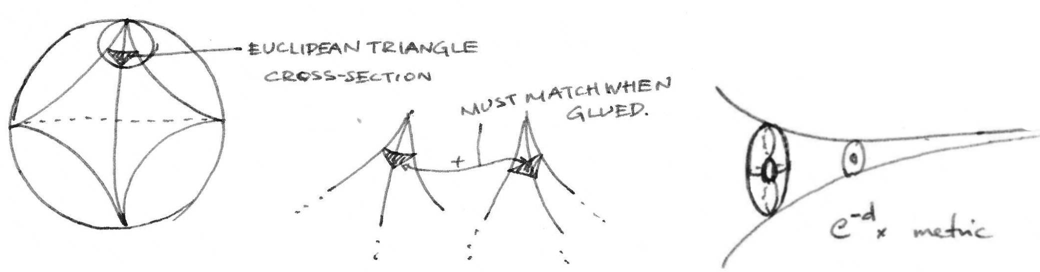
The program SnapPea, written by Jeff Weeks, is able to calculate the volume of a specified knot complement to high precision. It uses a procedure outlined by Thurston for taking a knot and producing an ideal triangulation of its complement. By an *ideal triangulation*, we mean a way to divide the knot complement into tetrahedra whose vertices are ideal. Such a tetrahedron is homeomorphic to a compact tetrahedron with its vertices removed.

The algorithm to find the hyperbolic structure of a three-manifold can be broken down into two steps.

- Use Thurston's procedure to obtain a topological ideal triangulation of your manifold.
- Find hyperbolic structures for these ideal tetrahedra so that they fit together correctly to give a complete hyperbolic metric on your manifold.

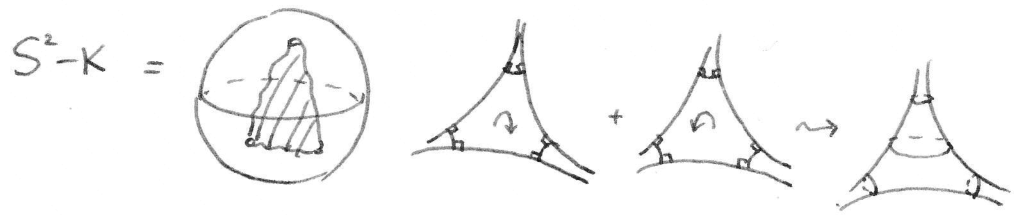
To obtain a complete hyperbolic metric, the hyperbolic ideal tetrahedra obtained from the second step need to satisfy the following conditions.

- *Edge condition*
  The sum of the dihedral angles at each edge is equal to 2π, and when we go around an edge, we return to the same point rather than a point translated along the edge. Note that this is a local condition around an edge in the ideal triangulation.
- *Completeness condition*
  It must be possible to choose a horospherical triangle around each ideal vertex so that they fit together to give a closed Euclidean surface. Such surfaces have Euler characteristic zero by the Gauss-Bonnet theorem, so must be tori — or Klein bottles, in the non-orientable case — whose Euclidean metrics shrink exponentially fast as you move towards the cusp. Note that this is a global condition on the way that the ideal tetrahedra are glued together.



To finish, we consider an example of a lower-dimensional analogue of this procedure.

Example. Consider the 0-dimensional submanifold $K = \{\text{three points}\} \subseteq S^2$. There exists a complete hyperbolic metric on $S^2 \setminus K$ which can be constructed as follows. The basic idea is to topologically decompose $S^2 \setminus K$ into two ideal triangles. Geometrically, we can take two ideal triangles in $\mathbb{H}^2$ and glue them together by *doubling*. More precisely, take two ideal triangles with a choice of horocycle at each of their three cusps. Then glue these triangles together so that the first forms the front of $S^2 \setminus K$ and the second forms the back.



6 Some references

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