THE UNIVERSITY OF MELBOURNE

DOCTORAL THESIS

Moduli Spaces of Surfaces

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Abstract

We give a generalisation of λ-length coordinates for the Teichmüller space of bordered hyperbolic surfaces, before defining moduli spaces and Techmüller spaces for crowned surfaces and establishing mixed λ-length and Fenchel-Nielsen coordinates for the latter. We then define a mapping class group invariant Weil-Petersson 2-form and find a presentation for it in terms of mixed coordinates. We also prove McShane identities for crowned surfaces, closed surfaces with one marked point and quasi-Fuchsian representations of the thrice-punctured projective plane. Finally, we geometrically interpret Bowditch-type proofs of the McShane identity using the ideal Ptolemy relation.

Declaration

This is to certify that:

i) the thesis comprises only my original work towards the degree of Doctor of Philosophy, except where indicated in the Preface,

ii) due acknowledgement has been made in the text to all other material used,

iii) the thesis is fewer than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices

Signed: ______________________________

Date: ______________________________

Acknowledgements

First of all, I thank my parents and grandparents. Life has been extraordinarily good to me, and I trust that their manifold sacrifices have had a hand in this.

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[1] On the other hand, feel free to name me if I do something bad or stupid. Go all out!

Preface

Chapter 1 provides some of the background for Riemann surface theory and hyperbolic geometry. We walk through a proof of the uniformisation theorem given in [OPS88], highlighting a conjectured generalisation (Conjecture 1.1) of the uniformisation theorem.

Chapter 2 defines the various moduli spaces we'll be working with and begins with a review of some classical coordinate systems before introducing novel coordinate systems on various families of moduli spaces. Everything starting from Subsubsection 2.3.2.2 is original work.

Chapter 3 begins with a quick review of some well-known presentations for the Weil-Petersson form in different coordinate systems. In Section 3.1, I define a mapping class group invariant Weil-Petersson 2-form for moduli spaces of crowned surfaces, and find an explicit presentation for this form in terms of coordinates. Section 3.2 is also the product of my own research, although only Proposition 3.15 is possibly new — the goal of this section is to give an appreciation for Section 3.3: where we explain Mirzakhani's strategy for computing general moduli space volumes.

Chapter 4 is a combination of my own research (Section 4.2, Section 4.3), results obtained from my collaboration with Paul Norbury (Subsection 4.4.2) as well as a collection of my thoughts on existing results in the literature (Section 4.1, Subsection 4.4.1, Section 4.5). Note that Section 4.3 and Subsection 4.4.2 are essentially statements of the main results of my preprints [Hua12, HN13] accompanied by rough sketches of how these results are proved.

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Introduction

0.1 A Little History

The study of Riemann's moduli problem — the problem of how one might parametrise the space $\mathcal{M}(R)$ of complex structures on a given topological surface R, led naturally to the development of Teichmüller theory. Specifically, the Teichmüller space $\mathcal{T}(R)$ of a surface R is the universal cover of Riemann's moduli space $\mathcal{M}(R)$.

Teichmüller originally conjectured and "proved" that the Teichmüller space $\mathcal{T}(R)$ is homeomorphic to an open ball $\mathbb{R}^{3|\chi(R)|}$ [Tei82]. This was later rigorously proven by Ahlfors (and later, Bers) [Ahl54, AB61] using quasiconformal maps, the uniformisation theorem [Kle83, Koe09, Poi08] and the theory of Fuchsian groups. This identification has since been concretely expressed in terms of global coordinate charts such as the Fenchel-Nielsen coordinates [Abi80], Penner's λ-length coordinates [Pen87] and Thurston's shearing lengths [FLP12].

Ahlfors further proved in [Ahl60] that Teichmüller spaces are naturally complex manifolds, and that the Weil-Petersson metric [Wei58] is a natural Hermitian metric on $\mathcal{T}(R)$. Its induced Hermitian form — the Weil-Petersson form is a Kähler 2-form on $\mathcal{T}(R)$ [Ahl61], and simple expressions have been found for it in the three aforementioned coordinate systems [Wol83c, Pen92, SB01].

Since the Weil-Petersson form is a mapping class group invariant symplectic form, its top dimensional wedge product is a volume form on Teichmüller space $\mathcal{T}(R)$ that descends to the moduli space $\mathcal{M}(R)$. In [Mir07a], Mirzakhani shows that the Weil-Petersson volume of the moduli space $\mathcal{M}(R, \mathbf{L})$ of boundary length $\mathbf{L} = (L_1, \ldots, L_m)$ hyperbolic surfaces homeomorphic to R is a polynomial in $L_1^2, \ldots, L_m^2$ with coefficients in $\mathbb{Q}[\pi^2]$. Interpreting these coefficients as the intersection numbers of Mumford-Miller-Morita (cohomology) classes on compactified moduli spaces, she gave a new proof of Witten's conjecture [Mir07b]. Key to her volume integrations were McShane identities [McS98, Mir07a] — an infinite sum for hyperbolic surfaces R, whose summands depend on the geometry of R, but whose total does not.

0.2 Main Results

We list some of our main results in order of appearance. The first five results are proved in this thesis, whilst we only give sketches of proof for the remainder; the actual proofs for these latter results may be found in [Hua12] and [HN13]. Note that our main results differ slightly in presentation here compared with how they appear in the main body of the thesis, as we've attempted to avoid introducing some notation at this point in the thesis.

A natural generalisation of Penner's (horocycle-)decorated Teichmüller spaces of cusped hyperbolic surfaces [Pen87] to a decorated Teichmüller space of geodesic bordered hyperbolic surfaces is to attach equidistant curves to each boundary component. Such curves are called *hypercycles*, and the *(hypercycle-)decorated Teichmüller space* $\hat{\mathfrak{T}}(R)$ for a bordered hyperbolic surface R (with labelled borders) is defined as:

$$\hat{\mathfrak{T}}(R) := \left\{ (S,f,\eta) \;\middle|\; \begin{array}{l} S \text{ is a geodesic bordered hyperbolic surface, and} \\ f: R \to S \text{ is a label-preserving homeomorphism,} \\ \eta \text{ is a set consisting of one horocycle for each cusp} \\ \text{and one hypercycle for each closed border} \end{array} \right\}$$

identified under the equivalence $\sim_{\hat{\mathfrak{T}}}$, where $(S_1,f_1,\eta_1) \sim_{\hat{\mathfrak{T}}} (S_2,f_2,\eta_2)$ if and only if $f_2 \circ f_1^{-1} : S_1 \to S_2$ is homotopy equivalent to an isometry and the length of the horocycles and hypercycles of η_1 and η_2 agree for corresponding cusps and closed borders on S_1 and S_2.

Our first main result is a coordinate system on the above decorated Teichmüller space based on Penner's λ-length coordinates.

Definition 0.1. *We refer to simple, bi-infinite geodesics on a hyperbolic surface as* ideal geodesics.

Fix a collection $\triangle = \{\sigma_1, \ldots, \sigma_{6g-6+3m+3n}\}$ of ideal geodesics which decompose R into ideal triangles. We require that the ends of each σ_i either goes up a cusp or spirals into a geodesic boundary so that it agrees with the orientation of the boundary as imposed by the orientation of R. For an arbitrary element $[S,f,\eta]$ of the decorated Teichmüller space $\hat{\mathfrak{T}}(R)$, let $f_{\#}\triangle$ denote the collection of ideal geodesics on S obtained by isotopically "pulling-tight" each curve $f \circ \sigma_i$ to an ideal geodesic $f_{\#}\sigma_i$. The collection η of horocycles and hypercycles on S truncates the ideal geodesics in $f_{\#}\triangle$, and denote $\exp(\frac{1}{2}\cdot)$ of length of the truncated $f_{\#}\sigma_i$ by $\lambda_i([S,f,\eta])$. These functions $\lambda_i : \hat{\mathfrak{T}}(R) \to \mathbb{R}_+$ are referred to as λ-*lengths*. In addition, let $L_j : \hat{\mathfrak{T}}(R) \to \mathbb{R}_+$ assign to a decorated marked surface $[S,f,\eta]$ the length of the j-th geodesic border of S.

Theorem 2.19. *The following function is a homeomorphism:*

$$\begin{aligned} \Lambda_b : \hat{\mathfrak{T}}(R) &\to \mathbb{R}_+^{6g-6+3m+3n} \times \mathbb{R}_+^m \\ [S,f,\eta] &\mapsto (\lambda_1, \ldots, \lambda_{6g-6+3m+n}, L_1, \ldots, L_m), \end{aligned}$$

we call Λ_b the (generalised) λ-length coordinates.

Definition 0.2. *A* crowned hyperbolic surface R *is a complete finite-area (possibly non-closed) geodesic-bordered hyperbolic surface. We further require that crowned hyperbolic surface contain either*

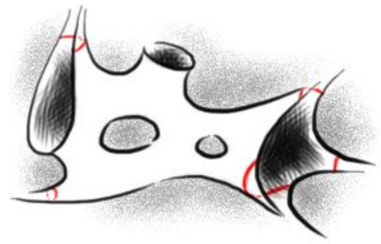


FIGURE 0.1: A crowned hyperbolic surface with decorated cusps and tines.

at least one cusp or a half-cusp called a tine. *Specifically, tines are ideal vertices of* R *and are locally modelled on a vertex of an ideal triangle. In addition, we refer to ideal geodesic boundaries joining (possibly distinct) tines as* arches.

Like cusps, the tines of a crowned hyperbolic surface may be decorated with horocyclic segments (Figure 0.1). And the *partially decorated Teichmüller space* $\acute{\mathfrak{T}}(R)$ of a crowned hyperbolic surface R (with labelled cusps, tines and closed borders) is defined as:

$$\acute{\mathfrak{T}}(R) := \left\{ (S, f, \eta) \;\middle|\; \begin{array}{l} S \text{ is a crowned hyperbolic surface,} \\ f : R \to S \text{ is a label-preserving homeomorphism, and} \\ \eta \text{ is a set consisting of one horocycle for each cusp} \\ \text{and one horocyclic segment for each tine} \end{array} \right\}$$

identified under the equivalence $\sim_{\acute{\mathfrak{T}}}$, where $(S_1, f_1, \eta_1) \sim_{\acute{\mathfrak{T}}} (S_2, f_2, \eta_2)$ if and only if $f_2 \circ f_1^{-1} : S_1 \to S_2$ is isotopy equivalent to an isometry, and the length of the horocycles and horocyclic segments of η_1 and η_2 agree for corresponding cusps and tines on S_1 and S_2.

We define and describe *mixed coordinates* for the partially decorated Teichmüller space $\acute{\mathfrak{T}}(R)$ — a coordinate system that is part Fenchel-Nielsen and part λ-length. Given a collection

$$(\Gamma, A) = ((\gamma_1, \ldots, \gamma_M), (\sigma_1, \ldots, \sigma_N))$$

of disjoint simple closed geodesics Γ and (simple) ideal geodesics A which decompose a crowned hyperbolic surface R into ideal triangles and pairs of half-pants, let $f_\# \gamma_j$ denote the simple closed geodesic representative of $f \circ \gamma_j$ on S.

Theorem 2.25. *The following function gives a global coordinate system on* $\acute{\mathfrak{T}}(R)$*:*

$$\begin{aligned} \Lambda_{\Gamma,A} : \acute{\mathfrak{T}}(R) &\to (\mathbb{R}_+ \times \mathbb{R})^M \times \mathbb{R}_+^N \\ [S, f, \eta] &\mapsto (\ell_1, \tau_1, \ldots, \ell_M, \tau_M, \lambda_1, \ldots, \lambda_N), \end{aligned}$$

where λ_i *is the* λ*-length for* $f_\# \sigma_i$ *on* S*, and* ℓ_j, τ_j *are respectively the length and twist parameters for* $f_\# \gamma_j$ *on* S.

We further show that mixed coordinates induce a real-analytic structure on the partially decorated moduli space $\acute{\mathcal{M}}(R)$ — which has $\acute{\mathfrak{T}}(R)$ as its (orbifold) universal cover.

Theorem 2.26. *The mapping class group* $\mathrm{Mod}(R)$ *acts real-analytically on* $\acute{\mathcal{T}}(R)$ *with respect to mixed coordinates. Equivalently: the partially decorated moduli space* $\acute{\mathcal{M}}(R)$ *is a real-analytic orbifold/manifold with respect to mixed coordinates.*

In Definition 3.4, we describe a mapping class group invariant 2-form $\omega_{WP}(R)$ on the Teichmüller space $\mathcal{T}(R)$. The partially decorated Teichmüller space $\acute{\mathcal{T}}(R)$ is a fiber bundle over $\mathcal{T}(R)$, this 2-form $\omega_{WP}(R)$ pulls back to a 2-form on $\acute{\mathcal{T}}(R)$ with the following presentation:

Corollary 3.6. *Let* $\Lambda_{\Gamma,A}$ *be a mixed coordinate system on the Teichmüller space* $\mathcal{T}(R) = \mathcal{M}(R,(\Gamma,A))$ *of a crowned surface. Let* $\ell_1,\ldots,\ell_M$ *denote the length parameters for the collection of closed geodesics* $\Gamma = (\gamma_1,\ldots,\gamma_M)$*, and let* $\tau_1,\ldots,\tau_M$ *denote a collection of corresponding twist parameters for* $\mathcal{T}(R)$*. Further let* $T_1,\ldots,T_p$ *be the resulting ideal triangles from cutting up* R *along the geodesic representatives of* A*. Then the (mapping class group invariant) Weil-Petersson form* $\omega_{WP}(R)$ *is given by:*

$$\begin{aligned}&2\sum_{i=1}^{p}\left(d\log\lambda_{i,1}\wedge d\log\lambda_{i,2}+d\log\lambda_{i,2}\wedge d\log\lambda_{i,3}+d\log\lambda_{i,3}\wedge d\log\lambda_{i,1}\right)\\&+\sum_{j=1}^{M}d\ell_j\wedge d\tau_j,\end{aligned}\tag{1}$$

where $\lambda_{i,1},\lambda_{i,2},\lambda_{i,3}$ *are the* λ*-lengths of the ideal geodesics constituting the sides of* T_i *ordered in the opposite order to the orientation.*

When R is a (non-strictly) crowned hyperbolic surface with only cusps and closed geodesic boundaries, the 2-form $\omega_{WP}(R)$ agrees with the Weil-Petersson form for the Teichmüller space $\mathcal{T}(R)$. This is our justification for regarding $\omega_{WP}(R)$ as the Weil-Petersson form on the Teichmüller space $\mathcal{T}(R)$ of a crowned hyperbolic surface R.

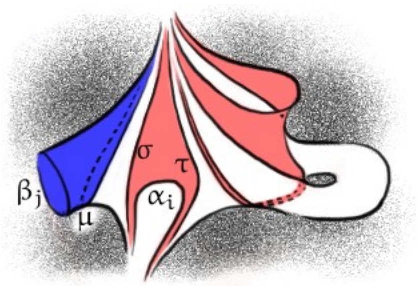


FIGURE 0.2: Examples of embedded half-pants and ideal triangles on a crowned surface.

We further prove a McShane identity for crowned hyperbolic surfaces:

Theorem 4.5. *Given a crowned surface* S *with* m *closed boundary geodesics* $(\beta_1,\ldots,\beta_m)$ *of lengths* $(L_1,\ldots,L_m)\in\mathbb{R}^m_{\geqslant 0}$ *and* k *boundary ideal geodesics* $(\alpha_1,\ldots,\alpha_k)$*. We partially decorate* S *with length 1 horocycles at the cusps of* S *and length* 1 *horocyclic segment at the tines of* S*; let*

- $\mathcal{S}_i$ *be the collection of embedded ideal triangles with the* α_i *opposite to tine 1 (left figure in Figure 4.3), each ideal triangle is denoted by the two bi-infinite geodesics* $\{\sigma,\tau\}$ *adjacent to tine* 1*;*

- $\mathcal{P}_j$ *be the collection of embedded half-pants with its tine based at tine 1 and with boundary* j *as a boundary component (center figure in Figure 4.3), each such pair of half-pants is denoted by its bi-infinite geodesic boundary* μ;
- $\mathcal{P}$ *be the collection of all*[1] *embedded half-pants with its tine based at tine 1 (center and right figures in Figure 4.3), each pair of half-pants is denoted by* $\{\gamma, \gamma_\infty\}$*, where* γ *is the closed geodesic boundary and* γ_∞ *is the bi-infinite geodesic boundary of this pair of half-pants.*

Then,

$$1 = \sum_{i=1}^{k} \sum_{\{\sigma,\tau\}\in\mathcal{S}_i} e^{\frac{1}{2}(\ell_{\alpha_i}-\ell_\sigma-\ell_\tau)} + \sum_{j=1}^{m} \sum_{\mu\in\mathcal{P}_j} 2e^{\frac{-1}{2}\ell_\mu}\sinh(\tfrac{L_j}{2}) + \sum_{\{\gamma,\gamma_\infty\}\in\mathcal{P}} 2e^{\frac{-1}{2}(\ell_\gamma+\ell_{\gamma_\infty})},$$

where for any ideal geodesic β*, the positive number* ℓ_β *denotes the length of* β *truncated at the length* 1 *horocycles at the cusps and tines of* S.

The above McShane identity differs from McShane's original identities in that we sum over embedded pairs of half-pants and ideal triangles instead of embedded pairs of pants. This idea that we can sum over pairs of half-pants is also the basis for our McShane-type identity [Hua12] for closed hyperbolic surfaces with one marked point p, or equivalently: closed hyperbolic surfaces with one 2π cone-point. The derivation of our identity involves classifying the 2π direction's worth of geodesics rays launched from p according to the behaviour of these geodesics. To do so, we first prove a generalisation (Theorem 4.10) of the Birman-Series geodesic sparsity theorem [BS85, TWZ06], hence showing that geodesics launched from p generically self-intersect:

Corollary 4.11. *Given a complete finite-volumed hyperbolic surface* S *and any countable collection of points* $C \subset S$*, the set of points which lie on geodesics possibly broken at points in* C *has zero Lebesgue measure.*

The entire statement of the following McShane identity is quite involved, and is given in Section 4.3. We presently confine ourselves to an expurgated version:

Theorem 4.12. *Given a closed hyperbolic surface* S *with marked point* p*, let* $\mathcal{HP}$ *denote the collection of (lasso-induced) immersed half-pants with at* p*. Define the real function* $\mathrm{Gap} : \mathcal{HP} \to [0,\pi]$ *to output the total angle of all the directions from* p *that shoot out geodesics which lie in* P *up to their first self-intersection. Then:*

$$\sum_{P\in\mathcal{HP}} \mathrm{Gap}(P) = 2\pi,$$

where the Gap *function generically (i.e. for* embedded *pairs of half-pants) takes the form:*

$$2\mathrm{arcsin}\left(\frac{\cosh(\frac{\ell_\gamma}{2})}{\cosh(\frac{\ell_{\gamma_P}}{2})}\right) - 2\mathrm{arcsin}\left(\frac{\sinh(\frac{\ell_\gamma}{2})}{\sinh(\frac{\ell_{\gamma_P}}{2})}\right).$$

[1] Including those in $\mathcal{P}_j$.

For immersed half-pants P*, the function* Gap *evaluates to a smaller value. As previously mentioned, please refer to this result in Section 4.3 for the precise expression of* Gap *for strictly immersed pairs of half-pants.*

The following results come from our paper with Paul Norbury [HN13] studying simple geodesics on 3-cusped projective planes. Our paper closely follows an algebraic (trace-based) approach taken by Bowditch [Bow98] to generalise McShane's original identity [McS91] to quasi-Fuchsian representations of the fundamental group of a 1-punctured torus.

Theorem 4.17. *For any 3-cusped hyperbolic projective plane, the length of its shortest geodesic is at most* $2\operatorname{arcsinh}(2)$*. Moreover, there is a unique 3-cusped hyperbolic projective plane that has a shortest geodesic of length* $2\operatorname{arcsinh}(2)$*.*

Theorem 4.19. *Let* X *be the hyperbolic manifold corresponding to a quasi-Fuchsian representation of* $\pi_1(S)$*, then:*

$$1 = \sum_{\gamma} \frac{2}{1+e^{\frac{1}{2}\ell_\gamma}},$$

where the sum is taken over the collection of 2-sided simple closed geodesics γ *on* X*.*

We also determined the structure of the moduli space of 3-cusped projective planes, although it is a bit of a mouthful to state.

Theorem 4.21. *The moduli space* $\mathcal{M}(S)$ *of 3-cusped projective planes is homeomorphic to an open 3-ball with an open hemisphere of order 2 orbifold points glued on, and a line of orbifold points running straight through the center of this 3-ball — joining two antipodal points of the glued on orbifold hemisphere. The orbifold points on this line are of order 2, except for the very center of the 3-ball, which is order 4.*

Chapter 1

Surfaces

A very basic understanding of Riemann surfaces (complex manifolds of $\mathbb{C}$-dimension 1) and hyperbolic surfaces (constant -1 Gaussian curvature surfaces) is assumed of the reader. And unless otherwise specified, all surfaces that we deal with in this thesis are connected, oriented and have (finite) negative Euler characteristic.

Definition 1.1. Riemann surfaces of finite type *are Riemann surfaces with finite genus and finitely many smooth boundary components and punctures, where the punctures may be on either the interior of the surface or on the boundary.*

We think of a Riemann surface as containing its boundary components but not its puncture points. In addition, we label these punctures and borders with distinct positive integers. The equivalence of categories between:

- the category $\mathcal{R}\mathrm{mn}$ of (boundary-labelled) Riemann surfaces of finite type, with holomorphic maps as morphisms and
- the category $\mathcal{H}\mathrm{yp}$ of (boundary-labelled) finite-area geodesic bordered hyperbolic surfaces, with isometries as morphisms

endows a Riemann surface with a canonical hyperbolic metric, and translates features of and objects on Riemann surfaces into the language of hyperbolic surfaces and vice versa. For example:

- interior punctures $\Leftrightarrow$ cusps,
- unpunctured smooth borders $\Leftrightarrow$ closed geodesic boundaries,
- essential homotopy classes of free loops $\Leftrightarrow$ closed geodesics. (Lemma 2.9)

The definition of finite type Riemann surfaces that we've adopted differs from convention [Abi80] in one important regard: we've allowed for Riemann surfaces with boundary punctures. The equivalent object in $\mathcal{H}\mathrm{yp}$ is a *crowned hyperbolic surface* (Definition 0.2): finite-area geodesic bordered hyperbolic surfaces with cusps, closed geodesic boundaries and bi-infinite geodesic boundaries. Recall also from Definition 0.2 that the arcs lying on a k-punctured border are uniformised to k bi-infinite boundary geodesics called *arches*, and the k boundary punctures themselves are uniformised to k *tines* on this crowned boundary (Figure 0.1).

In our dealings with hyperbolic surfaces, we speak of horocycles and hypercycles of arbitrarily large length. To properly define these objects, we consider the universal cover $\tilde{R}$ of a hyperbolic surface R as sitting in the Poincaré disk model of $\mathbb{H}$.

Definition 1.2. *On the hyperbolic plane* $\mathbb{H}$*, a* horocycle *is a complete constant geodesic curvature* 1 *path and a* hypercycle *is a complete constant geodesic curvature* $\kappa \in (0, 1)$ *path. Whereas, on a hyperbolic surface* R*,*

- horocycles *are immersed curves which travel once around a cusp of* R *and lift up to horocycles in the universal cover* $\tilde{R} \subset \mathbb{H}$ *of* R*;*
- hypercycles *are immersed curves which travel once around a closed geodesic boundary of* R *and lift up to a hypercycle in the universal cover* $\tilde{R} \subset \mathbb{H}$ *of* R*.*

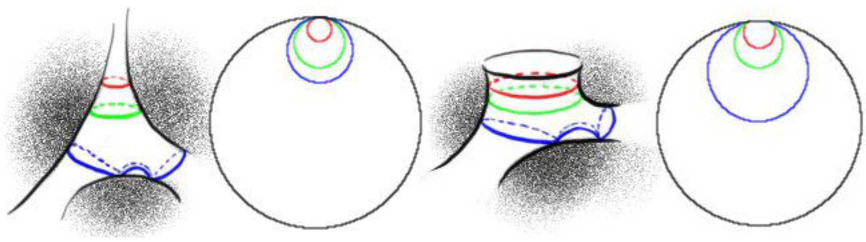


FIGURE 1.1: Horocycles (left) and hypercycles (right).

Strictly speaking, the above definition is incorrect because we wish to allow for horocycles (and hypercycles) to be arbitrarily long, and horocycles which are too long run the risk of running into the boundaries of R. We introduce an alternative description given in terms of the *Nielsen extension* $\overline{R}$ of R: classically constructed by attaching to each closed geodesic boundry of R an infinite hyperbolic trumpet as in Figure 1.2.

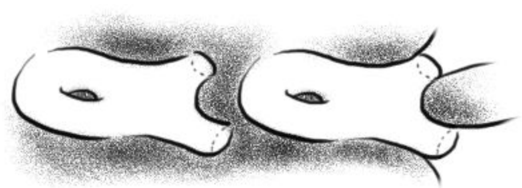


FIGURE 1.2: The Nielsen extension of a hyperbolic surface.

Definition 1.2′. *A* horocycle *around a cusp is an immersed (primitive) constant geodesic curvature* 1 *closed curve on* $\overline{R}$ *that may be homotoped into this cusp. Similarly, a* hypercycle *around a boundary is an immersed (primitive) constant geodesic curvature* $\kappa \in (0, 1)$ *closed curve on* $\overline{R}$ *that may be homotoped into this geodesic boundary component of* $R \subset \overline{R}$*.*

Further, when dealing with R with crowned boundary components, we can glue an orientation-reversed copy of the trumpet-attached $\overline{R}$ to itself along its crowned boundaries. The resulting surface $d\overline{R}$ contains R and is complete, hence has $\mathbb{H}$ as its universal cover.

Definition 1.2''. *A* horocyclic segment *around a tine on a crowned hyperbolic surface R is the restriction to R of a horocycle on the (Schottky-)doubled Nielsen-extension surface $d\overline{R}$.*

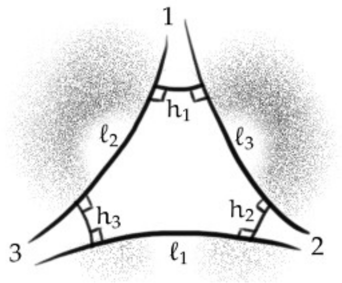


FIGURE 1.3: An ideal triangle with horocyclic segments decorating its vertices.

A working definition is that horocyclic segments around a tine are constant geodesic curvature 1 paths which separate this tine from the rest of the surface. Since we work a great deal with ideal triangles, we make a special mention of the following useful facts.

Proposition 1.3 (Penner, Proposition 2.8 of [Pen87])**.** *Given an ideal triangle with ideal vertices/tines labelled $1, 2, 3$ respectively surrounded by horocyclic segments of length h_1, h_2, h_3, and let ℓ_i be the horocycle-truncated length opposite to vertex i (Figure 1.3). Then*

$$h_1 = e^{\frac{1}{2}(\ell_1-\ell_2-\ell_3)},\ h_2 = e^{\frac{1}{2}(\ell_2-\ell_1-\ell_3)},\ \textit{and}\ h_3 = e^{\frac{1}{2}(\ell_3-\ell_1-\ell_2)}. \tag{1.1}$$

Proposition 1.4 (Penner, Proposition 2.6(a) of [Pen87])**.** *Given an ideal quadrilateral with tines cyclically labelled $1, 2, 3, 4$ respectively decorated by horocyclic segments of arbitrary length in $\mathbb{R}_+$. Let σ_{ij} denote the unique geodesic joining ideal vertices/tines i and j, and let ℓ_{ij} denote the* signed *horocycle-truncated length of σ_{ij}.*[1] *Then,*

$$e^{\frac{1}{2}\ell_{13}}e^{\frac{1}{2}\ell_{24}} = e^{\frac{1}{2}\ell_{12}}e^{\frac{1}{2}\ell_{34}} + e^{\frac{1}{2}\ell_{14}}e^{\frac{1}{2}\ell_{23}}. \tag{1.2}$$

The above identity is known as the ideal Ptolemy relation*.*

We occasionally refer to the act of replacing one of the diagonals of an ideal quadrilateral with the other as *diagonal-flipping*, or just *flipping*[2].

Note 1.1. The ideal Ptolemy relation may be equivalently stated as:

$$\frac{e^{\frac{1}{2}\ell_{24}}}{e^{\frac{1}{2}\ell_{12}}e^{\frac{1}{2}\ell_{14}}} = \frac{e^{\frac{1}{2}\ell_{23}}}{e^{\frac{1}{2}\ell_{12}}e^{\frac{1}{2}\ell_{13}}} + \frac{e^{\frac{1}{2}\ell_{34}}}{e^{\frac{1}{2}\ell_{13}}e^{\frac{1}{2}\ell_{14}}}. \tag{1.3}$$

Proposition 1.3 allows us to geometrically interpret equation (1.3) in terms of cutting the tine 1 horocyclic segment of length $e^{\frac{1}{2}(\ell_{24}-\ell_{12}-\ell_{14})}$ along σ_{13} into two horocyclic segments of lengths $e^{\frac{1}{2}(\ell_{23}-\ell_{12}-\ell_{13})}$ and $e^{\frac{1}{2}(\ell_{34}-\ell_{13}-\ell_{14})}$ (Figure 1.4).

[1] That is: ℓ_{ij} is the length of the segment of σ_{ij} joining the vertex i and the vertex j horocyclic segments. It's taken to be positive if these two horocycles are disjoint and negative if they intersect.

[2] Not to be confused with curve-flipping, which is described in Chapter 4.

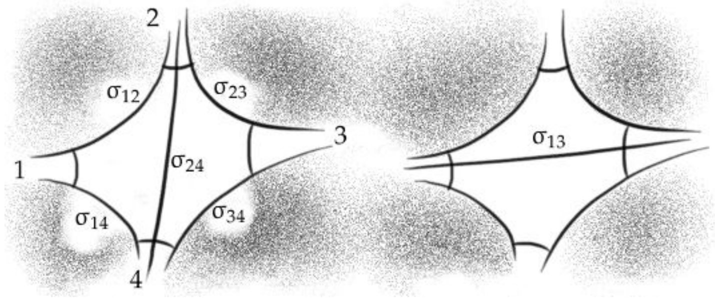


FIGURE 1.4: Two triangulations of an ideal quadrilateral related by a diagonal flip.

Ideal triangles are fundamental to hyperbolic geometry because they're "building-blocks" for other hyperbolic surfaces. Another building-block hyperbolic surface that we refer to a great deal is a *pair of half-pants*.

Definition 1.5. *A(n ideal)* pair of half-pants *is a crowned hyperbolic surface homeomorphic to an annulus with a puncture on one of its boundaries (blue subsurface in Figure 0.2). Thus, a pair of half-pants* P *has one closed geodesic boundary and one ideal geodesic boundary, we refer to them respectively as the* cuff *of* P *and the* zipper *of* P.

Note 1.2. Consider doubling a pair of half-pants P with cuff γ to a pair of pants dP by taking P and an orientation-reversed copy of P and gluing them along their zippers via the "identity" map. The resulting Pair of pants dP is uniquely determined by the length of its geodesic boundaries [Bus92, Theorem 3.1.7], which are $0, \ell_\gamma, \ell_\gamma$. Thus, the geometry of a pair of half-pants P is completely determined by the length ℓ_γ of its cuff γ.

1.1 Riemannian $\Rightarrow$ Riemann: Isothermal Coordinates

We now show how one might associate a canonical Riemann surface structures to a hyperbolic surface S, before considering how one might do this for a whole conformal class of Riemannian metrics. This gives us a chance to play around with conformally equivalent metrics prior to working with them in Section 1.2.

1. Any interior point $p_0 \in S$ has a small neighbourhood modelled on the hyperbolic plane. Specifically, there is an open set $U \ni p_0$ and a coordinate chart $(U, (u, v))$ around p_0,

$$(u, v) : U \to \mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} \subset \mathbb{R}^2,$$

such that the pullback of the usual hyperbolic metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2} \text{ on } \mathbb{H}$$

is the same as the metric on S restricted to U.

2. Consider the maximal atlas $\{(U,(u,v))\}$ of such neighbourhoods for X, and define the following complex atlas $\{(U,z)\}$ given by:

$$z : U \to \mathbb{C},\ p \mapsto z(p) := u(p) + iv(p).$$

3. The transition map between two neighbourhoods

$$(U_1,(u_1,v_1)) \text{ and } (U_2,(u_2,v_2))$$

is an isometry and a little computation shows that the condition that the coordinate change $(u_{12},v_{12}) := (u_2,v_2)\circ(u_1,v_1)^{-1}$ preserves the hyperbolic metric and has positive Jacobian determinant forces u_{12} and v_{12} to obey the Cauchy-Riemann equations. Thus, the transition function between the complex coordinates (U_1,z_1) and (U_2,z_2) is holomorphic and the complex atlas $\{(U,z)\}$ yields a natural Riemann surface structure on S.

For a general Riemannian surface (M, ds^2), we assign a canonical Riemann surface structure by constructing *isothermal coordinates* for M. That is: we cover M with an atlas whose coordinate charts $(U, z : U \to \mathbb{C})$ satisfy

$$z^*|dz|^2 = \rho ds^2 \text{ for some positive real function } \rho : U \to \mathbb{R}^+.$$

Given any two such isothermal coordinate charts (U_1,z_1) and (U_2,z_2),

$$z_1^*|dz_1|^2 = \rho_1 ds^2 = \tfrac{\rho_1}{\rho_2} z_2^*|dz_2|^2 \Rightarrow |dz_1|^2 = \frac{\rho_1}{\rho_2} z_{12}^*|dz_2|^2.$$

Thus, the Jacobian for z_{12} in real coordinates is $\rho := \frac{\rho_1}{\rho_2}$ times a rotation matrix, which in turn means that the transition function z_{12} satisfies the Cauchy-Riemann equations and is holomorphic.

As long as we're able to construct isothermal coordinate patches for every point in (M, ds^2), the maximal atlas consisting of all isothermal coordinates on M will form a canonical Riemann surface structure. It remains to show that isothermal coordinates exist around each point.

Consider a coordinate patch $(U,(x,y))$ of the Riemannian surface (M, ds^2), the restriction of the metric ds^2 on U is expressible as:

$$E(x,y)\,dx^2 + 2F(x,y)\,dxdy + G(x,y)\,dy^2,$$

where E, F and G are smooth real functions on U. Form the coordinate patch (U,z) by:

$$z : U \to \mathbb{C},\ z(p) = x(p) + iy(p),$$

then the differential forms dx and dy in this new coordinate are:

$$dx = \frac{1}{2}(dz + d\bar{z}) \text{ and } dy = \frac{1}{2i}(dz - d\bar{z}).$$

Expressing the metric ds^2 in terms of these complex coordinates yields:

$$\begin{aligned}
&\frac{1}{4}(E-G-2iF)dz^2+\frac{1}{2}(E+G)dz\,d\bar{z}+\frac{1}{4}(E-G+2iF)d\bar{z}^2,\\
=&\frac{1}{4}(E+G+2\sqrt{EG-F^2})\left|dz+\frac{E-G+2iF}{E+G+2\sqrt{EG-F^2}}d\bar{z}\right|^2.
\end{aligned}\tag{1.4}$$

Assume for the moment that an isothermal coordinate

$$(U,w),\ w:U\to\mathbb{C}\text{ exists,}$$

then the metric ds^2 takes the form:

$$\rho\,|dw|^2=\rho\left|\frac{\partial w}{\partial z}\right|^2\left|dz+\left(\frac{\partial w}{\partial\bar{z}}\right)\left(\frac{\partial w}{\partial z}\right)^{-1}d\bar{z}\right|^2.\tag{1.5}$$

Comparing the coefficients of (1.4) and (1.5), we see that an isothermal coordinate for (U,z) exists precisely when the following differential equation, called the *Beltrami equation* has a nonconstant solution on U:

$$\frac{\partial w}{\partial\bar{z}}=\mu\frac{\partial w}{\partial z}.$$

In this particular case, observe that

$$\mu=\frac{E-G+2iF}{E+G+2\sqrt{EG-F^2}}\text{ satisfies that }|\mu|<1.$$

In general, as long as μ is a measurable complex function with $|\mu|$ almost everywhere less than some constant $C<1$, the Beltrami equation admits a unique solution up to normalisation [Ahl66].

1.1.1 Solving the Beltrami Equation

Rather than tackle the Beltrami equation in very general settings [Ahl66, IT92], we give a simple construction (due to Gauss [Spi79]) of a solution at a point $p\in U$ where the function $\mu:U\to\mathbb{C}$ is locally expressible as a power series $\mu(z,\bar{z}):U\to\mathbb{C}$. Assuming that such expansions are possible at every point, connectedness allows these small neighbourhoods to patch together to give a Riemann surface structure. We assume without loss of generality that $z(p)=0$.

1. Find a power series $F(z,w)$ around p, satisfying:

$$\frac{\partial F}{\partial z}=-\mu(F(z,w),z),\text{ such that }F(0,w)=w.\tag{1.6}$$

2. The function $(z,w)\mapsto(z,F(z,w))$ is analytic around $(0,0)$, let

$$(z,w)\mapsto(z,G(z,w))$$

denote its inverse in a sufficiently small neighborhood around $(0,0)$.

3. Define the analytic function $h(z,w) := G(w,z)$.

Lemma 1.6. *The function $f_\mu(z) := h(z,\bar{z})$ is a non-constant solution to the Beltrami equation.*

Proof. We first note that h is non-constant as $(z, G(z,w))$ is the inverse function to $(z, F(z,w))$. In particular, since these two functions are inverse,

$$F(z, G(z,w)) = w \text{ and } F(w, G(w,z)) = z \tag{1.7}$$

Differentiating the right identity with respect to w and z yields:

$$\begin{aligned} \frac{\partial F}{\partial z}(w, G(w,z)) + \frac{\partial F}{\partial w}(w, G(w,z))\frac{\partial G(w,z)}{\partial w} &= 0, \\ \text{and } \frac{\partial F}{\partial w}(w, G(w,z))\frac{\partial G(w,z)}{\partial z} &= 1. \end{aligned}$$

Using equation (1.6) and the definition of $h(z,w)$, the above two lines become:

$$-\mu(F(w, G(w,z), w) + \frac{\partial F}{\partial w}(w, G(w,z))\frac{\partial h(z,w)}{\partial w} = 0, \tag{1.8}$$

$$\text{and } \frac{\partial F}{\partial w}(w, G(w,z))\frac{\partial h(z,w)}{\partial z} = 1. \tag{1.9}$$

Multiplying the leftmost term in equations (1.8) by (1.9) and substituting for equation (1.7):

$$\frac{\partial h}{\partial w}(z,w) = \mu(z,w)\frac{\partial h}{\partial z}(z,w),$$

which proves that $f_\mu(z) := h(z,\bar{z})$ is a solution to the Beltrami equation. □

Note 1.3. A crucial first step of this construction is to actually solve the differential equation (1.6). That a solution around $z(p) = 0$ exists, is a fundamental result of the theory of ordinary differential equations.

1.2 Riemann ⇒ Hyperbolic: Uniformisation

At the heart of the procedure taking Riemann surfaces to Hyperbolic ones is the uniformisation theorem [Kle83, Koe09, Poi08]:

Theorem 1.7 (Uniformisation)**.** *Every simply connected Riemann surface is biholomorphically equivalent to one of the following Riemann surfaces:*

- *the complex plane* $\mathbb{C}$*,*
- *the Riemann sphere* $\mathbb{C} \cup \{\infty\}$ *or*
- *the hyperbolic upper-half space* $\mathbb{H} := \{x + iy \in \mathbb{C} \mid y > 0\}$.

Uniformisation tells us that there is a biholomorphism between the universal cover $\tilde{R}$ of the interior R° of a Riemann surface R (with negative Euler characteristic) and the upper-half plane $\mathbb{H}$. The covering group acting on $\tilde{R}$ then corresponds to a Fuchsian subgroup of the Möbius transformations acting on $\mathbb{H}$ [IT92]. Since Möbius transformations are isometries of $\mathbb{H}$ endowed with the usual hyperbolic metric, this means that R° may be expressed as a hyperbolic quotient of $\mathbb{H}$. This process of assigning a canonical hyperbolic metric to a Riemann surface is what we think of as the uniformisation of a Riemann surface.

There are two standard ways of assigning a canonical hyperbolic metric to a Riemann surface. The first assigns to any Riemann surface R the unique complete hyperbolic metric g (often infinite volume) on R° such that (R°, g) as a hyperbolic surface induces the complex structure on R°. This establishes an equivalence of categories between the subcategory of $\mathcal{R}mn$ consisting of finite-type Riemann surfaces without boundary punctures and the category of *complete* hyperbolic surfaces without boundary.

Although this first method of assigning hyperbolic metrics doesn't quite give what we want[3], it does "correctly" assign finite-area cusped hyperbolic surfaces to punctured Riemann surfaces. In order to extend the process of uniformisation to any Riemann surface of finite type, we tweak the above method using the (Schottky) doubling-construction trick [Abi80]:

1. Glue a (bordered) Riemann surface R to copy of itself R' having the opposite orientation (i.e.: complex conjugated) along the boundaries of R and R' using the identity map. The resulting surface, denoted by dR, is a Riemann surface. And the complex structure of dR agrees with the complex structure of R and R'.

2. The resulting doubled Riemann surface dR is a punctured Riemann surface, and we uniformise it using the previous method to obtain a complete hyperbolic surface (dR, g).

3. Observe that ∂R (as a set) are the fixed points of the reflection symmetry on dR exchanging R and R', and that the unique geodesic representatives (Lemma 2.9) of the curves constituting ∂R are also fixed points of the reflection symmetry. Hence conclude that ∂R must consist of geodesics.

Note 1.4. This gives us the desired equivalence of categories between $\mathcal{R}mn$ and $\mathcal{H}yp$. In particular, unpunctured borders on Riemann surfaces become closed geodesic borders when uniformised, and punctured borders on Riemann surfaces become boundary arches on a uniformised crowned hyperbolic surface.

Various proofs for the uniformisation theorem exist in the literature, including:

- Klein's original *continuity method* based proof [Kle83] [4]
- Ahlfors' Green's function based proof [Ahl73]
- Bers' simultaneous uniformization [Ber60]

[3] Because we wish to uniformise to finite-area hyperbolic surfaces.

[4] Brouwer's work in dimension theory (e.g.: invariance of domain) is needed to make this into a proper proof.

- Osgood-Philips-Sarnak's elliptic operators based proof [OPS88]
- Chen-Lu-Tian's proof by Ricci flow [CLT06, AAR13].

We opt to focus on Osgood, Phillips and Sarnak's proof. It avoids the Schottky doubling trick, and also yields an interesting generalisation: it allows us to uniformise Riemann surfaces to Euclidean metrics with constant geodesic boundary curvatures. We present the strategy for their proof of the uniformisation theorem in the guise of a "pseudo-proof" for a conjectured generalisation of the uniformisation theorem.

Note 1.5. Strictly speaking, the Osgood-Phillips-Sarnak proof of the uniformisation theorem does *not* allow us to uniformise Riemann surfaces with punctures, and a little extra work is needed to obtain the cusped hyperbolic metrics for punctured Riemann surfaces.

Our goal is to look for a canonical hyperbolic metric within the collection of all Riemannian metrics on R whose isothermal coordinate charts yield the Riemann surface structure on R. We call these *compatible Riemannian metrics*, and begin by showing that the collection of compatible Riemannian metrics is non-empty.

1.2.1 Compatible Riemannian Metrics

The complex plane $\mathbb{C}$ is naturally equipped with the usual Euclidean metric:

$$ds^2 = dz\,d\bar{z} = |dz|^2 = dx^2 + dy^2,$$

and any coordinate chart (U, φ) for a Riemann surface R may be equipped with the pullback metric $\varphi^* ds^2$. Each Euclidean coordinate patch obtained in this way is compatible with the underlying Riemann surface structure of U (and hence of R), and may be patched together to form a compatible Riemannian metric over all of R using a *partition of unity* for R. That is:

1. a countable collection $\{(U_\alpha, f_\alpha)\}$ of pairs comprised of open sets U_α and functions $f_\alpha :$ $R \to [0, 1]$ such that,
2. the open sets $\{U_\alpha\}$ form an open cover of R where every point $p \in R$ is covered by only finitely many U_α's, and
3. the functions $f_\alpha : R \to [0, 1]$ are smooth, 0 outside of U_α, and the sum of all these f_α's

$$\sum_\alpha f_\alpha \equiv 1$$

is the constant function 1 on R.

Lemma 1.8. *Given a partition of unity* $\{(U_\alpha, f_\alpha)\}$ *where each* U_α *arises as the domain of a coordinate chart* $(U_\alpha, \varphi_\alpha)$ *for the given Riemann surface structure on* R, *the metric*

$$g := \sum_\alpha f_\alpha \cdot \varphi_\alpha^*(dx^2 + dy^2)$$

is compatible with the Riemann surface structure of R.

Proof. We may assume without loss of generality that the open cover $\{U_\alpha\}$ satisfies the additional property that any open set U_α only intersects finitely many other sets $U_{\beta_1}, \dots, U_{\beta_k}$ in this cover. Since any transition map $\varphi_{\alpha\beta_i} := \varphi_\beta \circ \varphi_\alpha^{-1}$ is holomorphic, the pullbacks of the 1-forms dz and $d\bar{z}$ on U_{β_i} are given by:

$$\begin{aligned}
\varphi_{\alpha\beta_i}^*(dz) &= d\varphi_{\alpha\beta_i} = \partial_z\varphi_{\alpha\beta_i}dz + \partial_{\bar{z}}\varphi_{\alpha\beta_i}d\bar{z} = \partial_z\varphi_{\alpha\beta_i}dz, \text{ and}\\
\varphi_{\alpha\beta_i}^*(d\bar{z}) &= d\overline{\varphi_{\alpha\beta_i}} = \partial_z\overline{\varphi_{\alpha\beta_i}}dz + \partial_{\bar{z}}\overline{\varphi_{\alpha\beta_i}}d\bar{z} = \overline{\partial_z\varphi_{\alpha\beta_i}}d\bar{z}.
\end{aligned}$$

Therefore, the pullback of the Euclidean metric $|dz_{\beta_i}|^2$ on U_{β_i} is given by:

$$\varphi_{\alpha\beta_i}^*|dz_{\beta_i}|^2 = \partial_z\varphi_{\alpha\beta_i}\overline{\partial_z\varphi_{\alpha\beta_i}}dzd\bar{z} = |\partial_z\varphi_{\alpha\beta_i}|^2|dz_\alpha|^2. \tag{1.10}$$

Hence, on the open set U_α, the metric g is given by the pullback metric

$$\varphi_\alpha^*((f_\alpha \circ \varphi_\alpha^{-1} + \sum_{i=1}^k f_{\beta_i} \circ \varphi_\alpha^{-1}|\frac{\partial\varphi_{\alpha\beta_i}}{\partial z}|)\, dzd\bar{z}) = (f_\alpha + \sum_{i=1}^k f_{\beta_i}|\frac{\partial\varphi_{\beta_i}}{\partial\varphi_\alpha}|)\,|d\varphi_\alpha|^2.$$

Therefore, any coordinate chart $(U_\alpha, \varphi_\alpha)$ is an isothermal coordinate chart for the Riemannian manifold (R, g) and the metric g is compatible with the Riemann surface structure of R. ☐

Note 1.6. The computation of equation (1.10) tells us that given a compatible metric g for a Riemann surface R, any holomorphic coordinate chart (U, φ) for the Riemann structure of R satisfies that the restriction of the metric g to U is of the form:

$$g|_U = \rho \cdot \varphi^*|dz|^2,$$

for some positive real function $\rho : U \to \mathbb{R}_+$. Thus, each holomorphic chart is an isothermal coordinate for g. In addition, for two compatible metrics g_1 and g_2, the ratio $\rho_1 : \rho_2$ of the induced positive real functions ρ_1 and ρ_2 is independent of which chart we use, since:

$$\frac{g_1}{g_2} = \frac{\rho_1 \cdot \varphi_\alpha^*|dz_\alpha|^2}{\rho_2 \cdot \varphi_\alpha^*|dz_\alpha|^2} = \frac{\rho_1 \cdot \varphi_\alpha^*|\partial_{z_\beta}\varphi_{\alpha\beta}|^2|dz_\alpha|^2}{\rho_2 \cdot \varphi_\alpha^*|\partial_{z_\beta}\varphi_{\alpha\beta}|^2|dz_\alpha|^2} = \frac{\rho_1 \cdot \varphi_\beta^*|dz_\beta|^2}{\rho_2 \cdot \varphi_\beta^*|dz_\beta|^2}.$$

Thus, $\frac{g_1}{g_2} : R \to \mathbb{R}^+$ is a well-defined smooth function.

1.2.2 Compatible Hyperbolic Metrics

The following uniformisation theorem due to Osgood-Phillips-Sarnak [OPS88] establishes our description of the equivalence of categories between the bordered Riemann surfaces in $\mathcal{R}$mn and the bordered hyperbolic surfaces in $\mathcal{H}$yp:

Theorem 1.9 (Osgood, Phillips, Sarnak)**.** *The conformal class of all compatible Riemannian metrics for a closed Riemann surface* R *with (finite) negative Euler characteristic contains a unique hyperbolic metric. Moreover, the conformal class of all compatible Riemannian metrics on a smoothly bordered Riemann surface* R *with no punctures contains:*

- *a unique type I metric: a hyperbolic metric with geodesic boundaries, and*
- *a unique type II metric: a flat metric with constant boundary geodesic curvature and with total boundary length* 1*.*

Note 1.7. We emphasise that every boundary has the *same* constant boundary geodesic curvature. To clarify: Euclidean pairs of pants obtained from excising two smaller discs from a larger Euclidean disc do not qualify as having constant boundary geodesic curvature.

We call metrics (such as type I and type II metrics) that have constant Gaussian curvature and constant geodesic curvature *uniform metrics*. We only consider uniform metrics with non-positive Gaussian and geodesic curvatures.

Osgood, Phillips and Sarnak showed that these two uniform metrics are the respective global minima of two functionals F_0 and F_1. We describe a 1-parameter family of functionals joining F_0 and F_1 such that:

Conjecture 1.1. Given a Riemann surface R, the class of compatible metrics on R contains an interval of uniform metrics g_τ, $\tau \in [0,1]$ such that any uniform metric on R is a constant multiple of a unique g_τ in this family.

Our *pseudo-proof* for this conjecture via variational principles may be broken into three parts:

1. define a strictly convex functional on a subspace of smooth metrics compatible with the complex structure of R,
2. show that the functional is bounded below and that a minimum for the functional exists when extended to a subspace of $W^{1,p}(R)$,
3. perform first variation analysis and invoke elliptic regularity to show that this minimum is a uniform metric.

We highlight some technical issues with such an approach, whilst noting that that these arguments do actually hold in the special case of type I uniform metrics ($\tau = 0$).

Pseudo-proof of Conjecture 1.1 part 1. We showed in Lemma 1.8 that the space of compatible Riemannian metrics on R is non-empty. Let ∇, Δ, K, κ, dA and ds respectively denote the *gradient*, *Laplacian*, *Gaussian curvature*, *boundary geodesic curvature*, *area form* and *boundary length form* for some fixed metric g. Since multiplying g by a smooth positive real function results in another compatible metric, we may choose g to have area $A = 1$ and boundary length $s = 1$.

Note 1.6 tells us that the space of all compatible Riemannian metrics is in bijection with the set $C^\infty(R)$ of smooth real functions on R via:

$$(\rho : R \to \mathbb{R}) \mapsto e^{2\rho} g.$$

Thus, we identify the space of compatible metrics with $C^\infty(R)$ and define the following family of functionals $F_\tau : C^\infty(R) \to \mathbb{R}$,

$$\begin{aligned} F_\tau(\varphi) :=& \frac{1}{2}\int_R |\nabla\varphi|^2 dA + \int_R K\varphi dA + \int_{\partial R} \kappa\varphi ds \\ & - \pi\chi(R)(1-\tau)\log\left(\int_R e^{2\varphi} dA\right) - 2\pi\chi(R)\tau\log\left(\int_{\partial R} e^{\varphi} ds\right). \end{aligned}$$

Ideally, we would like to extend the domain of F_τ to the Sobolev space $W^{1,2}(R)$, on which the trace operator

$$T : W^{1,2}(R) \to L^2(\partial R)$$

telling us how to restrict a measurable function to its boundary is well-defined. To clarify, we mean that:

$$\int_{\partial R} \kappa\varphi\, ds := \int_{\partial R} \kappa T(\varphi)\, ds \text{ and } \int_{\partial R} e^{\varphi}\, ds := \int_{\partial R} e^{T(\varphi)}\, ds.$$

But since we don't have a 1-dimensional version of Trudinger's inequality, we don't know much about the convergence of this latter term and only know that F_τ has domain equal to $W^{1,2}(R)$ when $\tau = 0$ (the fourth term converges thanks to the Moser-Trudinger inequality [Tru67, Mos71]). And so we come to the first issue with this pseudo-proof:

Issue 1. It is unclear if we should be analyzing F_τ over $W^{1,2}(R)$. Although this worked fine for type I uniform metrics (i.e.: $\tau = 0$), but for type II uniform metrics, Osgood, Phillips and Sarnak shifted the analysis over to the space $W^{1,2}(\partial R)$.

As for properties of these functionals, the Gauss-Bonnet theorem tells us that F_τ is invariant under constant scaling. That is:

$$F_\tau(\varphi + a) = F_\tau(\varphi), \text{ for all } a \in \mathbb{R}. \tag{1.11}$$

This is desirable because constant scaled metrics impose fundamentally identical dynamics on R, and should be thought of as being the same. Thus, we restrict F to the following space of normalised metrics:

$$C^\infty(R)^\tau := \left\{ \varphi \in C^\infty(R) : (1-\tau)\int_R \varphi \, dA + \tau \int_{\partial R} \varphi \, ds \right\}, \tag{1.12}$$

and consider the extension of F_τ on the correspondingly restricted subspace $W^{1,2}(R)^\tau$. In this codimension 1 subspace, the functional F_τ is strictly convex:

$$F_\tau(\alpha\varphi_1 + (1-\alpha)\varphi_2) \leqslant \alpha F_\tau(\varphi_1) + (1-\alpha)F_\tau(\varphi_2).$$

Hence if F_τ is bounded below, there is at most one minimum in $W^{1,2}(R)^\tau$.

We now show this convexity term-by-term; for the first term in F_τ:

$$\begin{aligned}
&|\alpha\nabla\varphi_1 + (1-\alpha)\nabla\varphi_2|^2 \\
&\qquad \leqslant \alpha^2|\nabla\varphi_1|^2 + 2\alpha(1-\alpha)|\nabla\varphi_1||\nabla\varphi_2| + (1-\alpha)^2|\nabla\varphi_2|^2 \\
&\qquad \leqslant \alpha^2|\nabla\varphi_1|^2 + \alpha(1-\alpha)(|\nabla\varphi_1|^2 + |\nabla\varphi_2|^2) + (1-\alpha)^2|\nabla\varphi_2|^2 \\
&\qquad = \alpha|\nabla\varphi_1|^2 + (1-\alpha)|\nabla\varphi_2|^2.
\end{aligned}$$

The second and third terms in F_τ are linear and automatically convex. And since $-\chi(R)$ is positive by assumption, the fourth and fifth terms are convex thanks to Hölder's inequality:

$$\int_R e^{\alpha\varphi_1 + (1-\alpha)\varphi_2} \, dA \leqslant \left(\int_R e^{\varphi_1} \, dA\right)^\alpha \cdot \left(\int_R e^{\varphi_2} \, dA\right)^{1-\alpha}.$$

Finally, the strictness of this equality follows from the fact that Hölder's inequality is an equality if and only if:

$$e^{\varphi_1} = e^{\varphi_2 + \beta}.$$

Therefore, $\varphi_1 = \varphi_2 + \beta = \varphi_2$, where $\beta = 0$ is due to the constraint on $W^{1,2}(R)^\tau$ stated in equation (1.12).

This completes the first part of our pseudo-proof. □

Pseudo-proof of Conjecture 1.1 part 2. We now show that F_τ is bounded below. By Jensen's inequality,

$$\int_R e^{2\varphi} \, dA \geqslant \exp\left(\int_R 2\varphi \, dA\right) \text{ and } \int_{\partial R} e^{\varphi} \, ds \geqslant \exp\left(\int_{\partial R} \varphi \, ds\right). \tag{1.13}$$

Therefore, the sum of the last two terms of F_τ is greater than or equal to

$$\log\exp\left((1-\tau)\int_R \varphi \, dA + \tau\int_{\partial R} \varphi \, ds\right) = 0,$$

and may be ignored for the purposes of demonstrating the existence of a lower bound. The Trace theorem for Sobolev spaces tells us that:

$$\|T\varphi\|_{L_1(R)} \leqslant C\|\varphi\|_{W^{1,1}(R)} \leqslant 2C(\|\varphi\|_{L_1(R)} + \||\nabla\varphi|\|_{L_1(R)}), \tag{1.14}$$

for some constant C. Therefore, showing that $\frac{1}{2}\int|\nabla\varphi|^2 dA$ dominates $\||\nabla\varphi|\|_{L_1(R)}$ and $\|\varphi\|_{L_1(R)}$ when these terms are sufficiently large would suffice to show that F_τ is bounded below. The first comparison follows from Jensen's inequality,

$$\int_R |\nabla\varphi|^2\, dA \geqslant \left(\int_R |\nabla\varphi|\, dA\right)^2 =: \||\nabla\varphi|\|^2_{L_1(R)}. \tag{1.15}$$

The second comparison follows from combining the above comparison and the following version of the Poincaré inequality:

Lemma 1.10 (τ-Poincaré inequality)**.** *There exists a constant* $C = C(p)$ *such that any measurable function* $\varphi \in W^{1,p}(R)^\tau$ *satisfies :*

$$\|\varphi\|_{L_p(R)} \leqslant C\||\nabla\varphi|\|_{L_p(R)}.$$

Proof. We prove this by contradiction. Assume that there is sequence of functions $\{\varphi_k\}$ in $W^{1,p}(R)^\tau$ such that

$$\int_R |\varphi_k|^p\, dA > k\int_R |\nabla\varphi_k|^p\, dA.$$

This means that φ_k can't be 0, hence the norms $\|\varphi_k\|_p$ are nonzero and we may assume without loss of generality that the φ_k were normalised to have magnitude 1. Also note that these normalised functions still lie in $W^{1,p}(R)^\tau$.

We now have that

$$\||\nabla\varphi_k|\|_p < \frac{1}{k} \text{ and } \|\varphi_k\|_p = 1.$$

The Rellich-Kondrachov theorem then tells us that a subsequence of the $\{\varphi_k\}$ must converge strongly to some $\varphi \in L^p(R)$. Relabel $\{\varphi_k\}$ to be this subsequence, and let's show that $\varphi \in W^{1,p}(R)^\tau$.

We first show that the weak derivatives of φ are in $L^p(R)$. Taking an arbitrary compactly supported smooth function $\phi \in C_c^\infty(R)$:

$$\int_R \varphi\frac{\partial\phi}{\partial x_i}\, dA = \lim_{k\to\infty}\int_R \varphi_k\frac{\partial\phi}{\partial x_i}\, dA = -\lim_{k\to\infty}\int_R \frac{\partial\varphi_k}{\partial x_i}\phi\, dA = 0,$$

where the last limit follows from the fact that $\||\nabla\varphi_k|\|_p < \frac{1}{k}$.

Therefore, the weak derivative $\nabla\varphi$ exists and is the zero function, and $\varphi \in W^{1,p}(R)$. Now invoking (Problem 11 of [Eva98]), we see that φ must be constant. But we also know that φ has to satisfy the constraint placed upon $W^{1,p}(R)^\tau$ in equation (1.12). Therefore, $\varphi = 0$, which contradicts the fact that $\|\varphi\|_p = 1$. $\square$

Having now established that F_τ is bounded below, we may take a sequence $\{\varphi_k\}$ such that $F_\tau(\varphi_k)$ converges to the infimum of F_τ. Then $\||\nabla\varphi_k|\|_{L_2(R)}$ must be bounded above, or else by the τ-Poincaré inequality the values $F_\tau(\varphi_k)$ will become arbitrarily great. This in turn means that $\|\varphi_k\|_{W^{1,2}(R)}$ is a bounded sequence, and by the Banach-Alaoglu theorem (Section 3.5 in [Rud73]), we may replace $\{\varphi_k\}$ with a weakly convergent subsequence. In addition, by the Rellich-Kondrachov theorem, there is a strongly convergent subsequence of $\{\varphi_k\}$ in $L_2(R)$, and there is furthermore a pointwise convergent subsequence of this subsequence. Therefore, we may take $\{\varphi_k\}$ to be:

- a weakly convergent sequence in $W^{1,2}(R)$,
- a strongly convergent sequence in $L_2(R)$ and
- a pointwise convergent sequence of measurable functions.

And by the uniqueness of weak limits, the $W^{1,2}(R)$ and $L_2(R)$ limits for $\{\varphi_k\}$ is the same $\psi \in W^{1,2}(R)$. The fact that ψ is the pointwise limit of the sequence φ_k means that by Fatou's lemma, the value $F_\tau(\psi)$ is the infimum of F_τ, and hence ψ is the unique (by strict convexity) minimiser of F_τ in $W^{1,2}(R)^\tau$. $\square$

Pseudo-proof of Conjecture 1.1 part 3. Finally, we show that ψ yields a uniform metric. Since $W^{1,2}(R)^\tau$ is a subspace of constant scaling representatives of $W^{1,2}(R)$ and F_τ does not distinguish between constant scaled metrics, the function ψ minimises F_τ over all of $W^{1,2}(R)^\tau$. Thus, the first variation of F_τ at ψ must be zero:

$$\begin{aligned} 0 = \delta F_\tau(\psi)[\varphi] &= \int_R \nabla\psi \cdot \nabla\varphi \, dA + \int_R K\varphi \, dA + \int_{\partial R} \kappa\varphi \, ds \\ &\quad - \frac{2\pi\chi(R)(1-\tau)}{\int_R e^{2\psi} \, dA} \int_R e^{2\psi}\varphi \, dA - \frac{2\pi\chi(R)\tau}{\int_{\partial R} e^{\psi} \, ds} \int_{\partial R} e^{\psi}\varphi \, ds \qquad (1.16) \\ &= \int_R \left(-\Delta\psi + K - \frac{2\pi\chi(R)(1-\tau)e^{2\psi}}{\int_R e^{\psi} \, dA} \right) \varphi \, dA \\ &\quad + \int_{\partial R} \left(\partial_n\psi + \kappa - \frac{2\pi\chi(R)\tau e^{\psi}}{\int_{\partial R} e^{\psi} \, ds} \right) \varphi \, ds, \qquad (1.17) \end{aligned}$$

By choosing φ to be compactly supported in R, we see that:

$$0 = \delta F_\tau(\psi)[\varphi] = \int_R \left(-\Delta\psi + K - \frac{2\pi\chi(R)(1-\tau)e^{2\psi}}{\int_R e^{2\psi} \, dA} \right) \varphi \, dA, \qquad (1.18)$$

and elliptic regularity suffices to show that ψ is smooth on the interior of R and induces constant negative Gaussian curvature:

$$e^{-2\psi}(-\Delta\psi + K) = \frac{2\pi\chi(R)(1-\tau)}{\int_R e^{2\psi}\, dA}. \tag{1.19}$$

However, there are definite issues with the second summand:

Issue 2. We can't establish that ψ is well-defined on ∂R and induces constant negative geodesic curvature:

$$e^{-\psi}(\partial_n\psi + \kappa) = \frac{2\pi\chi(R)\tau}{\int_{\partial R} e^{\psi}\, ds}. \tag{1.20}$$

In fact, we don't actually know if we're actually allowed to apply Stokes' theorem to obtain equation (1.17).

□

An immediate consequence, should the above conjecture be true, is the following:

Conjecture 1.2. For any compact Riemann surface R with negative Euler characteristic and smooth boundary, there exists a unique uniform metric g compatible with its complex structure such that:

- the area of R is equal to its boundary length, and
- the ratio $[-K_g : -\kappa_g]$ between the Gaussian and boundary geodesic curvatures of (R, g) may be chosen to be any element in the space $\mathbb{P}\mathbb{R}_+$ of projective classes of positive real numbers.

Chapter 2

Moduli Spaces of Surfaces

The main goal of this chapter is to describe and construct global coordinate systems (equations (2.6) (2.10) (2.14) (2.15)) on various Teichmüller spaces. We begin with a little exposition on various moduli spaces (and Teichmüller spaces) of surfaces.

The *moduli spaces* that we study are all (subspaces of) fiber bundles over the *moduli space of Riemann surfaces* $\mathcal{M}(R)$, given (as a set) by:

$$\mathcal{M}(R) := \{S \mid S \text{ is a Riemann surface label-preserving homeomorphic to } R\} / \sim_M,$$

where two surfaces $S_1 \sim_M S_2$ if and only if there is a puncture and border label-preserving biholomorphism from S_1 to S_2. We denote equivalence classes of Riemann surfaces by $[S]$.

As sets, the moduli spaces that we define in Section 2.1 and Section 2.2 can each be expressed as the set of equivalence classes of pairs $(S, \mathrm{Str}(S))$, where S is a Riemann surface of fixed topological type and $\mathrm{Str}(S)$ is some topological or geometric object associated to S, identified up to biholomorphism and an induced action of that biholomorphism on $\mathrm{Str}(S)$. We then define topologies on these moduli spaces in Section 2.3 by constructing global coordinates on the universal covers of these moduli spaces.[1]

2.1 Moduli Spaces of Riemann Surfaces

Throughout this section, we set R to be a fixed Riemann surface (although topological surface would suffice) of finite type. And we denote by $\mathcal{R}\mathrm{mn}(R)$ the subcategory of $\mathcal{R}\mathrm{mn}$ whose objects are (boundary-labelled) Riemann surfaces label-preserving homeomorphic to R.

[1] For algebraic geometers: these moduli spaces are (possibly uncountably) infinite "covers" of moduli space of curves and we do not consider them as algebraic geometric objects such as Deligne-Mumford stacks [HM98].

Example 1 (Teichmüller Space). Pairing each Riemann surface $S \in \mathrm{Ob}_{\mathcal{R}mn(R)}$ with isotopy classes of homeomorphisms $f : R \to S$, we obtain the Teichmüller space $\mathcal{T}(R)$. Explicitly:

$$\mathcal{T}(R) := \{(S_1, f) \mid f : R \to S_1 \text{ is a label-preserving homeomorphism } \}/\sim_T,$$

and two pairs $(S_1, f_1) \sim_T (S_2, f_2)$ if and only if $f_2 \circ f_1^{-1} : S_1, \to S_2$ is isotopy equivalent to a biholomorphism. We call equivalence classes $[S, f]$ of these pairs *marked surfaces*.

As topological spaces (Section 2.3), the Teichmüller space $\mathcal{T}(R)$ is the orbifold universal cover of the moduli space $\mathcal{M}(R)$. And the covering group is the *mapping class group*:

$$\begin{aligned} \mathrm{Mod}(R) &:= \mathrm{Homeo}^+(R)/\mathrm{Homeo}_0^+(R) \\ &= \{ \text{ isotopy classes } [h] \text{ of orientation-preserving homeomorphisms } h : R \to R\}, \end{aligned}$$

where $\mathrm{Homeo}^+(R)$ is the group of orientation-preserving (and label-preserving) homeomorphisms of R. And $\mathrm{Homeo}_0^+(R)$ is the normal subgroup of orientation-preserving homeomorphisms isotopic to the identity map. The action of a mapping class $[h] \in \mathrm{Mod}(R)$ on the Teichmüller space is given by pre-composition:

$$[h] \cdot [S, f] := [S, f \circ h].$$

Note 2.1. There are a few common variations to how the mapping class group is defined. For example: *homeomorphism* may be replaced with *diffeomorphism* and/or *isotopy classes* of maps may be replaced with *homotopy classes* of maps [FM12]. It's a rather convenient fact that these definitions agree when we're dealing with the mapping class group of a surface, and the careful reader may find at times that we freely switch between homotopy equivalence and isotopy equivalence.

The next example of a moduli space arises in Mirzakhani's integration scheme for computing the Weil-Petersson volume of moduli spaces via McShane identities [Mir07a]. The set of homotopy classes of free loops in R

$$[\mathbb{S}^1, R] := \{ \text{ homotopy classes of continuous maps from } \mathbb{S}^1 \text{ to } R\}$$

is in natural bijection with the set of conjugacy classes

$$\pi_1(R, x)/\mathrm{Inn}(\pi_1(R, x))$$

of the fundamental group $\pi_1(R, x)$. [2]

Definition 2.1. *We call a tuple $\Gamma = ([\gamma_1], \dots, [\gamma_n])$ of homotopy classes of free loops a* curve class. *Any homotopy class of a free loop $[\gamma_i]$ that can be represented by an* embedding $\gamma_i : \mathbb{S}^1 \to R$ *is called* simple, *and two distinct homotopy classes of free loops $[\gamma_i]$ and $[\gamma_j]$ which may be realised by maps with disjoint images are called* disjoint. *A curve class where each $[\gamma_i]$ is simple/disjoint is called a* simple/disjoint curve class.

[2] It doesn't actually matter which base point $x \in R$ we use since R is path-connected.

A homeomorphism $f : R \to S$ induces a map f_* by post-composition

$$\begin{aligned} f_* : [\mathbb{S}^1, R] &\to [\mathbb{S}^1, S] \\ [\gamma] &\mapsto f_*[\gamma] := [f \circ \gamma], \end{aligned}$$

we accordingly define $f_*\Gamma := (f_*[\gamma_1], \ldots, f_*[\gamma_n])$ for a curve class $\Gamma = ([\gamma_1], \ldots, [\gamma_n])$ on R. In particular, $\mathrm{Homeo}^+(R)$ acts on the set of curve classes on R with $\mathrm{Homeo}_0^+(R)$ acting trivially, hence the mapping class group $\mathrm{Mod}(R)$ also has a well-defined action on the collection of curve classes on R, allowing us to define the stabiliser subgroup $\mathrm{Stab}(\Gamma)$ of a curve class Γ.

Example 2 (Moduli Space of Γ-Surfaces). Fix a curve class Γ on R. Pairing each Riemann surface $S \in \mathrm{Ob}_{\mathcal{R}\mathrm{mn}(R)}$ with curve classes of the form $f_*\Gamma$, we obtain the *moduli space of* Γ*-surfaces* $\mathcal{M}(R, \Gamma)$. Explicitly:

$$\mathcal{M}(R, \Gamma) := \{(S, f_*\Gamma) \mid f : R \to S \text{ is a label-preserving homeomorphism } \}/\sim_\Gamma,$$

and two pairs $(S_1, f_{1*}\Gamma) \sim_\Gamma (S_2, f_{2*}\Gamma)$ if and only if there exists a label-preserving biholomorphism $h : S_1 \to S_2$ so that $h_*(f_{1*}\Gamma) = f_{2*}\Gamma$.

The moduli space $\mathcal{M}(R)$ is the simplest example of a Γ-surface moduli space, that is: where the simple curve class is $\Gamma = (\varnothing)$.

Proposition 2.2. *Given a moduli space* $\mathcal{M}(R, \Gamma)$ *of* Γ*-surfaces, the natural map*

$$\begin{aligned} \Pi_\Gamma : \mathcal{T}(R) &\to \mathcal{M}(R, \Gamma), \\ [S, f] &\mapsto [S, f_*\Gamma] \end{aligned}$$

is the quotient map for $\mathcal{T}(R)$ *quotiented by* $\mathrm{Stab}(\Gamma) \leqslant \mathrm{Mod}(R)$.

Proof. Let $[h] \cdot [S, f] = [S, f \circ h]$ be an arbitrary element in the $\mathrm{Stab}(\Gamma)$ orbit of a marked surface $[S, f]$, then

$$\Pi_\Gamma([S, f \circ h]) = [S, f_*(h_*\Gamma)] = [S, f_*\Gamma] = \Pi_\Gamma([S, f]).$$

Therefore, the $\mathrm{Stab}(\Gamma)$-orbits of any marked surface maps to a unique point in the target. On the other hand, given two points in $\mathcal{T}(R)$ mapping to the same point:

$$[S_1, f_1], [S_2, f_2] \mapsto [S_1, f_{1*}(\Gamma)] = [S_2, f_{2*}(\Gamma)],$$

there is a biholomorphism $g : S_1 \to S_2$, such that $g_*(f_{1*}(\Gamma)) = f_{2*}(\Gamma)$. Then, the mapping class

$$[h] := [f_2^{-1} \circ g \circ f_1] \in \mathrm{Stab}(\Gamma) \leqslant \mathrm{Mod}(R)$$

takes $[S_1, f_1]$ to $[S_2, f_2]$. Therefore, the preimage of any point in $\mathcal{M}(R, \Gamma)$ is a $\mathrm{Stab}(\Gamma)$-orbit. For surjectivity: any point in $\mathcal{M}(R, \Gamma)$ is definitionally of the form $[S, f_*\Gamma]$ and hence the image of $[S, f] \in \mathcal{T}(R)$. $\square$

To describe the next class of moduli spaces, we mentally fill in the punctures on R and think of these added labelled compactification points as lying on the boundary of R. [3] We use the notation $\hat{\partial} R$ to refer to this *extended border* of R containing these punctures as well as ∂R. If $\hat{\partial} R \neq \varnothing$, consider the relative homotopy class of paths in $R \cup \hat{\partial} R$

$$\begin{aligned}[(I, \partial I), (R \cup \hat{\partial} R, \hat{\partial} R)] := \{ &\text{ relative homotopy classes of} \\ &\text{puncture-relative paths in } R \cup \hat{\partial} R \} .\end{aligned}$$

By *puncture-relative*, we mean two things:

- although the end-points of the path may lie on the punctures of R, the interior points of our path must avoid them, and
- we may not homotope a path onto or through punctures.

Definition 2.3. *We call a tuple $A = ([\alpha_1], \ldots, [\alpha_n])$ of relative homotopy classes of paths an* arc class*. Any homotopy class of a path $[\alpha_i]$ that can be represented by a relative embedding $\alpha_i : I \to R \cup \hat{\partial} R$ is called* simple*, and two distinct relative homotopy classes of paths $[\alpha_i]$ and $[\alpha_j]$ which may be realised by maps with disjoint images are called* disjoint*. An arc class where each $[\alpha_i]$ is simple/disjoint is called a* simple/disjoint arc class*.*

As before, a homeomorphism $f : R \to S$ induces a map f_* by post-composition

$$\begin{aligned} f_* : [(I, \partial I), (R \cup \hat{\partial} R, \hat{\partial} R)] &\to [(I, \partial I), (S \cup \hat{\partial} S, \hat{\partial} S)] \\ [\alpha] &\mapsto f_*[\alpha] := [f \circ \alpha], \end{aligned}$$

we accordingly define $f_* A := (f_*[\alpha_1], \ldots, f_*[\alpha_n])$ for an arc class A on R. Just as with curve classes, the mapping class group $\mathrm{Mod}(R)$ has a well-defined action on the collection of arc classes on R, allowing us to define the stabiliser subgroup $\mathrm{Stab}(A)$ of an arc class A.

Example 3 (Moduli Spaces of A-Surfaces). Fix an arc class A on R. Pairing each Riemann surface $S \in \mathrm{Ob}_{\mathcal{R}mn(R)}$ with arc classes of the form $f_* A$, we obtain the *moduli space of A-surfaces* $\mathcal{M}(R, A)$. Explicitly:

$$\mathcal{M}(R, A) := \{(S, f_* A) \mid f : R \to S \text{ is a label-preserving homeomorphism } \} / \sim_A ,$$

and two pairs $(S_1, f_{1*} A) \sim_A (S_2, f_{2*} A)$ if and only if there exists a label-preserving biholomorphism $h : S_1 \to S_2$ so that $h_*(f_{1*} A) = f_{2*} A$.

The same proof for Proposition 2.2 applies for moduli spaces $\mathcal{M}(R, A)$ of A-surfaces. Therefore:

Proposition 2.4. *Given a moduli space $\mathcal{M}(R, A)$ of A-surfaces, the natural map*

$$\begin{aligned} \Pi_A : \mathcal{T}(R) &\to \mathcal{M}(R, A), \\ [S, f] &\mapsto [S, f_* A] \end{aligned}$$

is the quotient map for $\mathcal{T}(R)$ quotiented by $\mathrm{Stab}(A) \leqslant \mathrm{Mod}(R)$.

[3] This is including interior punctures that have been *mentally* filled in.

We call a disjoint simple arc class with maximally many entries an *ideal triangulation*. We adopt the notation:

$$\mathrm{Tri}(R) = \{\text{ ideal triangulations } \triangle \text{ on } R\ \},$$

to refer to the set of ideal triangulations on a given Riemann surface R. In fact, ideal triangulations may be used to give an alternative description of the Teichmüller space whenever R has non-empty extended boundary (i.e.: it has punctures).

Proposition 2.5. *Given an ideal triangulation* $\triangle = ([\alpha_1], \ldots, [\alpha_n])$ *of a Riemann surface* R*, the map* $\Pi_\triangle$ *gives a bijection between* $\mathcal{T}(R)$ *and the moduli space* $\mathcal{M}(R, \triangle)$ *of* $\triangle$*-surfaces.*

Proof. By Proposition 2.4, it suffices to show that $\mathrm{Stab}(\triangle)$ is trivial. Fix disjoint simple representatives α_i for each $[\alpha_i]$. Then, given a mapping class $[h] \in \mathrm{Stab}(\Gamma)$, we may isotope $h : R \to R$ to another homeomorphism h' that fixes each α_i set-wise. Since the paths α_i are disjoint, we may further isotope on small neighbourhoods around each α_i to obtain a homeomorphism $h'' : R \to R$ that fixes each α_i point-wise. Since the complements of the α_i in R are disks, we may isotope h'' to the identity map on R. □

If a Riemann surface R has no punctures on its (possibly empty) boundary components (i.e.: R is a cusped/bordered Riemann surface), then *Fenchel-Nielsen classes* give a similar type of characterisation for the Teichmüller space $\mathcal{T}(R)$. To construct a Fenchel-Nielsen class, we start with a *pants decomposition* P — a disjoint simple curve class with maximally many entries. We then add to $P = ([\gamma_1], \ldots, [\gamma_n])$ one simple homotopy class $[\gamma_i']$ of free loops for each $[\gamma_i]$ such that $[\gamma_i']$ is

1. disjoint from $[\gamma_j]$ for all $j \neq i$, and
2. intersects $[\gamma_i]$ minimally among the set of all homotopy classes of simple free loops which satisfy the first condition.

Note that the $\{[\gamma_i']\}$ we choose may (and do) intersect. Note also that condition 2 is equivalent to requiring that there are representative simple free loops γ_i and γ_i' which intersect once if γ_i is an essential curve on an embedded 1-holed torus in R whose border curve corresponds to a homotopy class in $\mathcal{P}$, and which intersect twice if γ_i is an essential curve on an embedded 4-holed sphere in R whose border curves correspond to homotopy classes in $\mathcal{P}$.

We add another simple homotopy class $[\gamma_i'']$ of free loops for each $[\gamma_i]$ such that γ_i'' is obtained from either Dehn-twisting γ_i' along γ_i if they intersect once, or half Dehn-twisting if they intersect twice.

We call the resulting curve class

$$([\gamma_1], \ldots, [\gamma_n], [\gamma_1'], \ldots, [\gamma_n''], [\gamma_1''], \ldots, [\gamma_n''])$$

a *Fricke-Klein* class.

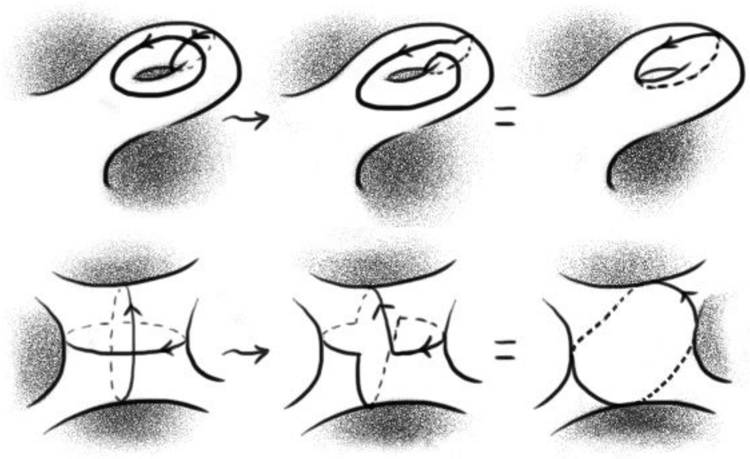


FIGURE 2.1: Dehn-twisting (top) and half Dehn-twisting (bottom) to get γ_i''.

Proposition 2.6. *Given a Fenchel-Nielsen class* $\Gamma = ([\gamma_1], \ldots, [\gamma_n], [\gamma_1'], \ldots, [\gamma_n'])$ *on a cusped/bordered Riemann surface* R*, the map* Π_Γ *gives a bijection between* $\mathcal{T}(R)$ *and the moduli space* $\mathcal{M}(R, \Gamma)$ *of* Γ*-surfaces.*

Proof. By Proposition 2.2, we only need to show that $\mathrm{Stab}(\Gamma)$ is trivial. As with the proof of Proposition 2.5, a mapping class $[h]$ may be represented by a homeomorphism $h : R \to R$ that fixes $\gamma_1, \ldots, \gamma_n$ setwise. Then, by isotoping away disks bounded by $h(\gamma_1')$ and γ_1' while preserving $\gamma_1, \ldots, \gamma_n$, results in a homeomorphism $h_1 \in \mathrm{Homeo}(R)^+$ that preserves both $\gamma_1, \ldots, \gamma_n$ and γ_1'. Repeating this process for $\gamma_2', \ldots, \gamma_n'$, we obtain a homeomorphism $h_n : R \to R$ isotopy equivalent to h that setwise preserves all of these representative free loops. We take h to be h_n without loss of generality.

Since h preserves each γ_i and γ_j', it must preserve the collection of intersection points between any two free loops. In particular, if γ_i and γ_i' consists of just one point, then h trivially preserves that point. And if γ_i and γ_i' consists of two points, then h still preserves both of these points because h preserves the directions on these curves, and γ_i crosses γ_i' from the left at one point and from the right at the other. Thus, at least one intersection point on each γ_i or γ_j' is fixed by h. This then fixes all possible intersection points on each of these curves, as the alternative would require h to cyclically permute these intersection points. Thus, h fixes all of the intersection points among the free loop representatives in Γ, and by a small neighbourhood isotopy, we obtain a homeomorphism h' of R that fixes each γ_i or γ_j' point-wise. Since Fenchel-Nielsen curve classes cut up R into disks and annuli containing at most one boundary loop, the homeomorphism h' is isotopy equivalent to the identity map on R. $\square$

Corollary 2.7. *Given a Fricke-Klein class* Γ *on* R*, the map* Π_Γ *is a bijection.*

Note 2.2. Proposition 2.5 may be thought of as a "topological" version of Penner's λ-length coordinates in [Pen87, Theorem 3.1], which says that it suffices to just keep track of the horocycle-normalised λ-lengths of an ideal triangulation. Similarly, Corollary 2.7 is a weaker form of the Fricke-Klein embedding theorem [IT92, Theorem 3.12] or [Kee71, Theorem 7], which says that it suffices to keep track of only the lengths of the unique geodesic representatives of each homotopy class of free loops in a Fricke-Klein class.

Definition 2.8. *A pair* (Γ, A) *consisting of a curve class* Γ *and an arc class* A *is called a* mixed class. *A mixed class is* simple *if* Γ *is a simple curve class and* A *is a simple arc class. And a mixed class is* disjoint *if there are disjoint representatives for all the (relative) homotopy classes of* Γ *and* A*.*

Example 4 (Moduli space of (Γ, A)-surfaces)*.* As with the two previous examples, the *moduli space of* (Γ, A)*-surfaces* $\mathcal{M}(R, (\Gamma, A))$is given by:

$$\begin{aligned}&\mathcal{M}(R, (\Gamma, A))\\&\quad:=\{(S, (f_*\Gamma, f_*A)) \mid f : R \to S \text{ is a label-preserving homeomorphism }\} / \sim_{(\Gamma,A)},\end{aligned}$$

and two pairs $(S_1, (f_{1*}\Gamma, f_{1*}A)) \sim_{(\Gamma,A)} (S_2, (f_{2*}\Gamma, f_{2*}A))$ if and only if there exists a label-preserving biholomorphism $h : S_1 \to S_2$ so that $h_*(f_{1*}\Gamma) = f_{2*}\Gamma$ and $h_*(f_{1*}A) = f_{2*}A$.

2.2 Moduli Spaces of Hyperbolic Surfaces

The remaining examples of moduli spaces that we study are best described in terms of hyperbolic surface structures. Before introducing them, we first use the uniformisation theorem and curve shortening to translate the examples of moduli spaces of Riemann surfaces that we've seen so far into the language of hyperbolic geometry.

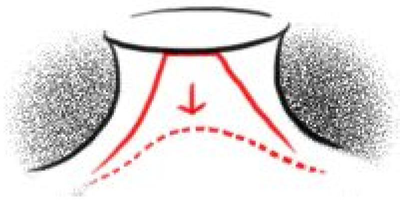
2.2.1 Dictionary

We begin with the necessary *unique geodesic representative* lemmas:

Lemma 2.9. *Given a homotopy class* $[c]$ *of free loops on a hyperbolic surface* S*, there is a unique (up to parametrisation) geodesic loop* γ *representing this homotopy class. Moreover, the geodesic* γ *is simple if* $[c]$ *is simple.*

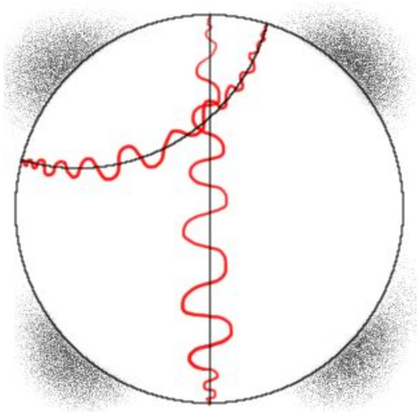
Proof. There is a classical approach to proving these results [Bus92, Theorem 1.6.6] by considering universal covers and noting that homotopy equivalent paths have identical ideal points on the boundary of $\mathbb{H}$, and concluding that the unique geodesic joining these ideal points covers the desired geodesic. Alternatively, an Arzelà-Ascoli theorem based proof may be found in [Bus92, Theorem A.19]. The basic idea is:

1. Take a length-parametrised family of smooth curves γ_n representing $[c]$ with lengths converging to the infimum L length of paths in $[c]$.
2. The Arzelà-Ascoli theorem produces a finite length L curve γ which is geodesic for the sub-arcs of γ which lie on the interior of S. For non-peripheral free loops, this means that γ is a closed geodesic. Because any path that hits a boundary component can be isotoped to a shorter path.



3. Prove the uniqueness of γ by arguing that having two distinct homotopic geodesics γ_1 and γ_2 would either result in hyperbolic 2-gons or a geodesic bordered hyperbolic annulus; both of which are impossible thanks to the Gauss-Bonnet theorem.

The fact that starting with a simple homotopy class $[c]$ results in its geodesic representative γ being a simple geodesic, can be seen by developing the homotopy between c and γ on the universal cover $\tilde{S} \subseteq \mathbb{H}$ of S. In particular, the fact that any lift of γ (and hence of c) is simple and cuts $\tilde{S}$ into two connected components means that c must self-intersect if γ does.



□

Lemma 2.10. *Given a homotopy class $[a]$ of puncture-relative paths on a hyperbolic surface R, there is a unique (up to parametrisation) shortest geodesic path α representing this relative homotopy class, and this shortest geodesic meets boundary geodesics perpendicularly. Moreover, the geodesic α is simple if $[a]$ is simple.*

Proof. The same proof works for geodesic bordered hyperbolic surfaces, and may be found in [Bus92, Theorems 1.52,1.53, A.19]. To adapt this proof for surfaces with cusps, truncate the cusps (or partial cusps) at small horocycles and invoke the theorem. The resulting shortest geodesics will meet these horocycles at right-angles, so upon adding back the excised cusp tips, one may extend these shortest geodesics by orthogonally extending them up the cusp. Alternatively, we could just apply this trick for a family of shorter and shorter horocycles and argue that the family of shortest geodesics produced in this way converges to a bi-infinite geodesic. Just as for curve-shortening, uniqueness may be established using the Gauss-Bonnet theorem. Simplicity may also be derived using the same arguments as above. □

Lemma 2.11. *The geodesic representatives of a disjoint curve class $\Gamma = ([c_1], \ldots, [c_n])$ will be disjoint.*

Proof. We assume without loss of generality that Γ consists of just two homotopy classes of free loops. The proof of this lemma is essentially the same as how we might show that geodesic representatives of simple curve/arc classes are simple. We write this out in a little detail for completeness.

For $i = 1, 2$, let γ_i denote geodesic representatives of the curve class $[c_i]$ and let c_i denote representatives such that c_1 and c_2 are non-intersecting. By lifting the homotopy between γ_i and c_i, we construct a homotopy between a lift $\tilde{\gamma}_i$ and a lift $\tilde{c}_i$ of c_i. If γ_i is closed, then $\tilde{\gamma}_i$ and $\tilde{c}_i$ share ideal points in $\partial_\infty \mathbb{H}$; otherwise, $\tilde{\gamma}_i$ and $\tilde{c}_i$ have their end points lying on the same two lifts of some component of ∂R. This means that if $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are chosen to be intersecting, then the end-points of $\tilde{\gamma}_2$ lie in both components of $\tilde{R} - \tilde{\gamma}_1$. This in turn means (by the Jordan curve theorem) that the end-points of $\tilde{c}_2$ lie in both components of $\tilde{R} - \tilde{c}_1$, thereby contradicting our assumption that c_1 and c_2 are non-intersecting. □

The same proof works for disjoint mixed (Γ, A)-classes. Hence,

Lemma 2.12. *The geodesic representatives of a disjoint mixed class are disjoint.*

To summarise the above lemmas, we have the following correspondences:

- (simple/disjoint) homotopy classes of free loops $\Leftrightarrow$ (simple/disjoint) closed geodesics;
- (simple/disjoint) homotopy classes of paths joining punctures $\Leftrightarrow$ (simple/disjoint) bi-infinite ideal geodesics arcs joining cusps (and/or tines);
- (simple/disjoint) homotopy classes of paths joining a puncture and a border $\Leftrightarrow$ (simple/disjoint) infinite geodesic rays with one end up a cusp or tine and the other end orthogonal to a geodesic boundary component;
- (simple/disjoint) homotopy classes of paths joining two borders $\Leftrightarrow$ (simple/disjoint) geodesic arcs with both ends orthogonal to geodesic boundary components.

2.2.2 Decorations and Boundary Lengths

A key advantage with working in the hyperbolic category is being able to easily specify lengths of geodesics, horocycles and hypercycles (Definition 1.2). Throughout this section, we set R to be a fixed hyperbolic surface of finite type. Also recall that although we, strictly speaking, defined horocycles and hypercycles in terms of immersed curves on the Nielsen extension of R (Definition 1.2′), we nevertheless think of and refer to horocycles and hypercycles as being "*on* R". Note that while examples 1 to 4 have been fiber bundles over the moduli space of Riemann surfaces with discrete fibers, the following examples are fiber bundles with fibers made up of (countably many) copies of $\mathbb{R}_+^k$.

Example 5 (Decorated Moduli Space). Pairing each hyperbolic surface $S \in \mathcal{H}yp(R)$ with horocycles and hypercycles η, we obtain the *decorated moduli space* $\hat{\mathcal{M}}(R)$. Explicitly, $\hat{\mathcal{M}}(R)$ is given by:

$$\left\{ (S,\eta) \;\middle|\; \begin{array}{l} S \text{ is label-preserving homeomorphic to } R, \\ \eta \text{ is a set consisting of one horocycle for each cusp,} \\ \text{one horocyclic segment for each tine and} \\ \text{one hypercycle for each closed border} \end{array} \right\}$$

identified under the equivalence $\sim_{\hat{\mathcal{M}}}$, where $(S_1, \eta_1) \sim_{\hat{\mathcal{M}}} (S_2, \eta_2)$ if and only if there exists an isometry $h : S_1 \to S_2$ and the length of the horocycles and hypercycles of η_1 and η_2 are of the same length for each cusp, tine and closed border. We call these equivalence classes $[S, \eta]$ *decorated surfaces*.

Example 6 (Decorated Teichmüller Space). Pairing each hyperbolic surface $S \in \mathcal{H}yp(R)$ with isotopy classes of homeomorphisms $f : R \to S$ as well as horocycles and hypercycles η, we obtain the *decorated Teichmüller space* $\hat{\mathcal{T}}(R)$. Explicitly, $\hat{\mathcal{T}}(R)$ is given by:

$$\left\{ (S,f,\eta) \;\middle|\; \begin{array}{l} f : R \to S \text{ is a label-preserving homeomorphism,} \\ \eta \text{ is a set consisting of one horocycle for each cusp,} \\ \text{one horocyclic segment for each tine and} \\ \text{one hypercycle for each closed border} \end{array} \right\}$$

identified under the equivalence $\sim_{\hat{\mathcal{T}}}$, where $(S_1, f_1, \eta_1) \sim_{\hat{\mathcal{T}}} (S_2, f_2, \eta_2)$ if and only if $f_2 \circ f_1^{-1} : S_1 \to S_2$ is isotopy equivalent to an isometry and the length of the horocycles and hypercycles of η_1 and η_2 agree for each cusp, tine and closed border. We call these equivalence classes $[S, f, \eta]$ *decorated marked surfaces*.

Example 7 (Partially Decorated Moduli/Teichmüller Space). We at times find it useful to keep track of additional horocycle and horocyclic segments at the cusps and tines but not hypercycles for a given hyperbolic surface $S \in \mathcal{H}yp$. We use $\acute{\mathcal{T}}(R)$ and $\acute{\mathcal{T}}(R)$ to respectively denote the *partially decorated moduli space* and the *partially decorated Teichmüller space* of R.

Example 8 (Boundary Length Specifications). For hyperbolic surfaces with (labelled) m closed geodesic borders, we may specify subspaces $\mathcal{M}(R, \mathbf{L})$ of the moduli space $\mathcal{M}(R)$ by only considering isometry classes of hyperbolic surfaces in $\mathcal{H}yp$ with boundary lengths $\mathbf{L} \in \mathbb{R}_+^m$. We similarly define *boundary length specified* Teichmüller spaces $\mathcal{T}(R, \mathbf{L})$, moduli spaces $\mathcal{M}(R, (\Gamma, A), \mathbf{L})$ of (Γ, A)-surfaces and (partially) decorated moduli and Teichmüller spaces.

Note 2.3. Although we've specified that these boundary lengths $\mathbf{L}$ should all be strictly positive, it is natural to consider length 0 geodesic boundaries as cusps. In this way, moduli spaces of cusped hyperbolic surfaces may be thought of as lying in the "boundary" of moduli spaces of bordered hyperbolic surfaces.

2.3 Topology and Coordinates

Having described a few moduli spaces as sets, we now endow them with topologies using coordinate charts. We start with the Fricke coordinates and the Fenchel-Nielsen coordinates on Teichmüller spaces of cusped and bordered surfaces, generalising to crowned surfaces in Section 2.4.

2.3.1 Closed, Cusped and Bordered Surfaces

For the whole of this subsection, let R be a Riemann surface with g genera, n punctures and m (non-puncture) boundary holes.

2.3.1.1 Fricke Coordinates

In Chapter 1, we saw that any Riemann surface $S \in \mathrm{Ob}_{\mathcal{R}mn(R)}$ may be equipped with a canonical hyperbolic metric so that punctures become cusps and closed loop borders become geodesic boundaries. Extend the hyperbolic surface S to a complete unbordered hyperbolic surface $\overline{S}$ by applying the Nielsen extension, given by attaching hyperbolic trumpets to each geodesic boundary component (Figure 1.2). The resulting surface $\overline{S}$ has the same fundamental group as S and has $\mathbb{H}$ as its universal cover. Since the fundamental group acts as the group of deck-transformations on $\mathbb{H}$, it embeds as a subgroup of $\mathrm{Aut}(\mathbb{H}) = \mathrm{PSL}_2(\mathbb{R})$, unique up to conjugation. The condition that that this subgroup acts properly discontinuously on $\mathbb{H}$ is equivalent to the condition that it's a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$. We call discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$ *Fuchsian*.

Given a Riemann surface R with genus g, m holes and n punctures, fix an ordered collection of homotopy classes of based loops [4]

$$\Gamma_* := ([\alpha_1], \ldots, [\alpha_g], [\beta_1], \ldots, [\beta_g], [\gamma_1], \ldots, [\gamma_n], [\delta_1], \ldots, [\delta_m])$$

which generate the fundamental group $\pi_1(R)$, where the $[\alpha_i]$ and $[\beta_i]$ have representative (unbased) curves which intersect once and all other pairings have representative (unbased) curves which don't intersect. In particular, we choose Γ_* to be arranged as, for example, in Figure 2.2. Specifically, we demand that the fundamental group be explicitly generated as:

$$\pi_1(R) = \langle\, [\alpha_i], [\beta_j], [\gamma_k], [\delta_l] \mid \prod^{g} [[\alpha_i], [\beta_i]] \prod^{n} [\gamma_j] \prod^{m} [\delta_k] = 1 \,\rangle.$$

Homeomorphisms $f : R \to R$ act on homotopy classes by pushforward $f_*[\alpha] := [f \circ \alpha]$. The fact that generators of the fundamental group $\pi_1(R)$ determine all homotopy classes on R and hence determine some Fenchel-Nielsen class on R means that the Teichmüller space $\mathcal{T}(R)$ is in canonical bijection with the following moduli space:

$$\mathcal{M}(R, \Gamma_*) := \{(S, f_*\Gamma_*) \mid f : R \to S \text{ is a label-preserving homeomorphism }\}/\sim_{\Gamma_*},$$

[4] Noting that we've, slightly unfortunately, used the same notation as for homotopy classes of free loops. We use the asterisk subscript of Γ_* to emphasise that this is a collection of homotopy classes of based loops.

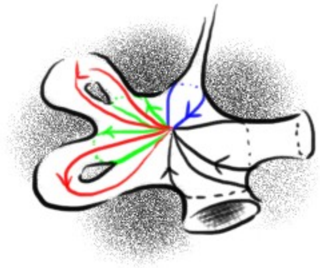


FIGURE 2.2: An example configuration of generators for the fundmental group.

where $(S_1, f_{1*}\Gamma_*) \sim_{\Gamma_*} (S_2, f_{2*}\Gamma_*)$ if and only if there exists a label-preserving isometry $h : S_1 \to S_2$ so that $h_*(f_{1_*}\Gamma_*) = h_*(f_{2*}\Gamma_*)$.

Just as a hyperbolic surface S corresponds to a unique $PSL_2(\mathbb{R})$-conjugacy class of Fuchsian subgroups isomorphic to $\pi_1(S)$, a pair $(S, f_*\Gamma_*)$ corresponds to a unique (up to conjugation by some element of $PSL_2(\mathbb{R})$) discrete faithful representation,

$$\rho : \pi_1(S) \to PSL_2(\mathbb{R})$$

that takes boundary loops around cusps to elliptic matrices — such representations are called *type-preserving*. This gives us an injective map:

$$\begin{aligned} \iota : \mathcal{T}(R) = \mathcal{M}(R, f_*\Gamma) \to & \mathrm{Rep}(\pi_1(R), PSL_2(\mathbb{R})) := \mathrm{Hom}(\pi_1(R), PSL_2(\mathbb{R}))/PSL_2(\mathbb{R}) \\ & = (\text{ the } PSL_2(\mathbb{R})\text{-representation variety of } \pi_1(R)\ ). \end{aligned}$$

The character variety $\mathrm{Rep}(\pi_1(R), PSL_2(\mathbb{R}))$ inherits a quotient topology from the algebraic variety $\mathrm{Hom}(\pi_1(R), PSL_2(\mathbb{R}))$. Thus, we may topologise $\mathcal{T}(R)$ as a subspace of $\mathrm{Rep}(\pi_1(R), PSL_2(\mathbb{R}))$. By normalising the presentation of each conjugacy class of representations, we show that the Teichmüller space is actually a semi-algebraic set (i.e.: it satisfies a collection of polynomials and a collection of polynomial inequalities), with global coordinates called the *Fricke coordinates*:

The conjugacy class of any type-preserving representation $\rho : \pi_1(R) \to PSL_2(\mathbb{R})$ contains a unique *normalised* representative satisfying that:

$$\rho([\alpha_g]) = \pm \begin{bmatrix} a_g & b_g \\ c_g & d_g \end{bmatrix} \text{ so that 1 is an attracting fixed point, and}$$
$$\rho([\beta_g]) = \pm \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \text{ for } \lambda \in (1, \infty).$$

This is true because we can take any triple of ideal points to $(0, 1, \infty)$ via Möbius transformations. Note that 1 being a fixed point of $\rho([\alpha_g])$ means that $a_g + b_g = c_g + d_g$.

No two distinct nontrivial homotopy classes constituting Γ commute, hence the above normalisation for $\rho([\beta_g])$ means that no other elements of $\rho(\Gamma)$ may be in diagonal form. In fact, apart from $\rho[\beta_g]$, the other matrices $\rho([\epsilon])$ constituting $\rho(\Gamma)$ can't be upper or lower-triangular because the sequence $\left\{\rho([\epsilon][\beta_g]^s[\epsilon]^{-1}[\beta_g]^{-s})\right\}_{s\in\mathbb{Z}}$ violates discreteness [IT92, Lemma 2.20]. Thus, we may assign the other elements of Γ to take the following forms:

$$\rho([\alpha_i]) = \pm\begin{bmatrix} a_i^{11} & a_i^{12} \\ a_i^{21} & a_i^{22} \end{bmatrix} \text{ for } i = 1,\dots,g-1 \text{ and where } a_i^{12} > 0,$$
$$\rho([\beta_j]) = \pm\begin{bmatrix} b_j^{11} & b_j^{12} \\ b_j^{21} & b_j^{22} \end{bmatrix} \text{ for } j = 1,\dots,g-1 \text{ and where } b_j^{12} > 0,$$
$$\rho([\gamma_k]) = \pm\begin{bmatrix} c_k^{11} & c_k^{12} \\ c_k^{21} & c_k^{22} \end{bmatrix} \text{ for } k = 1,\dots,n \text{ and where } c_k^{11} + c_k^{22} = 2,$$
$$\rho([\delta_l]) = \pm\begin{bmatrix} d_l^{11} & d_l^{12} \\ d_l^{21} & d_l^{22} \end{bmatrix} \text{ for } l = 1,\dots,m \text{ and where } d_l^{12} > 0.$$

Given the above normalisation conditions, a Fricke coordinate $F : \mathcal{T}(R) \to \mathbb{R}^{6g-6+2n+3m}$ takes a marked surface $[S, f]$ to its unique normalised type-preserving representation and then to:

$$\begin{aligned}(a_2^{11}, a_2^{12}, a_2^{22}, b_2^{11}, b_2^{21}, b_2^{22}, \dots, a_g^{11}, a_g^{21}, a_g^{22}, b_g^{11}, b_g^{21}, b_1^{22},& \\ c_1^{11}, c_1^{21}, \dots, c_g^{11}, c_g^{21}, d_g^{11}, d_g^{21}, d_g^{22}, \dots, d_g^{11}, d_g^{21}, d_g^{22})&\end{aligned}$$

and we have the following classical result [FK65]:

Theorem 2.13 (Fricke Coordinates)**.** *The Fricke coordinate $F : \mathcal{T}(R) \to \mathbb{R}^{6g-6+2n+3m}$ is a global coordinate chart.*

Note 2.4. Although this is a classical theorem, we nevertheless provide a proof here, as one of the main sources [IT92, Theorem 2.25] for a proof of this result appears to be incomplete with regards to determining the relative signs of the entries of $\rho([\alpha_g])$. Moreover, we've generalised slightly so that the result applies for non-closed surfaces.

Proof. The matrices $\rho([\alpha_i])$ are in $\mathrm{PSL}_2(\mathbb{R})$ and hence we may recover the whole matrix from $a_i^{11}, a_i^{21}, a_i^{22}$ via:

$$a_i^{12} = \frac{a_i^{11}a_i^{22} - 1}{a_i^{21}}, \tag{2.1}$$

and the same argument holds for $\rho([\beta_j])$ and $\rho([\delta_l])$. Furthermore, since $\rho([\gamma_k])$ must be elliptic, we know that $c_k^{22} = 2 - c_k^{11}$. Hence, we now know how to recover every matrix in $\rho(\Gamma)$ except for $\rho([\alpha_g])$ and $\rho([\beta_g])$.

The relation among the matrices in $\rho(\Gamma)$ tell us that:

$$\begin{bmatrix} a_g & b_g \\ c_g & d_g \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}\begin{bmatrix} a_g & b_g \\ c_g & d_g \end{bmatrix}^{-1}\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}^{-1}$$

may be expressed in terms of a matrix in $\mathrm{PSL}_2(\mathbb{R})$ completely determined by the Fricke coordinates, that is:

$$\begin{bmatrix} 1+b_g c_g(1-\lambda^{-2}) & a_g b_g(1-\lambda^2) \\ c_g d_g(1-\lambda^{-2}) & 1+b_g c_g(1-\lambda^2) \end{bmatrix} = \pm \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{2.2}$$

where a, b, c, d are functions in the Fricke coordinates. We show that this sign is forced if we are to obtain (real) solutions satisfying our normalisation conditions. Since $\rho([\alpha_g])$ can't be triangular, both $b_g c_g(1-\lambda^2)$ and $b_g c_g(1-\lambda^{-2})$ are non-zero, and hence $\pm a$ and $\pm d$ can't be 1. Thus:

$$\lambda^2 = -\frac{\pm d - 1}{\pm a - 1}. \tag{2.3}$$

Since λ must be a real number, we conclude that if $\operatorname{sgn}(a) = \operatorname{sgn}(d) \neq 0$, then we must choose the sign in (2.2) to be $\operatorname{sgn}(a)$. By the same argument, if $\operatorname{sgn}(a) = 0$, we must choose the sign in (2.2) to agree with $\operatorname{sgn}(d)$ and vice versa. Note that a and d can't both be 0.

When $\operatorname{sgn}(a) = -\operatorname{sgn}(d)$ either:

$$\lambda^2 = \frac{|d|-1}{|a|+1} \text{ or } \frac{|d|+1}{|a|-1}.$$

If only one of these is positive, we take the sign in equation (2.2) according to whether equation (2.3) is realised by taking $\operatorname{sgn}(a)$ or $\operatorname{sgn}(d)$ to replace the $\pm$s. If they're both positive, then we obtain that

$$1 < |a| < |d| - 2. \tag{2.4}$$

Since ad is negative, bc must also be negative and we see that $1 + |ad| = |bc|$. If:

$$\operatorname{sgn}(a) = \operatorname{sgn}(b) = -\operatorname{sgn}(c) = -\operatorname{sgn}(d),$$

then we must take $\operatorname{sgn}(a)$ for the sign in (2.2). Assume not, then:

$$\begin{aligned} a_g b_g = \frac{|b|}{\lambda^2 - 1} > 0,\ b_g c_g = \frac{|a|^2+1}{\lambda^{-2}-1} < 0,\ c_g d_g = \frac{|c|}{1-\lambda^{-2}} > 0 \\ \Rightarrow a_g = \frac{-|b| c_g}{\lambda^2(|a|+1)},\ b_g = \frac{-d_g(|a|+1)}{|c|}, \end{aligned}$$

and by the fact that $a_g + b_g = c_g + d_g$, we see that:

$$c_g^2 = c_g d_g \cdot \frac{c_g}{d_g} = \frac{-|c|^2(|a|+1)}{(1-\lambda^{-2})(|a|+\lambda^{-2}|b|^2+1)} < 0,$$

which gives us the desired contradiction. For the last case, the signs satisfy:

$$\operatorname{sgn}(a) = -\operatorname{sgn}(b) = \operatorname{sgn}(c) = -\operatorname{sgn}(d),$$

we need make use of the fact that 1 is an attracting fixed point of $\rho([\alpha_g])$. In particular, we show that if both sign choices admit real solutions for (2.2), then choosing $\operatorname{sgn}(d)$ results in 1 being a repelling fixed point of $\rho([\alpha_g])$.

Choosing the sign to be $\operatorname{sgn}(d)$, we know that $\lambda^2 = \frac{|d|-1}{|a|+1}$ and so:

$$a_g b_g = \frac{|b|(|a|+1)}{|a|+2-|d|},\ b_g c_g = \frac{(|a|+1)(|d|-1)}{|a|+2-|d|},\ c_g d_g = \frac{|c|(|d|-1)}{|a|+2-|d|}$$
$$\Rightarrow \frac{a_g}{c_g} = \frac{|b|}{|d|-1} > 0 \text{ and } \frac{b_g}{d_g} = \frac{|a|+1}{|c|} > 0.$$

Since $c_g d_g$ is negative, we require that $\frac{d_g}{c_g}$ also be negative. Which in turn means that $\frac{a_g}{c_g} - 1$ and $\frac{b_g}{d_g} - 1$ must have the same sign. Thus, we obtain that one of the following conditions must hold:

$$(D1) : |b| + 1 > |d| \text{ and } |a| + 1 > |c|, \text{ or}$$
$$(D2) : |b| + 1 < |d| \text{ and } |a| + 1 < |c|.$$

Similar analysis for choosing $\operatorname{sgn}(a)$ instead of $\operatorname{sgn}(d)$ produces the following two conditions:

$$(A1) : |b| > |d| + 1 \text{ and } |a| < |c| + 1, \text{ or}$$
$$(A2) : |b| < |d| + 1 \text{ and } |a| > |c| + 1.$$

It's straight forward to see that D2 is compatible with either A1 or A2 and must be discounted. Further, iff A2 and D1 both hold true, then:

$$|bc| < (|d|+1)(|a|-1) = |ad| + 1 - (|d| - |a| - 2) < |ad| + 1 = |bc|,$$

and so we must discount this combination too. This leaves us with A1 and D1, which tell us (combined with previous bounds) that:

$$1 < |a| < |c| + 1 < |a| + 2 < |d| < |b| - 1.$$

Having obtained these inequalities, we return to the $\operatorname{sgn}(d)$ case. Substituting a_g and b_g in $a_g + b_g = c_g + d_g$ yields that:

$$\frac{d_g}{c_g} = \frac{|c|(|b|+1-|d|)}{(|d|-1)(|c|-|a|-1)} < 0.$$

A quick check shows that that the non-1 fixed point of $\rho([\alpha_g])$ is given by:

$$\frac{-b_g}{c_g} = \frac{a_g}{c_g} - 1 - \frac{d_g}{c_g} = \frac{|a|+1}{|a|+1-|c|}\,\frac{|b|+1+|d|}{|d|-1} > 1.$$

Now, since $\rho([\alpha_g]) \cdot 0 = \frac{b_g}{d_g} = \frac{|a|+1}{|c|} > 1$, we see that 1 is a repelling fixed point of $\rho([\alpha_g])$ — contradicting our normalisation condition. Therefore, the sign choice for equation (2.2) is unique, and it's easy to solve for a_g, b_g, c_g, d_g (up to a choice of sign). $\square$

2.3.1.2 Fenchel-Nielsen Coordinates

By Propositions 2.6, the moduli space $\mathcal{M}(R, \Gamma)$ of Γ-surfaces is in natural bijection with the Teichmüller space $\mathcal{T}(R)$ when Γ is a Fenchel-Nielsen class on R:

$$\Gamma = ([\gamma_1], \dots, [\gamma_d], [\gamma_1'], \dots, [\gamma_d']), \text{ where } d = 3g + m + n - 3.$$

Define the *length parameters* $\ell_1, \dots, \ell_d$ of a point $[S, f]$ in Teichmüller space, to be the lengths of the geodesic representatives $f_\# \gamma_1, \dots, f_\# \gamma_d$ of the pants decomposition class

$$(f_*[\gamma_1], \dots, f_*[\gamma_d]).$$

Similarly define $L_1, \dots, L_m$ to be the length parameters for the geodesic boundaries of $[S, f]$. Knowing the geodesic boundary lengths of a hyperbolic pair of pants uniquely specifies its geometry [Bus92, Theorem 3.1.7]. Thus, knowing the lengths parameters completely specifies the geometry of the surface $S - \cup f_\# \gamma_j$.

In order to encode the full geometry of S, we need to also know how these pairs of pants are glued together — a number $\bar{\tau}_j \in \mathbb{R}/\ell_j\mathbb{Z}$ for each $f_\# \gamma_j$ suffices. However, to reconstruct the isotopy class of f, we need to also specify how many times we twist a "cuff" of a pair of pants before regluing. Intuitively, this is specifying a particular lift $\tau_j \in \mathbb{R}$ for $\bar{\tau}_j \in \mathbb{R}/\ell_j\mathbb{Z}$. We call τ_j the *twist parameter* of $[S, f]$ for $f_\# \gamma_j$ and now give a proper geometric description.

Case 1: If γ_i and γ_i' intersect once, then they lie on an embedded 1-bordered hyperbolic torus $S_i \subset S$ with some γ_j as its geodesic boundary. Cutting S_i along γ_i results in a hyperbolic pair of pants P_i, and let α_i denote the unique shortest geodesic arc joining the two boundaries of P_i which arise from cutting γ_i. We homotope γ_i' to a broken geodesic γ_i'' that travels (at unit speed) along α_i and then along γ_i. The twist parameter τ_i is defined to be the signed length for which γ_i'' travels along γ_i. We take τ_i to be negative if γ_i'' turns left (anticlockwise) by $\frac{\pi}{2}$ when turning from α_i into γ_i and positive if it turns right (clockwise).

Case 2: If γ_i and γ_i' intersect twice, then they lie on an embedded 4-bordered hyperbolic sphere $S_i \subset S$ with four (possibly distinct) γ_j as its geodesic boundaries [5]. Cutting S_i along γ_i results in two hyperbolic pairs of pants P_i and Q_i, and let α_i and β_i respectively denote the unique geodesic arc joining the boundaries of P_i and Q_i. We homotope γ_i' to a broken geodesic γ_i'' that travels (at unit speed) along α_i and then along γ_i and then along β_i and then along γ_i. The twist parameter τ_i is defined to be $\frac{1}{2}$ of the signed length for which γ_i'' travels along γ_i. We take τ_i to be negative if γ_i'' turns left (anticlockwise) by $\frac{\pi}{2}$ when turning from α_i or β_i into γ_i and positive if it turns right (clockwise).

Since hyperbolic pairs of pants may have any triple of boundary lengths in $\mathbb{R}_+^3$, the only constraints on the Fenchel-Nielsen coordinates are that length coordinates be positive. Thus:

Theorem 2.14 (Fenchel-Nielsen coordinates)**.** *Given a Riemann surface R with boundaries given by $(\beta_1, \dots, \beta_m)$ and a Fenchel-Nielsen class Γ, where $d = 3g - 3 + m + n$, the following map is a global*

[5] This isn't true for an arbitrary pair of twice-intersecting curves, but is true for Fenchel-Nielsen classes.

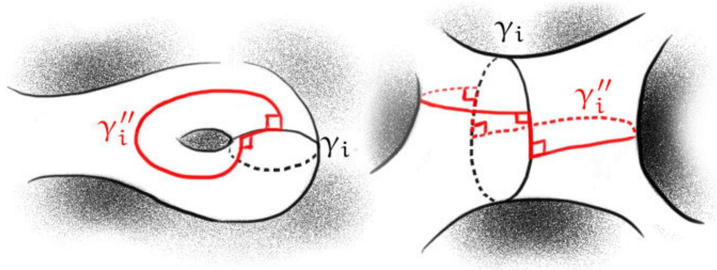


FIGURE 2.3: What γ_i'' is in case 1 (left) and case 2 (right).

coordinate chart for the Teichmüller space:

$$\begin{aligned} \mathrm{FN} : \mathcal{T}(R) &\to (\mathbb{R}_+ \times \mathbb{R})^d \times \mathbb{R}_+^m, \\ [S, f] &\mapsto (\ell_1, \tau_1, \ldots, \ell_d, \tau_d, L_1, \ldots, L_m), \end{aligned}$$

where $\ell_i := \ell_{f_\# \gamma_i}$ *is the length parameter for* γ_i*, and* $L_j := \ell_{f_\# \beta_j}$ *is the length of the* j*-th boundary geodesic and* τ_i *is the twist parameter obtained from* γ_i *and* γ_i'*.*

Note 2.5. The Fricke coordinates and the Fenchel-Nielsen coordinates each endow $\mathcal{T}(R)$ with the structure of a real-analytic open ball. It's classically known [Abi80] that these structures are real-analytically compatible.

The above result gives the Teichmüller space the structure of a fibration over $\mathbb{R}_+^m$, where the fiber for a boundary length vector $\mathbf{L} \in \mathbb{R}_+^m$ is the Teichmüller space $\mathcal{T}(R, \mathbf{L})$ of all hyperbolic surfaces whose boundaries $(\beta_1, \ldots, \beta_m)$ are of length $\mathbf{L}$. This results in the following global coordinates for $\mathcal{T}(R, \mathbf{L})$:

$$\begin{aligned} \mathrm{FN} : \mathcal{T}(R, \mathbf{L}) &\to (\mathbb{R}_+ \times \mathbb{R})^d, \\ [S, f] &\mapsto (\ell_1, \tau_1, \ldots, \ell_d, \tau_d), \end{aligned}$$

which is what's usually meant by Fenchel-Nielsen coordinates.

2.3.1.3 Fricke-Klein Embedding

We won't provide a proof of this result. Intuitively, given the length parameters $\ell_1, \ldots, \ell_d$, it is possible to compute the twist parameter τ_i from the length parameter ℓ_i' and $\ell_1, \ldots, \ell_d$ up to two possible values. The length parameter ℓ_j'' of the extra homotopy classes of free loops $[\gamma_j'']$ added into a Fenchel-Nielsen class to make up a Fricke-Klein class enables us to choose the correct value of τ_i. Thus:

Theorem 2.15. *Given a Fricke-Klein class*

$$\Gamma = ([\gamma_1], \ldots, [\gamma_d], [\gamma_1'], \ldots, [\gamma_d'], [\gamma_1''], \ldots, [\gamma_d'']),$$

the Fricke-Klein *map given by assigning to* Γ *the the length parameters of these homotopy classes of free loops*

$$\begin{aligned}\ell : \mathcal{T}(R) &\to \mathbb{R}_+^{3d}\\ [S,f] &\mapsto (\ell_1,\ldots,\ell_d,\ell_1',\ldots,\ell_d',\ell_1'',\ldots,\ell_d'')\end{aligned}$$

is an embedding.

2.3.1.4 Thurston's Shearing Coordinates

We've already seen two coordinate systems for the Teichmüller space and we give one more — due to Thurston [FLP12]. The *shearing coordinates* for $\mathcal{T}(R)$ are naturally phrased in terms of ideal triangulations $\triangle \in \mathrm{Tri}(R)$ of R, and hence requires that R have cusps or geodesic boundaries.

Given a marked surface $[S, f_*\triangle] \in \mathcal{M}(R,\triangle) = \mathcal{T}(R)$, we orient each boundary geodesic so as to agree with the orientation on S. Lemma 2.10 applied to the ideal triangulation

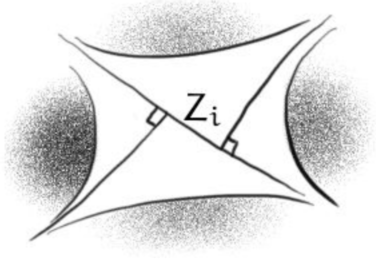
$$f_*\triangle = (f_*[\alpha_1],\ldots,f_*[\alpha_d]), \text{ where } d = 3|\chi(R)| = 6g-6+3m+3n$$

produces an *orthogeodesic triangulation* consisting of orthogeodesics and ideal geodesics, which we denote by $f_\#\triangle = (f_\#\alpha_1,\ldots,f_\#\alpha_d)$. This orthogeodesic triangulation in turn corresponds to a unique *geodesic ideal triangulation*

$$\triangle_f = (\alpha_{1,f},\ldots,\alpha_{d,f}) \tag{2.5}$$

where each $\alpha_{i,f}$ in $\triangle_f$ is a (simple) bi-infinite geodesic, with its ends either shooting up a cusp or spiralling toward a boundary geodesic in the same direction as that boundary is oriented. The geodesic ideal triangulation $\triangle_f$ cuts S up into ideal triangles, and for each arc α_i we define a *shearing length* $Z_i : \mathcal{T}(R) \to \mathbb{R}$ as follows:

Observe that $\alpha_{i,f}$ borders two ideal triangles T_1, T_2. Then drop two perpendiculars from the ideal corners of T_1 and T_2 which aren't end-points of $\alpha_{i,f}$ so as to (perpendicularly) meet $\alpha_{i,f}$. We define Z_i to be the signed length of the segment on $\alpha_{i,f}$ between the two perpendiculars. We choose Z_i to be negative if we turn to the left (anticlockwise) by $\frac{\pi}{2}$ when going from a perpendicular to the segment between the two perpendiculars, and positive if we turn to the right (clockwise).



An interesting property of the shearing coordinates are that for the collection of the $\alpha_{i,f}$ which spiral around a given boundary component $f_\#\beta$ (possibly a cusp), the absolute value of the sum (added with multiplicity) of the corresponding shearing lengths Z_i is equal to the length of β. This can be proven either algebraically using a trace-based characterisation of the shearing coordinates [Pen12]; or geometrically using horocycles to project the midpoints of these α_i onto β as in the proof of Lemma 2.20. Thus, the shearing lengths for the geodesics with one end up a cusp sum to 0. And the shearing lengths for the geodesics with one end spirally around a boundary geodesic necessarily sum to a positive number. These are the only two constraints that we have to impose upon the shearing lengths:

Theorem 2.16 (Thurston's shearing coordinates)**.** *Given an ideal triangulation $\triangle$ on a Riemann surface R with punctured boundaries given by $\beta_1, \ldots, \beta_n$ and bordered boundaries given by $\bar{\beta}_1, \ldots, \bar{\beta}_m$. Let Z_i denote the shearing length of the ith entry of $\triangle$. Then the map*

$$\begin{aligned} Z : \mathcal{T}(R) &\to \mathbb{R}^{3|\chi(R)|} = \mathbb{R}^{6g-6+3m+3n} \\ [S, f] &\mapsto (Z_1, \ldots, Z_{6g-6+3m+3n}) \end{aligned}$$

is a homeomorphic embedding of the Teichmüller space into the subspace of $\mathbb{R}^{3|\chi(R)|}$ satisfying the linear conditions for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ that:

(added with multiplicity) the $\{Z_j\}$ with an end in β_i sum to 0 and

(added with multiplicity) the $\{Z_j\}$ with an end in $\bar{\beta}_i$ sum to a positive number.

Note 2.6. The shearing coordinates map the Teichmüller space $\mathcal{T}(R, \mathbf{L})$ of surfaces with borders $(\bar{\beta}_1, \ldots, \bar{\beta}_m)$ of length $\mathbf{L} = (L_1, \ldots, L_m)$ homeomorphically onto the linear subspace of $\mathbb{R}^{6g-6+3m+3n}$ satisfying the (additional) linear conditions for $i = 1, \ldots, m$ that:

(added with multiplicity) the $\{Z_j\}$ with an end in $\bar{\beta}_i$ sum to L_i.

Moduli spaces described in examples 1 to 4 all arise from quotienting the Teichmüller space $\mathcal{T}(R)$ by some subgroup $\mathrm{Stab}(\Gamma, A)$ of the mapping class group $\mathrm{Mod}(R)$. Since the mapping class group is a countably infinite group and acts almost always freely, with finite stabiliser whenever it doesn't [FM12, Theorem 6.4]. Moreover, the mapping class group $\mathrm{Mod}(R)$ is generated by Dehn twists [FM12, Chapter 4], which act real-analytically with respect to the Fenchel-Nielsen coordinates on $\mathcal{T}(R)$. Therefore, the resulting quotient space $\mathcal{M}(R, (\Gamma, A))$ is at worst a real-analytic manifold/orbifold.

2.3.2 Horocycle and Hypercycle Decorated Surfaces

We now work with horocycle and hypercycle decorated surfaces. The underlying hyperbolic surfaces that we're dealing with are permitted to have cusps and geodesic borders, but can't be closed or crowned hyperbolic surfaces. We first review Penner's λ-length coordinates for Teichmüller spaces of *cusped* hyperbolic surfaces, before generalising λ-coordinates to Teichmüller spaces of *bordered* hyperbolic surfaces.

2.3.2.1 Penner's λ-length Coordinates

Given an ideal triangulation $\triangle \in \mathrm{Tri}(R)$ of an n-cusped genus g surface $R = R_{g,0,n}$, recall that the decorated Teichmüller space is:

$$\hat{\mathcal{T}}(R) = \hat{\mathcal{M}}(R, \triangle) \cong \mathcal{T}(R) \times \mathbb{R}^n_+,$$

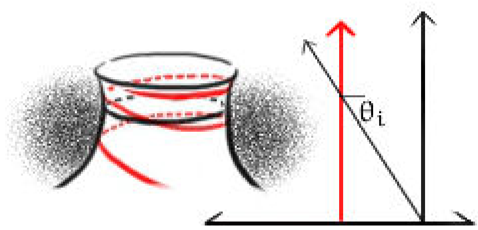
where $\mathbb{R}^n_+$ parameterises the lengths of a collection $\eta = (\eta_1, \ldots, \eta_n)$ of horocycles on a decorated marked surface $[S, f_*\triangle, \eta] \in \hat{\mathcal{T}}(R, \triangle)$. Moreover, for cusped hyperbolic surfaces, the orthogeodesic triangulation $f_\#\triangle$ agrees with the geodesic ideal triangulation described in $\triangle_f$ (2.5). Truncating $\triangle_f$ at the horocycles η yields $d = 6g - 6 + 3n = 3|\chi(R)|$ geodesics of signed (finite) lengths $s_1, s_2, \ldots, s_d$. We define λ-lengths to be:

$$(\lambda_1, \ldots, \lambda_d) := (\exp \tfrac{1}{2} s_1, \ldots, \exp \tfrac{1}{2} s_d).$$

In [Pen87], Penner showed that λ-coordinates give a homeomorphism between the decorated Teichmüller space and $\mathbb{R}^d_+$. In terms of these λ-lengths, the Teichmüller space $\mathcal{T}(R)$ embeds in the decorated Teichmüller space $\hat{\mathcal{T}}(R)$ as the slice of all decorated marked surfaces $[S, f_*\triangle, \eta] \in \hat{\mathcal{T}}(R)$ with length 1 horocycles η. His remark after Corollary 3.4 in [Pen87][6] shows that $\mathcal{T}(R)$ is a "subvariety" of $\mathbb{R}^d_+$, or more accurately: a semi-algebraic set.

2.3.2.2 λ-lengths for Border Surfaces

To generalise Penner's λ-lengths to genus g surfaces $R = R_{g,m,n}$ with m borders $(\beta_1, \ldots, \beta_m)$ and n punctures, we similarly define the λ-length of a structured surface $[R, f_*\triangle, \eta]$ to be $\exp(\frac{1}{2}\cdot)$ of the signed lengths of the geodesic ideal triangulation $\triangle_f$ of S truncated at the collection of hypercycles $(\eta_1, \ldots, \eta_m)$ and horocycles $(\eta_{m+1}, \ldots, \eta_{m+n})$. Let $\theta_i \in (0, \frac{\pi}{2})$ denote $\frac{\pi}{2}$ minus the acute angle formed by the intersection of η_i and any of the geodesics spiralling about β_i truncated by η_i. The following figure shows that θ_i is well-defined:



In the above figure, the vertical y-axis is the lift of the boundary geodesic, and the diagonal ray is a lift of the hypercycle around the boundary. Since any spiralling geodesic forms the same angle with the diagonal ray, the angle θ_i is independent of which spiralling geodesic we choose.

The following theorem is our first generalisation of Penner's λ-lengths:

[6] He explains this is slightly more concretely in Lemma 3.4.2 of [Pen92].

Theorem 2.17. *The following function is a homeomorphism:*

$$\begin{aligned}\Lambda_h : \hat{\mathcal{T}}(R) &\to \mathbb{R}_+^{6g-6+3m+3n} \times (0, \tfrac{\pi}{2})^m \\ [S, f_*\triangle, \eta] &\mapsto (\lambda_1, \dots, \lambda_{6g-6+3m+3n}, \theta_1, \dots, \theta_m).\end{aligned} \tag{2.6}$$

Proof. Given a decorated marked surface $[S, f_*\triangle, \eta]$, the ideal triangulation $\triangle_f$ cuts up S into ideal triangles T with three hypercyclic (or horocyclic) segments truncating each vertex. Viewed in the universal cover, each hypercycle contains one end that meets an ideal vertex of T. We need the following fact regarding these *hypercycle-decorated ideal triangles*:

Lemma 2.18. *Any* 6*-tuple* $(\lambda_1, \lambda_2, \lambda_3, \theta_1, \theta_2, \theta_3) \in \mathbb{R}_+^3 \times (0, \frac{\pi}{2})^3$ *specifies a unique ideal triangle with vertex-truncating hypercyclic segments. In other words: the moduli space of such hypercycle-decorated ideal triangles is* $\mathbb{R}_+^3 \times (0, \frac{\pi}{2})^3$.

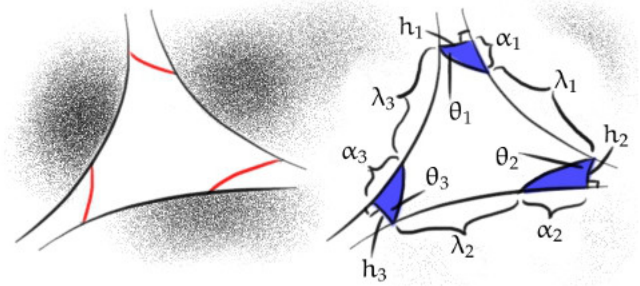


FIGURE 2.4: An ideal triangle T with hypercyclic segments (left) and with labels (right).

Proof. In Figure 2.4, besides the hypercyclic segments at each cusp, we also marked out horocyclic segments of length h_i at each cusp to align with the exterior endpoint of the hypercyclic segments. In this manner, we assign a small triangular region to each cusp, and we use α_i to denote the λ-length of the one geodesic side of this triangular region. Using Proposition 1.3, we know that:

$$h_1 = \frac{\alpha_2\lambda_2}{\alpha_3\lambda_3\alpha_1\lambda_1}, \quad h_2 = \frac{\alpha_3\lambda_3}{\alpha_1\lambda_1\alpha_2\lambda_2}, \quad h_3 = \frac{\alpha_1\lambda_1}{\alpha_2\lambda_2\alpha_3\lambda_3}. \tag{2.7}$$

The λ-lengths $(\lambda_1, \lambda_2, \lambda_3, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}_+^6$ gives a global coordinate system on the moduli space of hypercycle-decorated ideal triangles, because we can first determine from $\alpha_1\lambda_1, \alpha_2\lambda_2, \alpha_3\lambda_3$ where to position the horocyclic segments of length h_1, h_2, h_3 and from there determine the position of the necessary hypercyclic segments.

The following trigonometric identity may be shown to hold for T by explicit computation using Figure 2.5:

$$\alpha_i^{-2} = 1 - h_i \tan\theta_i. \tag{2.8}$$

This tells us that there's a well-defined map $\mathbb{R}_+^6 \to \mathbb{R}_+^3 \times (0, \frac{\pi}{2})^3$ that assigns to a 6-tuple $(\lambda_1, \lambda_2, \lambda_3, \alpha_1, \alpha_2, \alpha_3)$ the 6-tuple $(\lambda_1, \lambda_2, \lambda_3, \theta_1, \theta_2, \theta_3)$. Our strategy for proving this lemma will

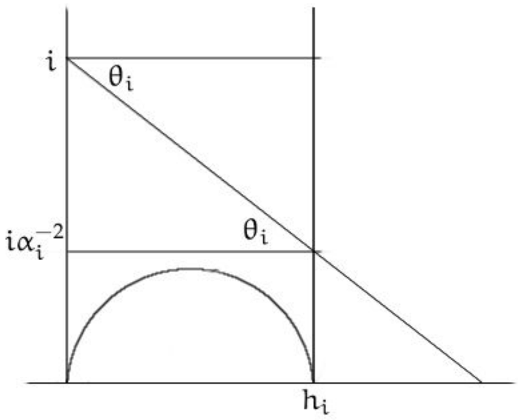


FIGURE 2.5: A figure for computing α_i in terms of θ_i and h_i.

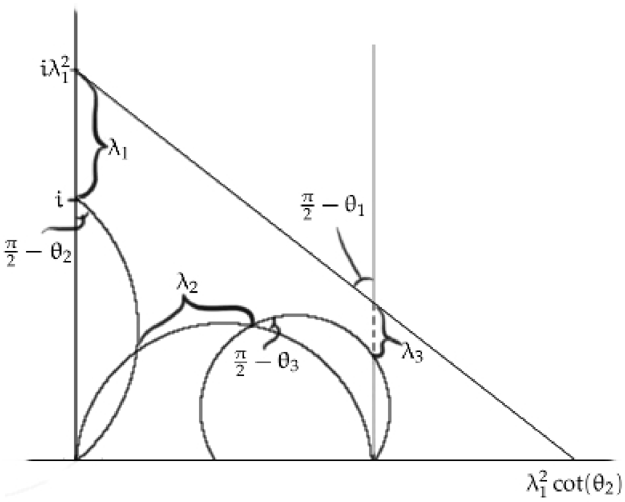


FIGURE 2.6: A figure to showing that λ_3 varies over all of $\mathbb{R}_+$.

be to first show that this map is surjective, and then to explicitly construct a real-analytic inverse.

Fix, for the moment, only the (arbitrary) positive real numbers $\lambda_1, \lambda_2, \theta_1, \theta_2, \theta_3$, we now construct a family of hypercycle-decorated ideal triangles in the hyperbolic plane so as to cover all possibilities for λ_3. We may, without loss of generality, position the geodesic segment of λ-length λ_1 on the y-axis of the Poincaré half-plane from i to $i\lambda_1^2$. The condition that the hypercycles corresponding to θ_1 and θ_2 must meet the ideal end-points of the λ_1-geodesic then uniquely determines these hypercycles and we obtain Figure 2.6. The geodesic on which λ_2 lies has one ideal point at $0 \in \overline{\mathbb{H}}$ and the other may lie anywhere to the left of the ideal point $\lambda_1^2 \cot(\theta_2) \in \mathbb{R} \subset \overline{\mathbb{H}}$ of the θ_2 hypercycle, thus we may imagine freely moving this ideal point within $(0, \lambda_1^2 \cot(\theta_2))$ and seeing the resulting λ_3 it produces. In particular, as this ideal point approaches 0, the value of λ_3 blows up and as this ideal point approaches $\lambda_1^2 \cot(\theta_2)$, the value of λ_3 approaches 0. By continuity, we obtain every possible $\lambda_3 \in \mathbb{R}_+$.

Combining (2.7) and (2.8), we see that:

$$\begin{aligned}(h_1 \tan\theta_1)(h_2 \tan\theta_2) &= \lambda_1^{-2} \tan\theta_1 \tan\theta_2 (1 - h_1 \tan\theta_1),\\ (h_2 \tan\theta_2)(h_3 \tan\theta_3) &= \lambda_2^{-2} \tan\theta_2 \tan\theta_3 (1 - h_2 \tan\theta_2),\\ (h_3 \tan\theta_3)(h_1 \tan\theta_1) &= \lambda_3^{-2} \tan\theta_3 \tan\theta_1 (1 - h_3 \tan\theta_3).\end{aligned} \tag{2.9}$$

Setting

$$c_1 := \lambda_1^{-2} \tan\theta_1 \tan\theta_2,\ c_2 := \lambda_2^{-2} \tan\theta_2 \tan\theta_3,\ c_3 := \lambda_3^{-2} \tan\theta_3 \tan\theta_1,$$

and solving for $h_1 \tan\theta_1$, we obtain the quadratic relation:

$$0 = c_3(1 + c_2)(h_1 \tan\theta_1)^2 + (c_1c_2 + c_1c_3 - c_2c_3 + c_1c_2c_3)h_1 \tan\theta_1 - c_1c_2(1 + c_3).$$

Since c_1, c_2, c_3 are all positive, the above quadratic in $h_1 \tan\theta_1$ has only one positive root:

$$\begin{aligned}h_1 \tan\theta_1 =& \frac{\sqrt{(c_1c_2 + c_1c_3 - c_2c_3 + c_1c_2c_3)^2 + 4c_1c_2c_3(1 + c_2)(1 + c_3)}}{2c_3(1 + c_2)}\\ &- \frac{(c_1c_2 + c_1c_3 - c_2c_3 + c_1c_2c_3)}{2c_3(1 + c_2)}\end{aligned}$$

and we see that h_1 can be expressed as a real analytic function in $\lambda_1, \lambda_2, \lambda_3, \theta_1, \theta_2, \theta_3$. This follows by symmetry for h_2 and h_3. Substituting this into (2.8) and using the fact that the α_i are exponentials (and hence positive), we obtain the desired inverse map. □

Returning to the proof of Theorem 2.17, a decorated marked hyperbolic surface $[S, f_*\triangle, \eta]$ may be reconstructed from its decomposition (along $\triangle_f$) into hypercycle-decorated ideal triangles. Simply take one copy of each constituent hypercycle-decorated ideal triangle and glue them together so that the hypercyclic segments align. In so doing, we reproduce S minus its boundary geodesics. Adding these boundary geodesics back in recovers S with a geodesic ideal triangulation $\triangle_f$ and hypercycles η. Since Lemma 2.18 enables us to uniquely specify these hypercycle-decorated ideal triangles in terms of the coordinates $(\lambda_1, \ldots, \lambda_{6g-6+3m+3n}, \theta_1, \ldots, \theta_m)$, we can construct $[S, f_*\triangle, \eta]$ from these λ-lengths and horocycle angles. The continuity of this inverse map follows from the continuous variation of the structure of each hypercycle-decorated ideal triangle as we vary these parameters. □

We now give an alternative λ-length-based parametrisation of the decorated Teichmüller space $\hat{\mathfrak{T}}(R)$ for a bordered surface R. The θ_i angle coordinates are replaced by L_i — the length of the i-th boundary of a given decorated marked surface $[S, f_*\triangle, \eta]$.

Theorem 2.19. *The following function is (also) a homeomorphism:*

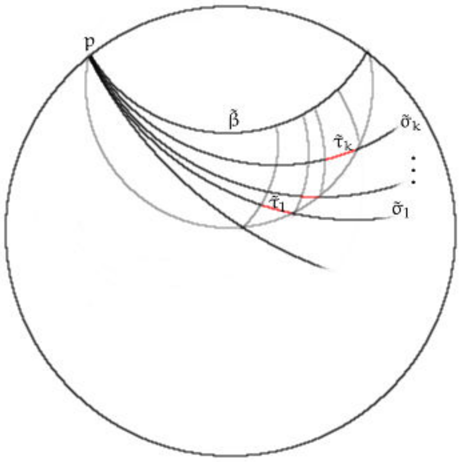
$$\begin{aligned}\Lambda_b : \hat{\mathfrak{T}}(R) &\to \mathbb{R}_+^{6g-6+3m+3n} \times \mathbb{R}_+^m\\ [S, f_*\triangle, \eta] &\mapsto (\lambda_1, \ldots, \lambda_{6g-6+3m+3n}, L_1, \ldots, L_m),\end{aligned} \tag{2.10}$$

we call Λ_b *the* (generalised) λ-length coordinates.

Proof. Any hypercycle-decorated marked surface $[S, f_*\triangle, \eta]$ may be decomposed into $2|\chi(R)|$ ideal triangles by cutting it along the $3|\chi(R)|$ ideal geodesics constituting geodesic ideal triangulation $\triangle_f$ composed of ideal geodesics which spiral into the boundaries of S. Using the notation we introduced in Figure 2.4, these triangles give us $6|\chi(R)|$ many α_i terms: one for each spiralling end of an ideal geodesic. Note that $\alpha_i = 1$ precisely when the end of an ideal geodesic goes up a cusp, otherwise $\alpha_i > 1$.

Lemma 2.20. *The product of all of the* $\{\alpha_j\}$ *corresponding to an end of an ideal geodesic spiralling into the boundary* β *is equal to* $\exp \frac{\ell_\beta}{2}$.

Proof. Given a sequence $\alpha_1, \ldots, \alpha_k$ corresponding to geodesic segments lying on the ideal geodesics $\sigma_1, \ldots, \sigma_k$ spiralling around (and into) the boundary β. Consider a collection of lifts of these geodesics in the universal cover $\tilde{S} \subset \mathbb{H}$ as shown in the following figure.



The lifts $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_k$ all converge at the ideal endpoint p of $\tilde{\beta}$. We may *horocyclically project* these geodesic segments $\tilde{\tau}_i$ of λ-length α_i on $\tilde{\sigma}_i$ onto $\tilde{\beta}$ by marking in the two horocycles based at $p \in \mathbb{H}$ which meet the start and the end of $\tilde{\tau}_i$. We call the geodesic segment on $\tilde{\beta}$ the *horocyclic projection* of $\tilde{\tau}_i$ onto $\tilde{\beta}$. Since horocyclic projections are length-preserving, the horocyclic projection of $\tilde{\tau}_i$ onto $\tilde{\beta}$ also has λ-length α_i. The hypercyclic segments on the ideal triangles corresponding to these geodesic segments $\tilde{\tau}_1, \ldots, \tilde{\tau}_k$ join to cover a segment of the lift $\tilde{\eta}$ of the hypercycle η around β. In particular, since the start and end point of this segment map to the same point in S, these hypercyclic segments join to give a segment that covers η precisely once. Thus, the horocyclic projections of $\tilde{\tau}_1, \ldots, \tilde{\tau}_k$ onto $\tilde{\beta}$ also join to form a geodesic segment of $\tilde{\beta}$ that covers β precisely once. Hence the product of the α_j must be the λ-length of β, which is $\exp \frac{\ell_\beta}{2}$.

□

Taking the logarithm of the three equations in (2.9), differentiating with respect to one of the angle coordinates θ_j and solving simultaneous equations then yields:

$$\begin{aligned}\frac{\partial 2\log \alpha_1}{\partial \theta_j} =& \frac{h_1 \tan\theta_1}{1+(1-h_1\tan\theta_1)(1-h_2\tan\theta_2)(1-h_3\tan\theta_3)} \\ &\times\Bigg((1+(1-h_2\tan\theta_2)(1-h_3\tan\theta_3))\frac{\partial \log\tan\theta_1}{\partial\theta} \\ &+h_2\tan\theta_2\frac{\partial\log\tan\theta_2}{\partial\theta_j}-h_3\tan\theta_3(1-h_2\tan\theta_2)\frac{\partial\log\tan\theta_3}{\partial\theta_j}\Bigg),\end{aligned} \tag{2.11}$$

as well as the two other equations for $\frac{\partial 2\log\alpha_2}{\partial\theta_j}$ and $\frac{\partial 2\log\alpha_3}{\partial\theta_j}$ related by cyclically shifting indices.

Lemma 2.21. *The following function is an immersion:*

$$\begin{aligned}\Lambda := \Lambda_b\circ\Lambda_h^{-1} : \hat{\mathfrak{T}}(R) = \mathbb{R}_+^{3|\chi(R)|}\times(0,\tfrac{\pi}{2})^m &\to \mathbb{R}_+^{3|\chi(R)|}\times\mathbb{R}_+^m \\ (\lambda_1,\ldots,\lambda_{3|\chi(R)|},\theta_1,\ldots,\theta_m) &\mapsto (\lambda_1,\ldots,\lambda_{3|\chi(R)|},L_1,\ldots,L_m).\end{aligned}$$

Proof. The Jacobian of Λ takes the form:

$$D\Lambda = \begin{bmatrix} I & * \\ 0 & X\end{bmatrix},$$

where I is the $3|\chi(R)|\times 3|\chi(R)|$ identity matrix. To show that $D\Lambda$ is invertible, and hence Λ is an immersion, we show that X is a strictly diagonally dominant matrix and invoke the *(strictly) diagonally dominant matrix theorem* [Tau49].

From our previous calculations (2.11), we obtain that:

$$\begin{aligned}\frac{\partial 2\log\alpha_1}{\partial\theta_1} &= \frac{h_1\tan\theta_1[1+(1-h_2\tan\theta_2)(1-h_3\tan\theta_3)]}{\sin\theta_1\cos\theta_1[1+(1-h_1\tan\theta_1)(1-h_2\tan\theta_2)(1-h_3\tan\theta_3)]}\\ \frac{\partial 2\log\alpha_2}{\partial\theta_1} &= \frac{-h_1\tan\theta_1 h_2\tan\theta_2(1-h_3\tan\theta_3)}{\sin\theta_1\cos\theta_1[1+(1-h_1\tan\theta_1)(1-h_2\tan\theta_2)(1-h_3\tan\theta_3)]}\\ \frac{\partial 2\log\alpha_3}{\partial\theta_1} &= \frac{h_1\tan\theta_1 h_3\tan\theta_3}{\sin\theta_1\cos\theta_1[1+(1-h_1\tan\theta_1)(1-h_2\tan\theta_2)(1-h_3\tan\theta_3)]}.\end{aligned}$$

And since

$$\begin{aligned}1+(1-h_2\tan\theta_2)(1-h_3\tan\theta_3)-h_2\tan\theta_2(1-h_3\tan\theta_3)-h_3\tan\theta_3\\ =2(1-h_2\tan\theta_2)(1-h_3\tan\theta_3)>0,\end{aligned}$$

for each ideal triangle,

$$\frac{\partial 2\log\alpha_1}{\partial\theta_1} > |\frac{\partial 2\log\alpha_2}{\partial\theta_1}|+|\frac{\partial 2\log\alpha_3}{\partial\theta_1}|. \tag{2.12}$$

Let $I_i \ni j$ be the index set of all of the α_j that correspond to the end of an ideal geodesic spiralling into the i-th boundary. By Lemma 2.20,

$$L_i = \sum_{j \in I_i} 2 \log \alpha_j,$$

and hence:

$$\frac{\partial L_i}{\partial \theta_1} = \sum_{j \in I_i} \frac{\partial 2 \log \alpha_j}{\partial \theta_1} > \sum_{k \notin I_i} \frac{\partial 2 \log \alpha_k}{\partial \theta_1} = \sum_{l \neq i} \frac{\partial L_l}{\partial \theta_1},$$

where the above inequality is partially due to (2.12) applied to triangles which contain one of the $\alpha_j, j \in I_i$, and partially because α_k terms which don't arise on a triangle spiralling into the i-th boundary are not affected by the change in θ_1.

Since the i-th row of the matrix X consists of partial derivatives of the form $\frac{\partial L_i}{\partial \theta_j}$, (the transpose of) the matrix L is therefore strictly diagonally dominant and the Jacobian of Λ is hence always invertible. Thus, Λ is an immersion. □

Using Lemma 2.21, we now prove by contradiction that Λ is injective. Since λ-lengths are preserved, two surfaces with distinct λ-lengths necessarily map to distinct points. Therefore, we consider two distinct decorated surfaces

$$(\lambda_1, \ldots, \lambda_{3|\chi(R)|}, \theta_1^i, \ldots, \theta_m^i) \in \hat{\mathfrak{T}}(R), \; i = 1, 2,$$

which don't differ in their λ-lengths, but both map to the same point $(\lambda_1, \ldots, \lambda_{3|\chi(R)|}, L_1, \ldots, L_m)$. Then, the Euclidean straight path $\mathcal{I}$ joining these surfaces maps to a closed loop in the following subspace of the codomain:

$$\{(\lambda_1, \ldots, \lambda_{3|\chi(R)|})\} \times \mathbb{R}_+^m \subset \mathbb{R}_+^{3|\chi(R)|} \times \mathbb{R}_+^m.$$

We peturb $\mathcal{I}$ near its end points so that the image of $\mathcal{I}$ is a smooth, closed loop $\gamma_{\mathcal{I}}$ lying within the same subspace of the codomain. Since Λ is a local diffeomorphism, the preimage of $\gamma_{\mathcal{I}}$ in $\hat{\mathfrak{T}}(R)$ consists of 1-dimensional submanifolds: closed loops and paths. If the path $\mathcal{I}$ were to lie on a closed loop γ in the preimage of $\gamma_{\mathcal{I}}$, then γ must be a finite n-cover of $\gamma_{\mathcal{I}}$ for $n > 1$, and hence any disk bounded by γ must contain a ramification point. But this would then contradict the local diffeomorphicity of Λ. Thus, we conclude that $\mathcal{I}$ lies on some arc $\mathcal{J}$.

By continuity, the preimage of γ is closed, hence the local diffeomorphicity of Λ ensures that the arc $\mathcal{J}$ cannot have endpoints in $\hat{\mathfrak{T}}(R)$. Since the first $3|\chi(R)|$ coordinates of $\mathcal{J}$ don't vary, this means that (wlog) the θ_1-coordinate of the arc $\mathcal{J}$ either approaches 0 or $\frac{\pi}{2}$. In the first case, since L_i is bounded above 0 on γ and only finitely many ideal triangles spiral into the i-th boundary, Lemma 2.20 ensures that there is a sequence of decorated marked surfaces for which (wlog) α_1 remains bounded above some $\epsilon > 1$. Using previous notation (e.g.: Figure

2.4 or equation (2.7)),

$$h_1 = \frac{\alpha_2\lambda_2}{\alpha_1\lambda_1\alpha_3\lambda_3} > \frac{1-\epsilon^{-2}}{\tan\theta_1}$$

must become arbitrarily large as $\theta_1 \to 0$. This in turn means that $\alpha_2 \to \infty$, which is impossible as it's bounded by $\max_j\{\exp \frac{1}{2}L_j\}$. Similarly, if $\theta_1 \to \frac{\pi}{2}$, then the fact that $h_1 \tan\theta_1$ is bounded above by 1 means that $h_1 \to 0$. Thus, $\alpha_1\alpha_3 \to \infty$, which is impossible as it's bounded above by $\max_j\{\exp L_j\}$.

Given a $3|\chi(R)|$-tuple $\boldsymbol{\lambda}^0 := (\lambda_1^0, \ldots, \lambda_{3|\chi(R)|}^0)$, consider the $\boldsymbol{\lambda}^0$-slice of the decorated Teichmüller space given by:

$$\left\{(\lambda_1^0, \ldots, \lambda_{3|\chi(R)|}^0, \theta_1, \ldots, \theta_m) \mid \theta_i \in (0, \tfrac{\pi}{2})\right\} \subset \hat{\mathfrak{T}}(R).$$

Note that showing that the $\boldsymbol{\lambda}^0$-slice maps surjectively onto

$$\left\{(\lambda_1^0, \ldots, \lambda_{3|\chi(R)|}^0, L_1, \ldots, L_m) \mid L_i \in \mathbb{R}_+\right\}$$

for each $\boldsymbol{\lambda}^0 \in \mathbb{R}_+^{3|\chi(R)|}$ suffices to prove the surjectivity of Λ. If the restriction of Λ to a given $\boldsymbol{\lambda}^0$-slice isn't surjective, then there exists a collection of lengths $\mathbf{L}^0$ so that $(\boldsymbol{\lambda}^0, \mathbf{L}^0)$ is on the boundary of the image of Λ. Hence, there's a sequence of points approaching the boundary of the $\boldsymbol{\lambda}^0$-slice that maps to $(\boldsymbol{\lambda}^0, \mathbf{L}^0)$. This means that we have a sequence of points where (wlog) θ_1 approaches either 0 or $\frac{\pi}{2}$ that correspond to surfaces with boundary lengths bounded away from 0 and ∞ — this is precisely what we showed was impossible as a part of our proof of the injectivity of Λ.

Since Λ and Λ_h are homeomorphisms, the map $\Lambda_b = \Lambda \circ \Lambda_h$ is a homeomorphism. □

We conclude this section with several remarks.

Note 2.7. The smoothness of the horocyclic lengths h_i (as found in equation (2.8)) with respect to these coordinates means that we may smoothly map between these generalised λ-lengths and (shear-coordinate)×(boundary-length coordinates) for the decorated Teichmüller space. Therefore, the mapping class group acts smoothly on these generalised λ-length coordinates and any decorated moduli space $\hat{\mathfrak{M}}(R, (\Gamma, A))$ hence inherits a smooth orbifold structure.

Note 2.8. As previously mentioned in Note 2.3, it is natural to regard cusps as length 0 boundaries and hence regard decorated Teichmüller spaces $\hat{\mathfrak{T}}(R_{g,m,n})$ as lying in the boundary of $\hat{\mathfrak{T}}(R_{g,m+n,0})$. We denote by $\overline{\mathfrak{T}}(R)$ the space constructed from $\hat{\mathfrak{T}}(R)$ by gluing on these boundary decorated Teichmüller spaces, and call it the *closure* the decorated Teichmüller space $\hat{\mathfrak{T}}(R)$. The maps Λ_b and Λ_h then extend naturally to give the following homeomorphisms:

$$\Lambda_h : \overline{\mathfrak{T}}(R) \to \mathbb{R}_+^{3|\chi(R)|} \times \left[0, \tfrac{\pi}{2}\right)^m,$$
$$\Lambda_b : \overline{\mathfrak{T}}(R) \to \mathbb{R}_+^{3|\chi(R)|} \times [0, \infty)^m.$$

Note 2.9. By projecting down to only the λ-lengths, Theorem 2.19 gives us a natural way of identifying decorated Teichmüller spaces $\hat{\mathcal{T}}(R, \mathbf{L})$ with different boundary length specifications $\mathbf{L}$. These identifications descend to the Teichmüller space and give us a way of identifying the Teichmüller spaces $\mathcal{T}(R, \mathbf{L})$. However, these identifications are not mapping class group equivariant, and hence do not descend to give us a natural way of identifying the moduli spaces $\mathcal{M}(R, \mathbf{L})$ as we vary $\mathbf{L} \in \mathbb{R}_+^m$.

Note 2.10. Whereas in Penner's λ-length coordinates, it's quite easy to compute the lengths of the decorating horocycles around each cusp. We've found it difficult to determine hypercycle lengths from our generalised λ-length coordinates. We have been able to do it for simple surfaces, but nothing in general. For example, we can use Proposition 3.9 to determine the hypercycle length on a 1-bordered torus:

Given a decorated marked surface $(\lambda_1, \lambda_2, \lambda_3, L) \in \hat{\mathcal{T}}(R_{1,1,0})$, we can decrease the length of the ideal geodesics corresponding to $\lambda_1, \lambda_2, \lambda_3$ by some length $d \in \mathbb{R}$ so that the resulting λ-length are obtained from truncating at the $L\sqrt{1 + \frac{1}{4}\operatorname{cosech}^2(\frac{L}{4})}$ hypercycle. Then, by Proposition 2.19:

$$0 = 4\sinh^2(\tfrac{L}{4})(e^{\frac{d}{2}})^3 + (\lambda_1^2 + \lambda_2^2 + \lambda_3^3)e^{\frac{d}{2}} - \lambda_1\lambda_2\lambda_3. \tag{2.13}$$

It's straightforward to see that (2.13) has precisely one positive real root as a cubic polynomial in $e^{\frac{d}{2}}$. Solving it with Cardano's cubic formula yields:

$$\begin{aligned} e^{\frac{d}{2}} =& \frac{1}{2\sinh(\frac{L}{4})}\left(2\sinh(\tfrac{L}{4})\lambda_1\lambda_2\lambda_3 + \sqrt{4\sinh^2(\tfrac{L}{4})\lambda_1\lambda_2\lambda_3 + \tfrac{1}{27}(\lambda_1^2+\lambda_2^2+\lambda_3^2)^3}\right)^{\frac{1}{3}} \\ &+ \frac{1}{2\sinh(\frac{L}{4})}\left(2\sinh(\tfrac{L}{4})\lambda_1\lambda_2\lambda_3 - \sqrt{4\sinh^2(\tfrac{L}{4})\lambda_1\lambda_2\lambda_3 + \tfrac{1}{27}(\lambda_1^2+\lambda_2^2+\lambda_3^2)^3}\right)^{\frac{1}{3}}. \end{aligned}$$

Thus, using the last line of the proof of Proposition 2.19, the hypercycle is of length:

$$L\sqrt{1 + \tfrac{1}{4}e^{-d}\operatorname{cosech}^2(\tfrac{L}{4})}.$$

Knowing this hypercycle length and the boundary length L allows us to recover the angle parameter θ, and hence this gives a more straight-forward proof that $(\lambda_1, \lambda_2, \lambda_3, L)$ form coordinates over the hypercycle decorated Teichmüller space $\hat{\mathcal{T}}(R_{1,1,0})$.

2.4 Moduli Spaces of Crowned Surfaces

In this section, we construct global coordinates for Teichmüller spaces of crowned hyperbolic surfaces. In order to more easily refer to different topological-types of crowned surfaces, we introduce the following notation:

$$R^{\mathbf{a}}_{g,n,m}, \text{ where } g, m, n \in \mathbb{Z}_{\geqslant 0} \text{ and } \mathbf{a} = (a_1, \dots, a_k) \in \mathbb{Z}^k_+,$$

to denote a crowned surface with genus g, m labelled closed geodesic borders, n labelled cusps and k crowned boundary components with a_1 arches and a_1 labelled tines for the first crown, a_2 arches and a_2 labelled tines for the second crown and so forth. For example: $R^{(i)}_{0,0,0}$ is an ideal i-gon and $R^{(i)}_{0,1,0}$ is an i-pointed crown [CB88, Figure 4.2]. There is a unique crowned hyperbolic surface homeomorphic to $R^{(1)}_{0,0,1}$, and we sometimes refer to it as a *1-cusped monogon*.

We first observe that the definition of ideal triangulations given for cusped and bordered hyperbolic surfaces are also well-defined for crowned surfaces. Thus, we can easily define shearing coordinates for the Teichmüller space of crowned surfaces using the doubling construction. Given an ideal triangulation $\triangle$ on a crowned hyperbolic surface R, let $\triangle^\circ$ denote the *interior ideal triangulation* consisting of the non-peripheral entries of $\triangle$. Then:

Corollary 2.22 (Generalised shearing coordinates)**.** *Given an interior ideal triangulation $\triangle^\circ$ on a crowned surface R with cusps given by $\beta_1, \ldots, \beta_n$ and closed geodesic borders given by $\bar\beta_1, \ldots, \bar\beta_m$. Let Z_i denote the shearing length of the ith entry of $\triangle^\circ$. Then the map*

$$\begin{aligned} Z_\triangle : \mathcal{T}(R) &\to \mathbb{R}^{3|\chi(R)|} = \mathbb{R}^{6g-6+3m+3n} \\ [S,f] &\mapsto (Z_1, \ldots, Z_{6g-6+3m+3n}) \end{aligned} \tag{2.14}$$

is a homeomorphic embedding of the Teichmüller space into the subspace of $\mathbb{R}^{3|\chi(R)|}$ satisfying the linear conditions for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ that:

(added with multiplicity) the $\{Z_j\}$ with an end in β_i sum to 0 and

(added with multiplicity) the $\{Z_j\}$ with an end in $\bar\beta_i$ sum to a positive number.

Proof. Let dR denote the double of R obtained by gluing R with an opposite-oriented copy R' of itself *only along its crowned boundary components* using the identity map. The interior ideal triangulations

$$\triangle^\circ = ([\sigma_1], \ldots, [\sigma_N]) \text{ and } (\triangle')^\circ = ([\sigma'_1], \ldots, [\sigma'_N])$$

respectively sitting in R and R' combined with the arc class $\boldsymbol{\alpha} = ([\alpha_1], \ldots, [\alpha_M])$, where $M = \sum a_i$, corresponding to the boundary arches ∂R of $R \subset dR$ give an ideal triangulation of dR:

$$d\triangle := ([\sigma_1], \ldots, [\sigma_N], [\sigma'_1], \ldots, [\sigma'_N], [\alpha_1], \ldots, [\alpha_M]).$$

The Teichmüller space $\mathcal{T}(R)$ sits as a (linear) subspace of $\mathcal{T}(dR)$ consisting of marked surfaces $[S,f] \in \mathcal{T}(dR)$ with a reflection automorphism taking $\triangle$ to $\triangle'$ whilst fixing $\boldsymbol{\alpha}$. Explicitly, this subspace is given by constraining the shearing coordinates $Z_{\boldsymbol{\alpha}}$ corresponding to α to be $\mathbf{0}$ and the shearing coordinates $Z_\triangle = -Z_{\triangle'}$.[7] The result follows from projecting down to the first N coordinates. □

Note 2.11. By the same arguments, the generalised λ-length coordinates that we developed in theorems 2.17 and 2.19 from the previous subsection also apply to the decorated Teichmüller

[7] We've actually chosen the geodesic boundaries of R' to spiral in the opposite direction to that induced by the orientation on R'. This is perfectly reasonable, but strictly speaking, doesn't follow from the statement of Thurston's shearing coordinates that we gave in Theorem 2.16.

space $\hat{\mathcal{T}}(R)$ of a crowned hyperbolic surface R.

2.4.1 Mixed Coordinates

For the purposes of later establishing various presentations for a mapping class group invariant closed 2-form on the Teichmüller space $\mathcal{T}(R)$ of a crowned hyperbolic surface, we define:

Definition 2.23. *The* partially decorated Teichmüller space $\acute{\mathcal{T}}(R)$ *is the set*

$$\left\{ (S,f,\eta) \;\middle|\; \begin{array}{l} f : R \to S \textit{ is a (label-preserving) diffeomorphism, and} \\ \eta \textit{ is a set consisting of one horocycle for each cusp} \\ \textit{and one horocyclic segment for each tine} \end{array} \right\}$$

identified under the equivalence $\sim_{\acute{\mathcal{T}}}$, *where* $(S_1,f_1,\eta_1) \sim_{\acute{\mathcal{T}}} (S_2,f_2,\eta_2)$ *if and only if* $f_2 \circ f_1^{-1} : S_1 \to S_2$ *is isotopy equivalent to an isometry and the length of the respective labelled horocycles constituting* η_1 *and* η_2 *agree for each cusp and tine. We denote these equivalence classes (a little confusingly) by* $[S,f,\eta]$, *and refer to them as* partially decorated marked surfaces.

Partially decorated Teichmüller spaces $\acute{\mathcal{T}}(R)$ may be topologised as the Cartesian product of the Teichmüller space $\mathcal{T}(R)$ and $\mathbb{R}_+^{n+\sum a_i}$, the latter specifying the lengths of the decorating horocyclic segments. We construct another λ-length-based coordinate system for the partially decorated Teichmüller space $\acute{\mathcal{T}}(R)$ of a crowned surface R, and describe how we can explicitly give the Teichmüller space $\mathcal{T}(R)$ as a submanifold. Note that this is a small generalisation of Penner's partially decorated Teichmüller space in that we're allowing for closed geodesic boundaries.

2.4.1.1 Quasi-triangulation Coordinates

An *quasi-triangulation* of a crowned surface R is an arc class consisting of a maximal collection of disjoint simple ideal geodesics that decompose R into ideal triangles and (possibly) pairs of half-pants. Quasi-triangulations always exist, and a quasi-triangulation for $R^{\mathbf{a}}_{g,m,n}$ decomposes the surface into m pairs of half-pants (one for each closed geodesic boundary) and $2|\chi(R)| - m - \sum a_i = 4g - 4 + m + 2n + 2k + \sum a_i$ ideal triangles. This is a small generalisation of Penner's quasi-triangulations [Pen12, Definition 2.13]: we allow for closed geodesic boundary components.

Proposition 2.24. *For* $R = R^{\mathbf{a}}_{g,n,m}$, *fix a quasi-triangulation* $\triangle = ([\sigma_1], \ldots, [\sigma_N])$, *where*

$$N = 6g - 6 + 3n + 2m + 3k + 2\sum_{i=1}^{k} a_i,$$

then the following map gives a global coordinate system for the decorated Teichmüller space $\hat{\mathfrak{T}}(R)$*:*

$$\Lambda_\triangle : \hat{\mathfrak{T}}(R) \to \mathbb{R}_+^N \times \mathbb{R}_+^m, \\ [S, f, \eta] \mapsto (\lambda_1, \ldots, \lambda_N, L_1, \ldots, L_m)$$

where λ_i *is the* λ*-length of* $f_\#[\sigma_i]$ *truncated at* η *and* L_j *is the length of the* j*-th closed geodesic boundary.*

Proof. Let $P_1, \ldots, P_m$ be the m pairs of half-pants on R obtained after cutting R along the geodesic representatives of $\triangle$, and let R' denote the crowned subsurface of R obtained from removing $P_1, \ldots, P_m$. The fact that the geometry of a pair of (cuspidal) half-pants P_j is uniquely determined by the length L_j of its closed geodesic cuff boundary (Note 1.2), and that there's only one way to glue P_j back onto R' so that horocyclic segments align and border labels are preserved means that

$$\acute{\mathfrak{T}}(R) = \hat{\mathfrak{T}}(R') \times \mathbb{R}_+^m,$$

where the $\mathbb{R}_+^m$ parametrises the boundary lengths $L_1, \ldots, L_m$. Finally, invoking our observation in Note 2.11, gives the desired result. □

Note 2.12. A quasi-triangulation may be defined on any surface $R^{\mathbf{a}}_{g,m,n}$ where $n \neq 0$, and the above proof doesn't use the crowned assumption in any fundamental way. Therefore, Proposition 2.24 gives global coordinates for the partially decorated Teichmüller space of any hyperbolic surface (crowned or otherwise) with at least one cusp (or tine).

The Teichmüller space $\mathfrak{T}(R)$ embeds as the following subspace of $\acute{\mathfrak{T}}(R)$:

$$\left\{[S, f, \eta] \in \acute{\mathfrak{T}}(R) \mid \text{ every horocycle and horocyclic segment in } \eta \text{ is length } 1\right\},$$

and these horocyclic length conditions may be explicitly specified as follows:

Given a partially decorated marked surface $[S, f, \eta]$, the geodesic representatives $f_\#\triangle$ of $f_*\triangle$ decompose each horocycle/horocyclic segment η_i in η into horocyclic segments. Some of these horocyclic segments lie on ideal triangles, and some lie on half-pants. The horocyclic lengths which lie on an ideal triangle may be computed in terms of λ-lengths using Proposition 1.3, whilst the length of a horocyclic segment on a pair of half-pants is given by (3.7) in Note 3.4. The length condition that η_i is of length 1 is then given by these horocyclic lengths (expressed in our coordinates) summing to 1.

As an example, we do this for the quasi-triangulation $\triangle$ for $R = R^{(1)}_{0,1,1}$ given in Figure 2.7. Here, the length condition for cusp 1 is:

$$\frac{\lambda_1}{\lambda_2\lambda_3} + \frac{\lambda_4}{\lambda_2\lambda_3} = 1,$$

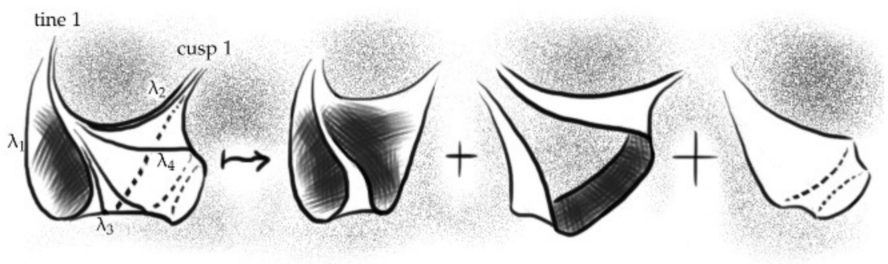


FIGURE 2.7: A quasi-triangulation for $R^{(1)}_{0,1,1}$.

and the length condition for tine 1 is:

$$\frac{2\cosh(\frac{L_1}{2})}{\lambda_4} + \frac{\lambda_2}{\lambda_1\lambda_3} + \frac{\lambda_3}{\lambda_1\lambda_2} + \frac{\lambda_2}{\lambda_3\lambda_4} + \frac{\lambda_3}{\lambda_2\lambda_4} = 1.$$

Thus, the Teichmüller space $\mathcal{T}(R)$ is given by:

$$\left\{ (\lambda_1, \lambda_2, \lambda_3, \lambda_4, L_1) \in \mathbb{R}^5_+ \;\middle|\; \begin{array}{l} \lambda_1 + \lambda_4 = \lambda_2\lambda_3, \\ 2\cosh(\frac{L_1}{2})\lambda_1\lambda_2\lambda_3 + (\lambda_1 + \lambda_4)(\lambda_2^2 + \lambda_3^2) = \lambda_1\lambda_2\lambda_3\lambda_4 \end{array} \right\}.$$

2.4.1.2 Mixed Coordinates

The quasi-triangulation coordinates that we've just defined is but a special subclass of what we call *mixed coordinates*, which combine λ-lengths and Fenchel-Nielsen coordinates. Given a maximal simple disjoint mixed class

$$(\Gamma, A) = (([\gamma_1], \ldots, [\gamma_M]), ([\sigma_1], \ldots, [\sigma_N]))$$

on a crowned hyperbolic surface R, the maximality of (Γ, A) ensures that the geodesic representatives of Γ and A decompose R into ideal triangles and pairs of half-pants.

Theorem 2.25. *The following function gives a global coordinate system on $\acute{\mathcal{T}}(R)$:*

$$\begin{aligned} \Lambda_{(\Gamma,A)} : \acute{\mathcal{T}}(R) &\to (\mathbb{R}_+ \times \mathbb{R})^M \times \mathbb{R}^N_+ \\ [S, f, \eta] &\mapsto (\ell_1, \tau_1, \ldots, \ell_M, \tau_M, \lambda_1, \ldots, \lambda_N), \end{aligned} \tag{2.15}$$

where λ_i is the λ-length for $f_\#[\sigma_i]$ on S, and ℓ_j, τ_j are respectively the length and twist parameters for $f_\#[\gamma_j]$ on S.

Proof. Cutting R along the geodesics representatives of Γ results in a collection of hyperbolic subsurfaces $R_1, \ldots, R_p$, such that the restriction of $\triangle$ to each of the R_i is a quasi-triangulation (we're including ideal triangulations). This gives us an embedding of $\acute{\mathcal{T}}(R)$ into $(\mathbb{R}_+ \times \mathbb{R})^M \times \acute{\mathcal{T}}(R_1) \times \ldots \times \acute{\mathcal{T}}(R_p)$ as the subset consisting of points

$$(\ell_1, \tau_1, \ldots, \ell_M, \tau_M, [S_1, f_1, \eta_1], \ldots, [S_p, f_p, \eta_p]),$$

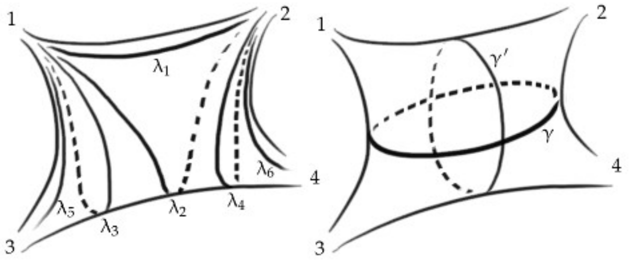


FIGURE 2.8: A triangulation of a 4-cusped sphere.

where the lengths of the boundaries of S_i are equal to the length parameters ℓ_j of the corresponding boundaries of R_i. By Proposition 2.24, the partially decorated Teichmüller space $\acute{\mathcal{T}}(R_i, \mathbf{L}_i)$, for constrained boundary lengths $\mathbf{L}_i$, is parametrised by some λ-lengths. Specifically, the λ-length of the geodesic representatives of the quasi-triangulation obtained from taking the entries in A which lie on R_i. Varying over the $\prod \acute{\mathcal{T}}(R_i, \mathbf{L})$ subject to these boundary conditions is tantamount to varying over all of $\mathbb{R}_+^N$. □

Note 2.13. As previously described in the paragraphs immediately following the proof of Proposition 2.24, we can recover the Teichmüller space $\mathcal{T}(R)$ as a subset of $\acute{\mathcal{T}}(R)$ by explicitly specifying horocyclic length conditions.

2.4.1.3 Real-analytic Structure of the Moduli Space

We finish off this chapter by showing that the mapping class group $\mathrm{Mod}(R)$ acts real-analytically on these mixed coordinates, and hence endows the partially decorated moduli space $\acute{\mathcal{M}}(R,(\Gamma, A))$ with the structure of a real-analytic orbifold. We show:

1. that any mixed coordinate is real-analytically compatible with some quasi-triangulation coordinate;
2. that all quasi-triangulation coordinates are real-analytically compatible.

Since the mapping class group acts on $\acute{\mathcal{T}}(R) = \acute{\mathcal{M}}(R, \triangle)$ by changing the quasi-triangulation $\triangle$ to another quasi-triangulation, these two statements suffice to show:

Theorem 2.26. *The mapping class group* $\mathrm{Mod}(R)$ *acts real-analytically on* $\acute{\mathcal{T}}(R)$ *with respect to mixed coordinates. Equivalently: the partially decorated moduli space* $\acute{\mathcal{M}}(R)$ *is a real-analytic orbifold/manifold with respect to mixed coordinates.*

Proof. Consider a 4-cusped sphere $R = R_{0,0,4}$, let $\triangle_{0,0,4} = ([\sigma_1], [\sigma_2], [\sigma_3], [\sigma_4], [\sigma_5], [\sigma_6])$ be an ideal triangulation on R as shown in Figure 2.8 and let $\Gamma_{0,0,4} = ([\gamma], [\gamma'])$ be a Fenchel-Nielsen class on R. This gives us λ-length coordinates for the decorated Teichmüller space $\hat{\mathcal{T}}(R)$ associated to $\triangle_{0,0,4}$, as well as (Fenchel Nielsen)$\times$(horocyclic length) coordinates $\{\lambda_i\}$ associated to $\Gamma_{0,0,4}$. It's fairly straight-forward to see that $\ell(\lambda_1, \ldots, \lambda_6)$ is independent of λ_5 and λ_6 because the simple closed geodesic γ representing $[\gamma]$ lies on the subsurface of R obtained from cutting off the 1-cusped hyperbolic monogons bordered by the ideal geodesics of $[\sigma_3]$ and $[\sigma_4]$. In particular, the horocyclic segment at the tine formed by cutting along the geodesic representative of $[\sigma_3]$ is of length $\frac{\lambda_4}{\lambda_1\lambda_2}+\frac{\lambda_1}{\lambda_2\lambda_3}+\frac{\lambda_2}{\lambda_1\lambda_3}$. Hence, by (3.7):

$$\frac{\lambda_3\lambda_4}{\lambda_1\lambda_2}+\frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_1}=2\cosh(\tfrac{\ell}{2}). \tag{2.16}$$

Since ℓ is necessarily positive, we can real-analytically express ℓ in terms of $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ using (2.16).

Let h_3 and h_4 respectively denote the horocyclic lengths at cusps 3 and cusp 4, the fact that

$$h_3=\frac{\lambda_3}{\lambda_5^2} \text{ and } h_4=\frac{\lambda_4}{\lambda_6^2}$$

means that τ may be expressed as a function of $\lambda_1,\lambda_2,\lambda_3,\lambda_4,h_3,h_4$. But since τ is geometrically unaffected by changes in horocycles, this means that $\tau=\tau(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$. We now go through a little trigonometry to show that τ is a real analytic function in these variables.

Since all Fenchel-Nielsen coordinates are smoothly (actually real-analytically) compatible, we may assume that $[\gamma']$ is disjoint from $[\sigma_1]$. Then, for fixed ℓ,λ_3,λ_4, the λ-length $\lambda_1:\acute{\mathfrak{T}}(R)\to\mathbb{R}_+$ is minimised (with respect to τ) when $\tau=0$. Let

$$\mu_1=\mu_1(\ell,\lambda_3,\lambda_4):=\lambda_1(\ell,\tau=0,\lambda_3,\lambda_4)$$

denote this minimum λ-length. We first explicitly compute μ_1 in terms of ℓ,λ_3,λ_4. Given an arbitrary decorated marked surface represented by $(\ell,\tau=0,h_1,h_2,h_3,h_4)$, normalise the horocycles at cusp 1 and 2 so that they're respectively of lengths $h_1=h_2=1+\operatorname{sech}(\frac{\ell}{2})$. By (2.16), this is equivalent to the condition that $\lambda_3=\lambda_4=2\cosh(\frac{\ell}{2})$. Excising the 1-cusped monogons bordered by the ideal geodesic representatives for $[\sigma_3]$ and $[\sigma_4]$ results in a $(1,1)$-crowned annulus (i.e.: it's homeomorphic to $R_{0,0,0}^{(1,1)}$) with boundary arches of λ-length $2\cosh(\frac{\ell}{2})$ and with horocyclic segments of length 1 at each of its two tines. Further cutting this crowned annulus along the geodesic representative of $[\gamma]$ and also along the shortest geodesic (path) joining the two boundary arches results into four isometric hyperbolic quadrilaterals. Three of the angles on such a quadrilateral are right-angles, and the remaining corner is an ideal vertex and is decorated by a horocyclic segment of length $\frac{1}{2}$. In addition, one of the two finite length sides is of length $\frac{\ell}{2}$ and the horocycle-truncated length of the ideal geodesic ray opposite to this side is of length $\log(2\cosh(\frac{\ell}{2}))$. By symmetry and the fact that ℓ may vary over all of $\mathbb{R}_+$ as we vary our initial surface, if the other finite side is length x, then its opposing horocycle-truncated side is of length $\log(2\cosh(x))$. Using formula 2.3.1(i) in the Formula Glossary at the back of [Bus92], we see that $x=\operatorname{arcsinh}(\frac{1}{\sinh(\frac{\ell}{2})})$. Using Figure 3.1, we see that horocycle-normalised $\hat{\mu}_1$ satisfies $\log(\hat{\mu}_1)=\log(2\cosh(x))$ and hence is given by:

$$\hat{\mu}_1=2\cosh(\operatorname{arcsinh}(\tfrac{1}{\sinh(\frac{\ell}{2})}))=2\coth(\tfrac{\ell}{2}).$$

Denormalising to have horocycles of arbitrary length, we have:

$$\mu_1 = \sqrt{\lambda_3\lambda_4}\,\operatorname{cosech}(\tfrac{\ell}{2}).$$

Similarly define the constrained λ-length function:

$$\mu_2 = \mu_2(\ell, \lambda_3, \lambda_4) := \lambda_2(\ell, \tau = 0, \lambda_3, \lambda_4).$$

By doing a little trigonometric bashing with Figure 3.2, we find that:

$$\mu_2 = \mu_1 \cosh(\tfrac{\ell}{2}) = \sqrt{\lambda_3\lambda_4}\,\coth(\tfrac{\ell}{2}).$$

Note 2.14. Alternatively, if we observe that (by symmetry) the diagonally opposite ideal geodesic to $[\sigma_3]$ is also of λ-length μ_2 when $\tau = 0$. Hence, by the ideal Ptolemy relation [Pen87], $\mu_2^2 = \mu_1^2 + \lambda_3\lambda_4 = \lambda_3\lambda_4 \coth^2(\tfrac{\ell}{2})$.

Now take a decorated surface $(\ell, \tau = 0, h_1, h_2, h_3, h_4)$ and mark the ideal geodesic representative σ_1 for $[\sigma_1]$ on this surface. Deforming (just) the twist parameter from 0 to some small $\tau > 0$ may be thought of as cutting the underlying surface along the geodesic representative γ of $[\gamma]$ and then twisting the pair of pants containing cusps 2 and 4 by τ before regluing. This process severs σ_1 into two ideal geodesic rays with their (non-ideal) endpoints displaced by length τ. The new ideal geodesic representative of $[\sigma_1]$, the two ideal rays of the old σ_1 and

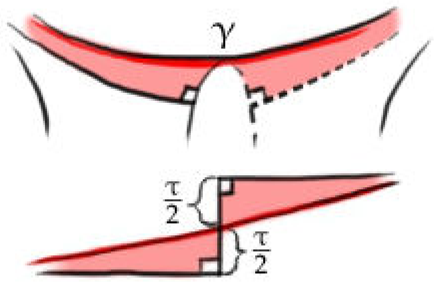


FIGURE 2.9: What happens to σ_1 after a small twist by τ.

the length τ segment on γ joining their endpoints bound two isometric hyperbolic triangles. Each of these triangles has one right angle, one ideal vertex and its finite side is of length $\frac{\tau}{2}$. Doing similar computations using Figure 3.2 as before, we obtain that:

$$\lambda_1 = \mu_1 \cosh(\tfrac{\tau}{2}) = \sqrt{\lambda_3\lambda_4}\cosh(\tfrac{\tau}{2})\,\operatorname{cosech}(\tfrac{\ell}{2}). \tag{2.17}$$

By lifting up these calculations onto the universal cover, we can see that the above formula applies for any $\tau \in \mathbb{R}$. And since $[\sigma_2]$ is related to $[\sigma_1]$ by a Dehn-twist (i.e.: a τ-twist of length ℓ) with respect to $[\gamma]$, its λ-length satisfies:

$$\lambda_2 = \mu_1 \cosh(\tfrac{\tau+\ell}{2}) = \sqrt{\lambda_3\lambda_4}\cosh(\tfrac{\tau+\ell}{2})\,\operatorname{cosech}(\tfrac{\ell}{2}). \tag{2.18}$$

Expanding out equation (2.18), we get:

$$\lambda_2 = \mu_1 \left(\cosh(\tfrac{\tau}{2})\cosh(\tfrac{\ell}{2}) + \sinh(\tfrac{\tau}{2})\sinh(\tfrac{\ell}{2})\right) = \lambda_1 \cosh(\tfrac{\ell}{2}) + \sqrt{\lambda_3\lambda_4}\sinh(\tfrac{\tau}{2}). \tag{2.19}$$

Putting equations (2.17) and (2.19) together, we obtain:

$$e^{\frac{\tau}{2}} = \frac{1}{\sqrt{\lambda_3\lambda_4}}(\lambda_2 - \lambda_1 e^{\frac{-\ell}{2}}). \tag{2.20}$$

And we see that τ too is a real-analytic function in $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

Conversely, equations (2.17) and (2.18) real-analytically express λ_1, λ_2 in terms of ℓ, τ, λ_3 and λ_4. Since $(\ell, \tau, \lambda_3, \lambda_4)$ and $(\lambda_1, \lambda_2, \lambda_3, \lambda)$ are both mixed coordinate systems on the partially decorated Teichmüller space of the $(1,1)$-crowned annulus obtained by excising the two 1-cusped monogons bordered by the geodesic representatives of $[\sigma_3]$ and $[\sigma_4]$. This tells us that these two mixed coordinates for $\hat{\mathfrak{T}}(R^{(1,1)}_{0,0,0})$ are real-analytically compatible.

Since every homotopy class of a simple closed loop $[\gamma_i]$ in a mixed class borders two pairs of half-pants, the geodesic representative of $[\gamma_i]$ lies on a $(1,1)$-crowned annulus. Replacing (for every i) the length and twist parameters ℓ_i and τ_i with λ-lengths associated to some ideal triangulation of the $(1,1)$-crowned annulus $[\gamma_i]$ lives in, results in some quasi-triangulation coordinate for $\acute{\mathfrak{T}}(R)$.

To complete this proof, we show that any two quasi-triangulation coordinates are compatible. Let $\triangle_1$ and $\triangle_2$ be two quasi-triangulations on R. Doubling R to a cusped hyperbolic surface dR by gluing R to an orientation reversed copy R' of itself along both its crowned boundaries and closed geodesic boundaries. The quasi-triangulation $\triangle_i$ then induces a quasi-triangulation $\triangle_i'$ on R', and the two arc classes combine to give an arc class $d\triangle_i$ on dR. Since $d\triangle$ is simple and disjoint, it can be extended to give an ideal triangulation $\overline{d\triangle_i}$ on dR, and using the ideal Ptolemy relation (Proposition 2.6 (a) of [Pen87]), the λ-length coordinates arising from $\overline{d\triangle_1}$ and $\overline{d\triangle_2}$ are smoothly compatible. In particular, the λ-length of $\overline{d\triangle_1}$ (and hence $\triangle_1$) are expressible in terms of the λ-lengths of $\overline{d\triangle_2}$. The doubling construction produces an embedding

$$\acute{\mathfrak{T}}(R) \hookrightarrow \acute{\mathfrak{T}}(dR) = \hat{\mathfrak{T}}(dR),\ [S, f, \eta] \mapsto [dS, df, \eta \cup \eta'].$$

On the sub-locus of $\hat{\mathfrak{T}}(dR)$ corresponding to $\acute{\mathfrak{T}}(R)$, the λ-lengths of the homotopy classes in $\triangle_i'$ are equal to those in $\triangle_i'$, therefore we need only show that the λ-lengths of the homotopy path classes added to $d\triangle_i$ to produce $\overline{d\triangle_i}$ may be expressed in terms of the λ-lengths in $d\triangle_i$ and the closed boundary length parameters of R. Observe that these added homotopy path classes in $\overline{d\triangle_2}$ lie on a $(1,1)$-crowned annulus cut up by the geodesic representatives of $\overline{d\triangle_2}$ and therefore may be expressed in terms of the λ-lengths of its crowned boundaries and Fenchel-Nielsen coordinates of the unique simple homotopy free loop class $[\gamma]$ on this annulus. Since $[\gamma]$ corresponds to the boundary of $R \subset dR$, and the twist parameter for $[\gamma]$ is 0 on the $\acute{\mathfrak{T}}(R)$ sub-locus of $\hat{\mathfrak{T}}(dR)$, we can smoothly express all of the λ-lengths of $\overline{d\triangle_2}$ in terms of the λ-length of $\triangle_2$. Thus, the λ-lengths of $\triangle_1$ are expressible in terms of the λ-lengths of $\triangle_2$. $\square$

Note 2.15. It's very tempting to want to rephrase equations like (2.16) and (2.20) in terms of "λ-lengths" $\lambda_\ell := e^{\frac{\ell}{2}}$ and $\lambda_\tau := e^{\frac{\tau}{2}}$ so that these relations are polynomial (or rational) in λ-lengths. Although, in some sense, this is perhaps just an artefact of the fact that we're doing a lot of hyperbolic trigonometry and exponentials are essentially unavoidable.

Note 2.16. Part 2 of the above proof — showing that quasi-triangulation coordinates are real-analytically compatible, can alternatively be shown by repeating Penner's proof of Theorem 2.15 in [Pen12] verbatim, and replacing his cuspidal *quasi-Ptolemy transformation*

$$a \mapsto \frac{(c+d)^2}{a}$$

with a generalised quasi-Ptolemy transformation for surfaces with boundary given by:

$$a \mapsto \frac{c^2 + d^2 + 2\cosh(\frac{L}{2})cd}{a}, \tag{2.21}$$

where L is the length of the boundary quasi-flipped through.

Note 2.17. An easy corollary of Theorem 2.26 for the decorated Teichmüller space $\hat{\mathfrak{T}}(R)$ of any cusped surfaces R which admits a pants decomposition where each constituent pair of pants contains a cusp, is that (Fenchel-Nielsen)$\times$(horocycle length) coordinates and Penner's λ-length coordinates are real-analytically compatible. This is an entirely unsurprising result, especially given that λ-lengths can more or less be thought of as "renormalised limiting length parameters" on the compactification locus of some larger Teichmüller space. Although, we've not been able to find a reference.

Chapter 3

Weil Petersson Forms and Volumes

The goal of Section 3.1 is to define mapping class group invariant Weil-Petersson 2-forms for moduli spaces of crowned surfaces (Definition 3.4) and to derive a presentation (Corollary 3.6) for this form in terms of mixed-coordinates — a presentation that also applies to the Weil-Petersson form of moduli spaces of (uncrowned) hyperbolic surfaces with at least one cusp. Sections 3.2 and 3.3 demonstrate Weil-Petersson volume computations: the top exterior product of a symplectic form is nondegenerate, hence the Weil-Petersson form gives a volume form for the moduli spaces $\mathcal{M}(R, \mathbf{L})$ of cusped/bordered hyperbolic surfaces R. Since the Weil-Petersson volume form extends smoothly to a compactification of $\mathcal{M}(R, \mathbf{L})$, the Weil-Petersson volume of these moduli spaces is finite. We perform a simple computation of this volume for $\mathcal{M}(R_{1,1}, L)$ using fundamental domains, as well as an integral of the systolic trace function. We also describe Mirzakhani's integration schemes for a special class of functions over $\mathcal{M}(R, (\Gamma, A), \mathbf{L})$.

The Teichmüller space $\mathcal{T}(R, \mathbf{L})$ of bordered surfaces (homeomorphic to R) with boundary lengths $\mathbf{L}$ is a covering space for the moduli space $\mathcal{M}(R, (\Gamma, A), \mathbf{L})$ of (Γ, A)-surfaces with boundary length $\mathbf{L}$. And the quotient group for this covering is a subgroup of the mapping class group $\mathrm{Mod}(R)$. Thus, mapping class group invariant forms on $\mathcal{T}(R, \mathbf{L})$ descend to $\mathcal{M}(R, (\Gamma, A), \mathbf{L})$. The *Weil-Petersson* form is one such mapping class group invariant 2-form. It is symplectic, and given by the following presentation in Fenchel-Nielsen coordinates:

Theorem 3.1 (Wolpert [Wol85])**.** *Given a Fenchel-Nielsen class Γ on $R = R_{g,m,n}$, the Weil-Petersson 2-form on $\mathcal{T}(R, \mathbf{L}) = \mathcal{M}(R, \Gamma, \mathbf{L})$ is given by*

$$\omega_{WP}(R) = d\ell_1 \wedge d\tau_1 + d\ell_1 \wedge d\tau_2 + \ldots + d\ell_k \wedge d\tau_k,$$

where ℓ_i and τ_i give the Fenchel-Nielsen coordinates corresponding to Γ, and $k = 3|\chi(R)| - m - n = 6g - 6 + 2m + 2n$.

Wolpert first obtained an angle sum formula for the Weil-Petersson form in [Wol83c], and used it to derive the above presentation in [Wol85]. In this latter paper, Wolpert also showed that the Weil-Petersson form extends smoothly with respect to extended Fenchel-Nielsen coordinates to the (Deligne-Mumford) compactification locus on the moduli space. In [Pen92], by thinking of the moduli space $\mathcal{M}(R)$ of a cusped hyperbolic surface R as lying in the compactification locus of the moduli space $\mathcal{M}(dR)$ of a "doubled" surface dR, Penner uses Wolpert's angle sum formula and some trigonometric facts about λ-lengths he gave in [Pen87] to derive the following presentation of the Weil-Petersson form in λ-length coordinates:

Theorem 3.2 (Penner [Pen92]). *Given a ideal triangulation $\triangle$ whose geodesic representatives decompose $R = R^{\mathbf{a}}_{g,0,n}$ into the ideal triangles $T_1, \dots, T_p$, the Weil-Petersson form $\omega_{WP}(R)$ on $\mathcal{T}(R)$ is given by:*

$$2\sum_{i=1}^{p} \left(d\log\lambda_{i,1} \wedge d\log\lambda_{i,2} + d\log\lambda_{i,2} \wedge d\log\lambda_{i,3} + d\log\lambda_{i,3} \wedge d\log\lambda_{i,1}\right),$$

where $\lambda_{i,1}, \lambda_{i,2}, \lambda_{i,3}$ are the λ-lengths of the three ideal geodesic sides of T_i labelled in the opposite order to the orientation of R.

Whilst Wolpert's presentation holds for closed hyperbolic surfaces as well as cusped and bordered hyperbolic surfaces, Penner's λ-length presentation for the Weil-Petersson form is for cusped hyperbolic surfaces and crowned surfaces without closed geodesic borders [Pen12]. Bonahon showed in [SB01] that Thurston's mapping class invariant form, defined in shearing coordinates, is (a constant multiple of) the Weil-Petersson form:

Theorem 3.3 (Thurston [Thu86], Bonahon [Bon96]). *Given a ideal triangulation $\triangle$ whose geodesic representatives decompose $R = R_{g,m,n}$ into the ideal triangles $T_1, \dots, T_p$, the Weil-Petersson 2-form $\omega_{WP}(R)$ on $\mathcal{T}(R)$ is given by:*

$$\omega_{WP}(R) = 2\sum_{T_i} \left(dZ_{i,1} \wedge dZ_{i,2} + dZ_{i,2} \wedge dZ_{i,3} + dZ_{i,3} \wedge dZ_{i,1}\right),$$

where $Z_{i,1}$, $Z_{i,2}$, $Z_{i,3}$ are the shearing-lengths of the three ideal geodesic sides of T_i labelled in the opposite *order to the orientation of R.*

3.1 Weil-Petersson Form for Crowned Surfaces

Given a crowned surface R, consider the cusped/bordered hyperbolic surface R' obtained by gluing the geometrically unique 1-cusped monogon $R^{(1)}_{0,0,1}$, to each boundary arch in R. We forget the labelling on the old tines and introducing new labels on the newly formed cusps. The uniqueness (up to our choice of cusp-labelling) of this construction means that

$$\mathcal{T}(R) \cong \mathcal{T}(R').$$

Definition 3.4. *Define the 2-form $\omega_{WP}(R)$ on $\mathcal{T}(R)$ as the pullback of the Weil-Petersson form $\omega_{WP}(R')$ on $\mathcal{T}(R')$ with respect to this map. We refer to $\omega_{WP}(R)$ as the* Weil-Petersson form *for the Teichmüller space $\mathcal{T}(R)$ whenever R is a crowned hyperbolic surface.*

Note 3.1. The boundary ∂R induces a mixed class (Γ, A) on R', and the mapping class group $\mathrm{Mod}(R)$ is the stabiliser subgroup $\mathrm{Stab}(\Gamma, A) \leqslant \mathrm{Mod}(R')$ of the mapping class group for R'.

Thus, the Weil-Petersson form we've defined is mapping class group invariant and descends to the moduli space $\mathcal{M}(R', (\Gamma, A)) \cong \mathcal{M}(R)$.

Note 3.2. Given an m-tuple $\mathbf{L}$ of boundary lengths for the borders of R, the Weil-Petersson form is symplectic on the Teichmüller space $\mathcal{T}(R', \mathbf{L})$ and hence also symplectic on $\mathcal{T}(R, \mathbf{L})$.

Theorem 3.5. *Given a quasi-triangulation $\triangle$ whose geodesic representatives decompose R into the ideal triangles $T_1, \ldots, T_p$ and m pairs of half-pants, the Weil-Petersson 2-form $\omega_{WP}(R)$ on $\mathcal{T}(R)$ is given by*

$$2\sum_{i=1}^{p} \left(d\log\lambda_{i,1} \wedge d\log\lambda_{i,2} + d\log\lambda_{i,2} \wedge d\log\lambda_{i,3} + d\log\lambda_{i,3} \wedge d\log\lambda_{i,1}\right), \tag{3.1}$$

where $\lambda_{i,1}, \lambda_{i,2}, \lambda_{i,3}$ are the λ-lengths of the three ideal geodesic sides of T_i labelled in the opposite order to the orientation of R.

Proof. Given a crowned surface R, we've already defined R' to be the unique hyperbolic surface obtained by gluing 1-cusped monogons along the boundary of R. Let R'' further denote a cusped hyperbolic surface obtained from by gluing pairs of pants with 2 cusps to the closed geodesic borders of R'. Although there are $(\mathbb{S}^1)^m$ ways to glue on these pants, our arguments are unaffected by this choice.

Given an embedding $\iota : R_1 \to R_2$, we define the induced surjection

$$\begin{aligned} \mathcal{T}(\iota) : \mathcal{T}(R_2) &\twoheadrightarrow \mathcal{T}(R_1) \\ [S, f] &\mapsto [(f \circ \iota)_*(R_1), f \circ \iota], \end{aligned}$$

where $(f \circ \iota)_*(R_1)$ is the unique geodesic bordered subsurface of S that's homotopy equivalent to $f \circ \iota(R_1)$. Then the following sequence of embeddings:

$$R \xhookrightarrow{\iota} R' \xhookrightarrow{\iota'} R''$$

induces a sequence of surjective submersions of Teichmüller spaces:

$$\mathcal{T}(R'') \xrightarrow{\mathcal{T}(\iota')} \mathcal{T}(R') \xrightarrow{\mathcal{T}(\iota)} \mathcal{T}(R).$$

Fix an arbitrary quasi-triangulation $\triangle$ on R and let $\triangle'$ be an ideal triangulation of R' that contains each homotopy path class in $\triangle$. And likewise let $\triangle''$ be an ideal triangulation of R'' that contains $\triangle'$. Cutting R'' along the geodesic representatives of $\triangle'$ results in ideal triangles $T_1, \ldots, T_p \subset R' \subset R''$, as well as m crowned surfaces $R_1, \ldots, R_m$ each homeomorphic to $R^{(1)}_{0,3,1}$. Since the λ-length on $\mathcal{T}(R'')$ for $[\sigma_i]$ in $\triangle$ is the pullback of the corresponding λ-length on $\mathcal{T}(R)$, we regard the same λ-lengths as being coordinates on all three of these Teichmüller spaces. Then, because $\omega_{WP}(R) = (\mathcal{T}(\iota)^{-1})^*\omega_{WP}(R')$ by definition, the λ-lengths based expression for the Weil-Petersson form $\omega_{WP}(R)$ on $\mathcal{T}(R)$ is the same expression as the pullback form

$$(\mathcal{T}(\iota) \circ \mathcal{T}(\iota'))^*\omega_{WP}(R) = \mathcal{T}(\iota')^*\omega_{WP}(R')$$

on $\mathcal{T}(R'')$. Thus, we can determine the expression for $\omega_{WP}(R)$ by showing that $\mathcal{T}(\iota')^*\omega_{WP}(R')$ takes the same λ-length based expression as (3.1).

Fix an arbitrary Fenchel-Nielsen class

$$\Gamma' = ([\gamma_1], \ldots, [\gamma_q], [\gamma_1'], \ldots, [\gamma_q']) \text{ on } R'$$

and let $[\beta_1], \ldots, [\beta_m]$ denote homotopy classes of free loops on R'' corresponding to the boundaries of $R' \subset R''$. We choose a Fenchel-Nielsen class on R''

$$\Gamma'' := ([\gamma_1], \ldots, [\gamma_q], [\gamma_1'], \ldots, [\gamma_q'], [\beta_1], \ldots, [\beta_m], [\beta_1'], \ldots, [\beta_m']).$$

Let ℓ_i, τ_i respectively denote the length and twist parameters for $[\gamma_i]$ on R' (and hence also on R''), the Weil-Petersson form on $\mathcal{T}(R')$ is given by:

$$\omega_{WP}(R') = d\ell_1 \wedge d\tau_1 + \ldots + d\ell_q \wedge d\tau_q, \tag{3.2}$$

and its pullback on $\mathcal{T}(R'')$ takes the same expression.

Since each of the R_i obtained from cutting R'' along $\triangle'$ is homeomorphic to $R^{(1)}_{0,3,0}$, the surface R_i' obtained from gluing a 1-cusped monogon along the boundary arch in R_i is a 4-cusped sphere. Then, $[\beta_i]$ gives a pants decomposition class of R_i', and may be extended to a Fenchel-Nielsen class. The Weil-Petersson form $\omega_{WP}(R_i')$ on $\mathcal{T}(R_i')$ with respect to the Fenchel-Nielsen coordinates arising from such a class is then given by

$$\omega_{WP}(R_i') = d\ell_i \wedge d\tau_i, \text{ where } \ell_i \text{ is the length parameter for } [\beta_i].$$

On the other hand, the restriction $\triangle_i$ of the ideal triangulation $\triangle''$ to R_i gives an an arc class on R_i'. Where the geodesic representatives of $\triangle_i$ cut up R_i' into ideal triangles and one 1-cusped monogon. There is a unique way to add a homotopy class of paths to $\triangle_i$ to obtain an ideal triangulation $\triangle_i'$ of R_i'. Let $T_1^i, T_2^i, T_3^i, T_4^i$ be the ideal triangles obtained from R_i' by cutting along the geodesic representatives of $\triangle_i'$, and let T_4^i be the ideal triangle obtained from cutting the 1-cusped monogon added to R_i to form R_i'. Since the λ-length based contribution of T_4^i to the Weil-Petersson form is:

$$d\log\lambda^i_{4,1} \wedge d\log\lambda^i_{4,2} + d\log\lambda^i_{4,2} \wedge d\log\lambda^i_{4,1} + d\log\lambda^i_{4,1} \wedge d\log\lambda^i_{4,1} = 0, \tag{3.3}$$

the Weil-Petersson form on $\mathcal{T}(R_i')$ is:

$$\begin{aligned}\omega_{WP}(R_i') &= d\ell_i \wedge d\tau_i \\ &= 2\sum_{j=1}^{3} \left(d\log\lambda^i_{j,1} \wedge d\log\lambda^i_{j,2} + d\log\lambda^i_{j,2} \wedge d\log\lambda^i_{j,3} + d\log\lambda^i_{j,3} \wedge d\log\lambda^i_{j,1}\right).\end{aligned} \tag{3.4}$$

The geodesic representatives of the ideal triangulation $\triangle''$ cut up R'' into the ideal triangles

$$T_1, \ldots, T_p, T_1^1, T_2^1, T_3^1, \ldots, T_1^m, T_2^m, T_3^m,$$

where m is the number of closed geodesic boundaries of R'. By the observation used to determine (3.3) that ideal triangles arising from cutting 1-cusped monogons don't contribute to the Weil-Petersson form, we obtain that the Weil-Petersson form $\omega_{WP}(R'')$ is:

$$\begin{aligned}
&2\sum_{i=1}^{m}\sum_{j=1}^{3}\left(d\log\lambda_{j,1}^{i}\wedge d\log\lambda_{j,2}^{i}+d\log\lambda_{j,2}^{i}\wedge d\log\lambda_{j,3}^{i}+d\log\lambda_{j,3}^{i}\wedge d\log\lambda_{j,1}^{i}\right)\\
&+2\sum_{k=1}^{p}\left(d\log\lambda_{k,1}\wedge d\log\lambda_{k,2}+d\log\lambda_{k,2}\wedge d\log\lambda_{k,3}+d\log\lambda_{k,3}\wedge d\log\lambda_{k,1}\right).
\end{aligned}$$

We know from (3.2) that the pullback of $\omega_{WP}(R')$ to $\mathcal{T}(R'')$ is

$$\mathcal{T}(\iota')^*(\omega_{WP}(R')) = \omega_{WP}(R'') - \sum_{i=1}^{m} d\ell_i \wedge d\tau_i. \tag{3.5}$$

Substituting each term on the right hand side of (3.5) using its λ-coordinates presentation, we see that $\mathcal{T}(\iota')^*\omega_{WP}(R')$ is given by the same presentation as (3.1), thus completing our proof. □

Note 3.3. It's actually quite straight-forward to show that (3.1) is mapping class group invariant, provided that we're willing to accept Corollary 5.1.9 of [Pen12] and equation (2.21). The idea is to explicitly show that the above form is invariant under the "flips" and "quasi-flips" described in [Pen12]. The check for quasi-flip-invariance uses equation (2.21).

Corollary 3.6. *Let $\Lambda_{\Gamma,A}$ be a mixed coordinate system on the Teichmüller space $\mathcal{T}(R) = \mathcal{M}(R, (\Gamma, A))$ of a crowned surface. Let $\ell_1, \ldots, \ell_M$ denote the length parameters for the curve class $\Gamma = ([\gamma_1], \ldots, [\gamma_M])$, and let $\tau_1, \ldots, \tau_M$ denote a collection of corresponding twist parameters for $\mathcal{T}(R)$. Further let $T_1, \ldots, T_p$ be the resulting ideal triangles from cutting up R along the geodesic representatives of A. Then the (mapping class group invariant) Weil-Petersson form $\omega_{WP}(R)$ is given by:*

$$\begin{aligned}
&2\sum_{i=1}^{p}\left(d\log\lambda_{i,1}\wedge d\log\lambda_{i,2}+d\log\lambda_{i,2}\wedge d\log\lambda_{i,3}+d\log\lambda_{i,3}\wedge d\log\lambda_{i,1}\right)\\
&+\sum_{j=1}^{M} d\ell_j\wedge d\tau_j,
\end{aligned} \tag{3.6}$$

where $\lambda_{i,1}, \lambda_{i,2}, \lambda_{i,3}$ are the λ-lengths of the ideal geodesics constituting the sides of T_i ordered in the opposite order to the orientation.

Proof. Let R' again denote the (unique) hyperbolic surface obtained from R by gluing 1-cusped monogons to the arch boundaries of R. Moreover, let R_i denote the resulting subsurfaces from cutting R along the simple closed geodesics representatives of Γ, then the collection of surfaces R_i' (obtained from gluing monogons) identifies with the collection of subsurfaces resulting from cutting R' along Γ. Finally, let $\iota : R_i \to R$ and $\iota' : R_i' \to R$ denote these embedding maps. We summarise all of these data with the following commutative diagram (for each i):

$$\begin{tikzcd} R_i \arrow[r, hook, "\iota_i"] \arrow[d, hook, "\iota|_{R_i}"] & R \arrow[d, hook, "\iota"] \\ R_i' \arrow[r, hook, "\iota_i'"] & R' \end{tikzcd}$$

which induces the following commutative diagram on Teichmüller spaces:

$$\begin{tikzcd} \mathcal{T}(R') \arrow[r, two heads, "\mathcal{T}(\iota_i')"] \arrow[d, two heads, "\mathcal{T}(\iota)"'] & \mathcal{T}(R_i') \arrow[d, two heads, "\mathcal{T}(\iota|_{R_i})"'] \\ \mathcal{T}(R) \arrow[r, two heads, "\mathcal{T}(\iota_i)"] & \mathcal{T}(R_i), \end{tikzcd}$$

where the vertical maps $\mathcal{T}(\iota)$ and $\mathcal{T}(\iota|_{R_i})$ are homeomorphisms. Since R' and R_i' are cusped/bordered hyperbolic surfaces, the Fenchel-Nielsen coordinates presentation of the Weil-Petersson form tells us that

$$\omega_{WP}(R') = \sum_{i=1}^{p} \mathcal{T}(\iota_i')^*(\omega_{WP}(R_i')) + \sum_{j=1}^{M} d\ell_j \wedge d\tau_j.$$

Using the definition of $\omega_{WP}(R)$,

$$\begin{aligned} \omega_{WP}(R) :=& (\mathcal{T}(\iota)^{-1})^*(\omega_{WP}(R')) \\ =& \sum_{i=1}^{p} (\mathcal{T}(\iota)^{-1})^*(\mathcal{T}(\iota|_{R_i}) \circ \mathcal{T}(\iota_i'))^*(\omega_{WP}(R_i')) + \sum_{j=1}^{M} d\ell_j \wedge d\tau_j \\ =& \sum_{i=1}^{p} \mathcal{T}(\iota_i)^*(\omega_{WP}(R_i)) + \sum_{j=1}^{M} d\ell_j \wedge d\tau_j. \end{aligned}$$

Finally, by invoking Theorem 3.5, we see that $\omega_{WP}(R)$ is equal to (3.6). $\square$

3.2 Integrating Over $\mathcal{M}(R_{1,1}, L)$

We adopt the notation $R_{1,1}$ to refer to both 1-cusped hyperbolic tori and 1-bordered hyperbolic tori. Moreover, we use $\mathcal{M}(R_{1,1}, 0)$ to denote the moduli space of 1-cusped tori and $\mathcal{M}(R_{1,1}, L)$ to denote the moduli space of 1-bordered tori with length L boundary. Similarly, we use $\mathcal{T}(R_{1,1}, 0)$ and $\mathcal{T}(R_{1,1}, L)$ to respectively refer to the Teichmüller spaces of 1-cusped tori and the Teichmüller space of 1-bordered tori with boundary length L. In fact, we use the notation $R_{g,m}$ whenever we wish to place emphasis on cusps being regarded as length 0 geodesics.

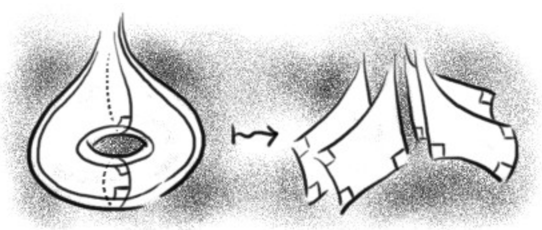
Given an ideal triangulation $\triangle = ([\sigma_1], [\sigma_2], [\sigma_3])$ of the 1-cusped torus $R_{1,1}$, their corresponding λ-lengths $(\lambda_1, \lambda_2, \lambda_3)$ give λ-length coordinates on the decorated Teichmüller space $\hat{\mathcal{T}}(R_{1,1})$. The (undecorated) Teichmüller space $\mathcal{T}(R_{1,1}) = \hat{\mathcal{M}}(R_{1,1}, \triangle)$ may be regarded as the subspace of

$\hat{\mathfrak{T}}(R_{1,1})$ consisting of all decorated marked surfaces $[S, f, \eta] = [S, f_*\triangle, \eta]$ whose horocycle η is of length 2.

There is a bijective correspondence between ideal triangulations $\triangle = ([\sigma_1], [\sigma_2], [\sigma_3])$ on R and simple curve classes $\Gamma = ([\gamma_1], [\gamma_2], [\gamma_3])$ consisting of three pairwise once-intersecting homotopy classes of free loops. This bijection is given by replacing each $[\sigma_i]$ with the unique simple homotopy class of free loops $[\gamma_i]$ disjoint from $[\sigma_i]$. Since homotopy classes of free loops are in natural bijection with conjugacy classes of the fundamental group $\pi_1(R_{1,1})$, we define the *trace* of $[\gamma_i]$ to be the trace of the corresponding conjugacy class with respect to a monodromy representation of $\pi_1(R_{1,1}) \to \mathrm{PSL}_2(\mathbb{R})$, lifted into $\mathrm{SL}_2(\mathbb{R})$. We choose this lift so that the traces of each $[\gamma_i]$ is positive.

Proposition 3.7. *Given any decorated marked surface* $[S, f_*\triangle, \eta] \in \mathfrak{T}(R_{1,1}, 0) \subset \hat{\mathfrak{T}}(R_{1,1}, 0)$*, the* λ*-length* λ_i *of the ideal geodesic* σ_i *representing* $[\sigma_i]$ *is equal to the (absolute value of the) trace of* $[\gamma_i]$*.*

Proof. Any 1-cusped torus S may be decomposed into four isometric tetrahedrals:



By positioning one of these tetrahedrals in the hyperbolic plane as in Figure 3.1, we've reduced this essentially to a problem in (Euclidean) coordinate geometry. Since

$$\frac{\ell_{\gamma_i}}{2} = \operatorname{arccosh}\left(1 + \frac{x_0^2 + (1-y_0)^2}{2y_0}\right) \text{ and } x_0^2 + y_0^2 = 1,$$

the corner point $x_0 + iy_0$ diagonally opposite to the ideal point of this quadrilateral is located at $\tanh(\frac{\ell_{\gamma_i}}{2}) + i\operatorname{sech}(\frac{\ell_{\gamma_i}}{2})$. Since the two "Euclidean" semicircles specifying the geodesics meeting at $x_0 + iy_0$ intersect orthogonally, so too do the normal vectors of these two curves at this intersection point. Using the fact that the centre of the left circle is at 0, we determine that the right circle is centred at $\coth(\frac{\ell_{\gamma_i}}{2})$ and hence

$$rh = \coth(\tfrac{\ell_{\gamma_i}}{2}).$$

Moreover, this circle is of radius $\operatorname{cosech}(\frac{\ell_{\gamma_i}}{2})$. And since the right vertical geodesic contains precisely half of the horocycle-truncated segment of σ_i,

$$h\lambda_i = \frac{\coth(\frac{\ell_{\gamma_i}}{2})}{r} \exp(2 \times \tfrac{1}{2} \log r \sinh(\tfrac{\ell_{\gamma_i}}{2})) = \cosh(\tfrac{\ell_{\gamma_i}}{2}).$$

As h is $\frac{1}{4}$ of the total horocyclic length of η, $h = \frac{1}{2}$. And the result follows because the trace corresponding to $[\gamma_i]$ is $2\cosh(\frac{\ell_{\gamma_i}}{2})$. $\square$

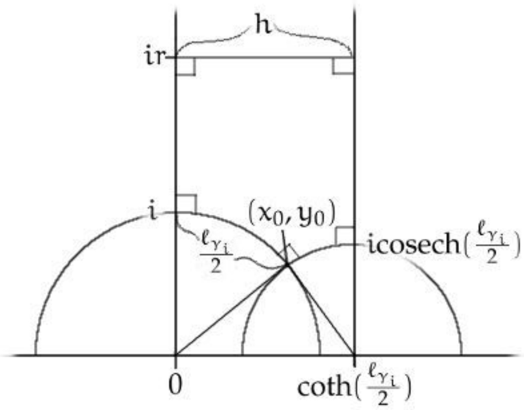


FIGURE 3.1: A figure for trigonometric computations on the 1-cusped torus.

Note 3.4. A porism of the above proposition is that given a pair of (cuspidal) half-pants P with cuff γ and with λ being the λ-length of the zipper of P truncated at the length H horocyclic segment, then:

$$H\lambda = 2\cosh(\tfrac{\ell_\gamma}{2}). \tag{3.7}$$

Take care that H here is $2h$ in the proof of the above proposition.

Using Proposition 1.3, the horocycle length associated to the λ-lengths $(\lambda_1, \lambda_2, \lambda_2)$ is

$$2\left(\frac{\lambda_1}{\lambda_2\lambda_3} + \frac{\lambda_2}{\lambda_1\lambda_3} + \frac{\lambda_3}{\lambda_1\lambda_2}\right) \tag{3.8}$$

The Teichmüller space $\mathcal{T}(R_{1,1}, 0)$ sitting in the decorated Teichmüller space $\hat{\mathcal{T}}(R_{1,1}, 0)$ as the collection of decorated marked surfaces with length 2 horocycles is then specified by requiring (3.8) to be equal to 2. Hence:

Proposition 3.8. *The collection of triples*

$$\left\{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}_+^3 \mid \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \lambda_1\lambda_2\lambda_3\right\} \tag{3.9}$$

gives a model of the Teichmüller space $\mathcal{T}(R_{1,1}, 0)$.

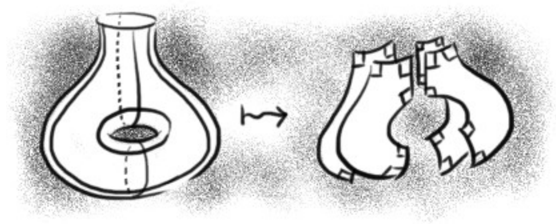
By the same argument, the sets:

$$\left\{(\lambda_1, \lambda_2, \lambda_2) \in \mathbb{R}_+^3 \mid \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \tfrac{H}{2}\lambda_1\lambda_2\lambda_3\right\}$$

for $H \in \mathbb{R}_+$ correspond to *leaves* of Teichmüller spaces $\mathcal{T}(R_{1,1}, 0)$ (embedded as the set of decorated marked surfaces of horocyclic length H) foliating the decorated Teichmüller space $\hat{\mathcal{T}}(R_{1,1})$. We now generalise to 1-bordered hyperbolic tori:

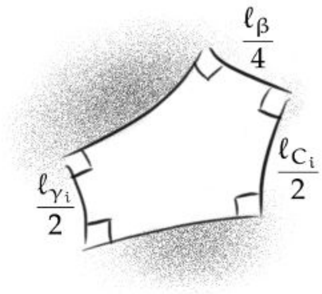
Proposition 3.9. *For hypercycle-decorated 1-bordered tori S with geodesic boundary β, such that the hypercycle is of length $\ell_\beta\sqrt{1 + \frac{1}{4}\mathrm{cosech}^2(\frac{\ell_\beta}{4})}$, the λ-length λ_i of the (spiralling) ideal geodesic σ_i is equal to the (absolute value of the) trace of $[\gamma_i]$.*

Proof. Decompose S into four isometric right-angled pentagons:

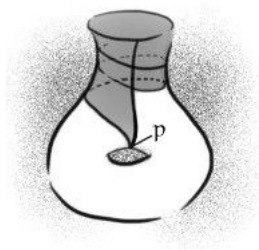


Formula 2.3.4 (i) in the Formula Glossary at the back of [Bus92] expresses the length ℓ_{C_i} of the shortest geodesic C_i boundary-relative homotopic to σ_i, in terms of ℓ_β and ℓ_{γ_i}:

$$\cosh(\tfrac{\ell_{\gamma_i}}{2}) = \sinh(\tfrac{\ell_\beta}{4})\sinh(\tfrac{\ell_{C_i}}{2}).$$



The hyperelliptic involution on S pair off all but one of the intersection points between C_i and σ_i. Thus, this unpaired intersection point $p \in S$ is the unique fixed point of the hyperelliptic involution. Since the hypercycle is preserved by the hyperelliptic involution, the point p is both the midpoint of C_i and the midpoint of the hypercycle-truncated segment of σ_i.



The two halves of C_i and σ_i depicted just above, may be thought of as the boundaries of some immersed hyperbolic triangle with one ideal vertex and one right-angled vertex that has been wrapped around β infinitely many times. Let H denote the length of the horocyclic segment joining the truncation point of this half-ray of σ_i by the hypercycle around β. A little coordinate geometry with Figure 3.2 shows that the λ-length

$$\lambda_i = \frac{\sinh(\frac{\ell_{C_i}}{2})}{H} = \frac{\cosh(\frac{\ell_\gamma}{2})}{H\sinh(\frac{\ell_\beta}{4})}. \tag{3.10}$$

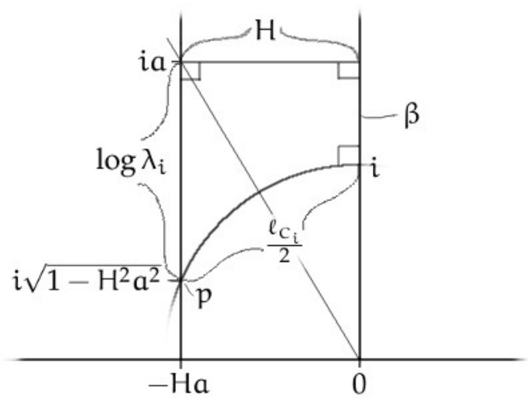


FIGURE 3.2: A figure for trigonometric computations on the 1-holed torus

Varying the decorating hypercycle allows us to arbitrarily vary H in $\mathbb{R}_+$, and choosing $H = \frac{1}{2}\mathrm{cosech}(\frac{\ell_\beta}{4})$ results in λ_i being equal to the trace of $[\gamma_i]$. So, the last step is simply to show that the necessary hypercycle to produce $H = \frac{1}{2}\mathrm{cosech}(\frac{\ell_\beta}{4})$ is of length $\ell_\beta\sqrt{1+\frac{1}{4}\mathrm{cosech}^2(\frac{\ell_\beta}{4})}$. This is a fairly elementary computation: the length of a hypercycle of horocyclic distance H away from a boundary component β is given by $\ell_\beta\sqrt{1+H^2}$. □

Note 3.5. Proposition 3.7 is obtained by taking the limit as $\ell_\beta \to 0$.

For 1-bordered tori, any triple of pairwise once-intersecting simple closed curves $\gamma_1, \gamma_2, \gamma_3$ satisfy the following trace relation:

$$\mathrm{tr}([\beta]) = -2\cosh(\tfrac{\ell_\beta}{2}) = \mathrm{tr}([\gamma_1])^2 + \mathrm{tr}([\gamma_2])^2 + \mathrm{tr}([\gamma_3])^2 - \mathrm{tr}([\gamma_1])\mathrm{tr}([\gamma_2])\mathrm{tr}([\gamma_3]) - 2.$$

Changing the above terms into λ-lengths using Proposition 3.9 gives us the following embedding of the Teichmüller space $\mathcal{T}(R_{1,1}, L)$ into the decorated Teichmüller space $\hat{\mathcal{T}}(R_{1,1}, L)$:

Corollary 3.10. *The collection of triples*

$$\left\{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3_+ \mid \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \lambda_1\lambda_2\lambda_3 - 4\sinh^2(\tfrac{L}{4})\right\}. \tag{3.11}$$

give a parametrisation of the Teichmüller space $\mathcal{T}(R_{1,1}, L)$.

Proof. The Teichmüller space $\mathcal{T}(R_{1,1}, L)$ consisting of length $L\sqrt{1+\frac{1}{4}\mathrm{cosech}^2(\frac{L}{4})}$ hypercycle-structured surfaces give λ-lengths $(\lambda_1, \lambda_2, \lambda_3)$ satisfying $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \lambda_1\lambda_2\lambda_3 - 4\sinh^2(\frac{L}{4})$. Conversely, any triple (μ_1, μ_2, μ_3) which come from an ideal triangulation truncated at a hypercycle *not* of length $L\sqrt{1+\frac{1}{4}\mathrm{cosech}^2(\frac{L}{4})}$ is related to those which come from the same ideal triangulation truncated at the hypercycle of this length by a multiplication by some positive number $r \in \mathbb{R}_+ - \{1\}$. Hence,

$$r^2\mu_1^2 + r^2\mu_2^2 + r^2\mu_3^2 = r^3\mu_1\mu_2\mu_3 - 4\sinh^2(\tfrac{L}{4}).$$

Therefore, this triple (μ_1, μ_2, μ_3) does not lie in our specified collection of triples, or else we'd have the impossible statement that:

$$0 = r^2\mu_1\mu_2\mu_3 + 4(1+r)\sinh(\tfrac{L}{4}) > 0.$$

□

Let (Z_1, Z_2, Z_3) to be the shear coordinates corresponding to $(\sigma_1, \sigma_2, \sigma_3)$, then by phrasing shear coordinates in terms of surface representation traces (equation (2.15) in [CP07]):

$$\begin{aligned}
\lambda_1 =& e^{\frac{1}{2}(Z_2+Z_3)} + e^{\frac{1}{2}(Z_2-Z_3)} + e^{\frac{-1}{2}(Z_2+Z_3)} \\
\lambda_2 =& e^{\frac{1}{2}(Z_3+Z_1)} + e^{\frac{1}{2}(Z_3-Z_1)} + e^{\frac{-1}{2}(Z_3+Z_1)} \\
\lambda_3 =& e^{\frac{1}{2}(Z_1+Z_2)} + e^{\frac{1}{2}(Z_1-Z_2)} + e^{\frac{-1}{2}(Z_1+Z_2)}.
\end{aligned}$$

Lemma 3.11. *The inverse map from shearing coordinates to the hypercycle-normalised λ-lengths on $\mathcal{T}(R_{1,1})$ is given by:*

$$\begin{aligned}
Z_1 &= 2\log\left(\frac{\lambda_2\lambda_3 + \lambda_1(e^{\frac{L}{2}} - 1)}{\lambda_2^2 + 4\sinh^2(\frac{L}{4})}\right) - \frac{L}{2}, \\
Z_2 &= 2\log\left(\frac{\lambda_3\lambda_1 + \lambda_2(e^{\frac{L}{2}} - 1)}{\lambda_3^2 + 4\sinh^2(\frac{L}{4})}\right) - \frac{L}{2}, \\
Z_3 &= 2\log\left(\frac{\lambda_1\lambda_2 + \lambda_3(e^{\frac{L}{2}} - 1)}{\lambda_1^2 + 4\sinh^2(\frac{L}{4})}\right) - \frac{L}{2}.
\end{aligned}$$

Proof. Although we originally geometrically derived the above formulas, they may be verified as an (fairly involved) algebraic exercise. Those who wish to make such an attempt might find it useful to first verify the following identities (and hence the identities from cyclically permutating the indices of the λ_i):

$$(\lambda_1^2 + 4\sinh^2(\tfrac{\ell_\beta}{4}))(\lambda_2^2 + 4\sinh^2(\tfrac{\ell_\beta}{4})) = (\lambda_1\lambda_2 + (e^{\frac{1}{2}\ell_\beta} - 1)\lambda_3)(\lambda_1\lambda_2 - (1 - e^{\frac{-1}{2}\ell_\beta})\lambda_3)$$

$$\begin{aligned}
e^{\frac{1}{2}\ell_\beta} &= \frac{(\lambda_1^2 + 4\sinh^2(\frac{\ell_\beta}{4}))(\lambda_2^2 + 4\sinh^2(\frac{\ell_\beta}{4}))(\lambda_3^2 + 4\sinh^2(\frac{\ell_\beta}{4}))}{(\lambda_1\lambda_2 - (1 - e^{\frac{-1}{2}\ell_\beta})\lambda_3)(\lambda_2\lambda_3 - (1 - e^{\frac{-1}{2}\ell_\beta})\lambda_1)(\lambda_3\lambda_1 - (1 - e^{\frac{-1}{2}\ell_\beta})\lambda_2)} \\
&= \frac{(\lambda_1\lambda_2 + (e^{\frac{1}{2}\ell_\beta} - 1)\lambda_3)(\lambda_2\lambda_3 + (e^{\frac{1}{2}\ell_\beta} - 1)\lambda_1)(\lambda_3\lambda_1 + (e^{\frac{1}{2}\ell_\beta} - 1)\lambda_2)}{(\lambda_1^2 + 4\sinh^2(\frac{\ell_\beta}{4}))(\lambda_2^2 + 4\sinh^2(\frac{\ell_\beta}{4}))(\lambda_3^2 + 4\sinh^2(\frac{\ell_\beta}{4}))}.
\end{aligned}$$

□

Next, we reparametrise the Teichmüller space $\mathcal{T}(R_{1,1}, L)$ as the unit triangle using the following map:

$$(\lambda_1, \lambda_2, \lambda_3) \mapsto \left(\tfrac{\lambda_1}{\lambda_2\lambda_3}, \tfrac{\lambda_2}{\lambda_1\lambda_3}\right) \in \left\{(x, y) \in \mathbb{R}_+^2 \mid x + y < 1\right\}$$

with the inverse given by:

$$(x,y) \mapsto \left(\sqrt{\frac{1+4\sinh^2(\frac{L}{4})xy}{y(1-x-y)}}, \sqrt{\frac{1+4\sinh^2(\frac{L}{4})xy}{x(1-x-y)}}, \sqrt{\frac{1}{xy}} \right).$$

Proposition 3.12. *In terms of this unit triangle coordinate system* (x,y) *for the Teichmüller space* $\mathcal{T}(R_{1,1},L)$*, the Weil-Petersson form is given by:*

$$\omega_{WP}(R_{1,1}) = \frac{dx \wedge dy}{xy(1-x-y)}.$$

This was obtained by direct computation (in this case, with a computer) using the shearing coordinates presentation for $\omega_{WP}(R_{1,1})$. It's slightly interesting to note that the presentation of the Weil-Petersson form is independent of the boundary length L.

Instead of finding a fundamental domain for the action of the mapping class group to integrate the volume of $\mathcal{M}(R_{1,1},L)$, we do so for the *extended mapping class group* $\mathrm{Mod}^{\pm}(R_{1,1})$ consisting of isotopy classes of (potentially orientation-changing) homeomorphisms of $R_{1,1}$. Since the mapping class group $\mathrm{Mod}(R_{1,1})$ is an order 2 subgroup of $\mathrm{Mod}^{\pm}(R_{1,1})$

$$1 \to \mathrm{Mod}(R_{1,1}) \to \mathrm{Mod}^{\pm}(R_{1,1}) \to \mathbb{Z}/2\mathbb{Z} \to 1,$$

the final volume that we obtain needs to be doubled. Now, the extended mapping class group $\mathrm{Mod}^{\pm}(R_{1,1})$ is the semi-direct product

$$\mathrm{Mod}^{\pm}(R_{1,1}) = ((\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}) * \mathbb{Z}/2\mathbb{Z} \rtimes \mathrm{Sym}(3)) \times \mathbb{Z}/2\mathbb{Z},$$

where the first three $\mathbb{Z}/2\mathbb{Z}$ are represented by orientation-reversing homeomorphisms of $R_{1,1}$. Specifically, these homeomorphisms take:

$$\begin{aligned}
([\gamma_1],[\gamma_2],[\gamma_3]) &\text{ to } ([\gamma_1'],[\gamma_2],[\gamma_3]) \\
([\gamma_1],[\gamma_2],[\gamma_3]) &\text{ to } ([\gamma_1],[\gamma_2'],[\gamma_3]) \\
([\gamma_1],[\gamma_2],[\gamma_3]) &\text{ to } ([\gamma_1],[\gamma_2],[\gamma_3']),
\end{aligned}$$

where $[\gamma_i']$ is the unique homotopy class of free loops that's disjoint from the opposing diagonal homotopy path class to $[\sigma_i]$; the $\mathrm{Sym}(3)$ is the subgroup of $\mathrm{Mod}^{\pm}(R_{1,1})$ that fixes the unordered triple $\{[\gamma_1],[\gamma_2],[\gamma_3]\}$ and the last $\mathbb{Z}/2\mathbb{Z}$ is the hyperelliptic involution. The standard $SL_2(\mathbb{C})$ trace identity that

$$\mathrm{tr}(AB^{-1}) = \mathrm{tr}(A)\mathrm{tr}(B) - \mathrm{tr}(AB)$$

shows us that the action of the first order 2 generators of the normal subgroup $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ of the extended mapping class group is:

$$(\lambda_1,\lambda_2,\lambda_3) \mapsto (\lambda_2\lambda_3 - \lambda_1, \lambda_2, \lambda_3), (\lambda_1, \lambda_3\lambda_1 - \lambda_2, \lambda_3), (\lambda_1, \lambda_2, \lambda_1\lambda_2 - \lambda_3)$$

when expressed in terms of λ-lengths. Hence,

$$(x,y) \mapsto (1-x, \tfrac{xy}{1-x}), (\tfrac{xy}{1-y}, 1-y), \left(\tfrac{x(1-x-y)}{x+y+4xy\sinh^2(\frac{L}{4})}, \tfrac{y(1-x-y)}{x+y+4xy\sinh^2(\frac{L}{4})}\right)$$

in terms of the unit triangle coordinates.

Lemma 3.13. *An order 12 cover of a fundamental domain for $\mathcal{T}(R_{1,1}, L)$ with respect to the extended mapping class group action is given by:*

$$D_L := \left\{(x,y) \in \mathbb{R}^2_+ \mid x, y \leqslant \tfrac{1}{2}, \operatorname{sech}^2(\tfrac{L}{4}) \leqslant (1+2x\sinh^2(\tfrac{L}{4}))(1+2y\sinh^2(\tfrac{L}{4})\right\}.$$

Proof. Given a triple of pairwise once-intersecting simple closed geodesic $(\gamma_1, \gamma_2, \gamma_3)$, let γ_1' denote the unique geodesic (asides from γ_1 which pairwise once-intersects γ_2 and γ_3. Similarly define γ_2' and γ_3'. The theory of Markoff triples and Farey triangulations described in [Bow96] [1] shows that every hyperbolic surface in $\mathcal{M}(R_{1,1}, L)$ has a triple of pairwise once-intersecting simple closed geodesics $(\gamma_1, \gamma_2, \gamma_3)$ where

$$\ell_{\gamma_i} \leqslant \ell_{\gamma_i'} \text{ for } i = 1, 2, 3.$$

Moreover, this triple of simple closed geodesics is generic: only a (Weil-Petersson) measure 0 set of surfaces admit more than one such triple. Proposition 3.9 translates the above fact into saying that almost every hyperbolic surface in $\mathcal{M}(R_{1,1}, L)$ contains a unique ideal triangulation $\triangle$ such that its corresponding λ-lengths $(\lambda_1, \lambda_2, \lambda_3)$ has the property that each λ_i is shorter than the diagonally flipped λ_i'. This gives a fundamental domain for the action of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \leqslant \mathrm{Mod}^{\pm}(R_{1,1})$, and the boundaries of this fundamental domain are given by points where $\lambda_i = \lambda_i'$, that is:

$$2\lambda_1 = \lambda_2\lambda_3 \text{ or } \ 2\lambda_2 = \lambda_1\lambda_3 \text{ or } 2\lambda_3 = \lambda_1\lambda_2.$$

Our desired result now follows from reparametrising these λ-lengths in terms of unit triangle coordinates, and observing that $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ is an order 12 subgroup of the extended mapping class group. $\square$

Proposition 3.14. *The Weil-Petersson volume $V_{1,1}(L)$ of $\mathcal{M}(R_{1,1}, L)$ is*

$$V_{1,1}(L) = \frac{\pi^2}{12} + \frac{L^2}{48}.$$

Proof. The Weil-Petersson volume $V_{1,1}(L)$ of the moduli space $\mathcal{M}(R_{1,1}, L)$ is given by the integral

$$V_{1,1}(L) = 2 \times \frac{1}{12}\int_{D_L} \frac{dy \wedge dx}{xy(1-x-y)} = \frac{1}{6}\int_0^{\frac{1}{2}}\int_{f(x,L)}^{\frac{1}{2}} \frac{dy \wedge dx}{xy(1-x-y)}$$

[1]Strictly speaking, Bowditch only shows this for 1-cusped tori, but the same argument holds for 1-bordered tori.

where $f(x,L) := \frac{\frac{1}{2}-x}{1+2x\sinh^2\frac{L}{4}}$. Differentiating this with respect to L, we obtain that:

$$\frac{d}{dL}V_{1,1}(L) = \frac{1}{6}\int_0^{\frac{1}{2}} \frac{-\frac{\partial f}{\partial L}(x,L)\,dx}{xf(x,L)(1-x-f(x,L))}$$
$$= \frac{1}{6}\int_0^{\frac{1}{2}} \frac{2\tanh(\frac{L}{4})\,dx}{1-4\tanh^2(\frac{L}{4})(x-\frac{1}{2})^2} = \frac{L}{24}.$$

Therefore, the Weil-Petersson volume is given by the polynomial $V_{1,1}(L) = \frac{L^2}{48} + V_{1,1}(0)$. The constant term $V_{1,1}(0)$ is given by the integral of the Weil-Petersson (volume) form over

$$D_0 = \left\{(x,y) \in \mathbb{R}^2_+ \mid x, y \leqslant \tfrac{1}{2} \leqslant x+y\right\}.$$

The substitution $(x,y) = (r\cos^2\theta, r\sin^2\theta)$ coupled with the reflection symmetry of the integral in the line $y = x$ yields that:

$$V_{1,1}(0) = \frac{1}{3}\int_0^{\frac{\pi}{4}} d\theta \int_{\frac{1}{2}}^{\frac{1}{2}\sec^2\theta} \frac{4\,dr}{r\sin(2\theta)(1-r)} = \frac{-4}{3}\int_0^{\frac{\pi}{4}} \frac{\log\cos(2\theta)}{\sin(2\theta)}d\theta.$$

Further using the substitution $u = -\log\cos(2\theta)$, we obtain that:

$$V_{1,1}(0) = \frac{2}{3}\int_0^{\infty} \frac{ue^{-u}du}{1-e^{-2u}} = \frac{2}{3}\sum_{k=0}^{\infty}\int_0^{\infty} ue^{-(2k+1)u}du = \frac{2}{3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{12},$$

where the final equality is given by (a variant form of) Euler's series. □

Note 3.6. In [Pen92], Penner describes a cell-decomposition of the moduli space $\mathcal{M}(R_{g,0,n})$ where each cell is a linear subset of some $3|\chi(R_{g,0,n})|$-dimensional simplex. In the punctured torus case, there is only one cell (up to hyperelliptic involutions) and Penner's simplicial coordinates are related to our (x,y) by the affine transformation $A = \frac{1}{2} - x$ and $B = \frac{1}{2} - y$. The setting up the integral for the $L = 0$ case computation of the Weil-Petersson volume is basically the same as that used in Penner's cell-decomposition based integration scheme for computing Weil-Petersson volumes of the Teichmüller spaces of cusped hyperbolic surfaces.

The Weil-Petersson volume of $\mathcal{M}(R_{1,1},0)$ and $\mathcal{M}(R_{1,1},L)$ exist in the literature [Wol83b, Wol83a, NN98, NN01]. In [Mir07a], Mirzakhani describes an algorithm for recursively computing the Weil-Petersson volume of any moduli space $\mathcal{M}(R_{g,m},\mathbf{L})$ of fixed boundary lengths $\mathbf{L} = (L_1,\ldots,L_m) \in \mathbb{R}^m_{\geqslant 0}$. In particular, she shows that the answer is always a polynomial in L_i^2 with $\mathbb{Q}[\pi^2]$ coefficients. Using symplectic reduction to interpret these coefficients as cohomological intersection numbers [Mir07b] led to a new proof of the celebrated Witten's conjecture [Wit91]. A biproduct of her work is an integration scheme for a general class of functions over the moduli space. We describe this in a tiny bit more detail in the next section. For now, we observe that:

Proposition 3.15. *The average systolic trace over the moduli space $\mathcal{M}(R_{1,1},0)$ of 1-cusped hyperbolic tori is $\frac{24}{\pi^2}$.*

Proof. Once again, we'll be integrating over D_0. Recall that the systolic trace is the trace of the shortest geodesic on a given surface. And over the domain D_0, the systolic trace is equal to:

$$\min\{\lambda_1, \lambda_2, \lambda_3\} = \min\left\{\sqrt{\tfrac{1}{y(1-x-y)}}, \sqrt{\tfrac{1}{x(1-x-y)}}, \sqrt{\tfrac{1}{xy}}\right\}.$$

Thus, the fundamental domain may be partitioned (up to a measure 0 set) into three regions

$$\begin{aligned}
D_0^1 &:= \{(x,y) \in D_0 \mid x \leqslant y, 1-x-y\}, \text{ where the systolic trace is } \sqrt{\tfrac{1}{y(1-x-y)}};\\
D_0^2 &:= \{(x,y) \in D_0 \mid y \leqslant x, 1-x-y\}, \text{ where the systolic trace is } \sqrt{\tfrac{1}{x(1-x-y)}};\\
D_0^3 &:= \{(x,y) \in D_0 \mid 1-x-y \leqslant x, y\}, \text{ where the systolic trace is } \sqrt{\tfrac{1}{xy}}.
\end{aligned}$$

Due to symmetry, the integral of the systolic trace is 3 times the integral of the systolic trace over D_0^3. Further using the symmetry between x and y reflected along $y = x$, we evaluate (with a little help from Mathematica) that the expection of the systole function is:

$$\frac{1}{V_{1,1}(0)} \int_{\frac{1}{4}}^{\frac{1}{2}} \int_{\max\{x, 1-2x\}}^{\frac{1}{2}} \frac{1}{x^{\frac{3}{2}} y^{\frac{3}{2}} (1-x-y)} \, dy \, dx = \frac{24}{\pi^2}.$$

□

Note 3.7. As far as we know, the above computation cannot be naturally obtained via Mirzakhani's integration scheme. To generalise to the moduli space $\mathcal{M}(R_{1,1}, L)$ of hyperbolic surfaces with geodesic border of length L, we would need to compute the following integral:

$$\frac{1}{V_{1,1}(L)} \int_{\frac{1}{4}\operatorname{sech}^2(\frac{L}{4})}^{\frac{1}{2}} \int_{\max\{x, \frac{1-2x}{1+4x\sinh^2(\frac{L}{4})}\}}^{\frac{1}{2}} \frac{1}{x^{\frac{3}{2}} y^{\frac{3}{2}} (1-x-y)} \, dy \, dx.$$

3.3 Mirzakhani's Integration Scheme

We begin with an example calculation of the Weil-Petersson volume of $\mathcal{M}(R_{1,2}, \mathbf{0}) = \mathcal{M}(R_{1,2}, (0,0))$. Using this as a launching pad, we then describe Mirzakhani's recursion-based integration scheme.

The following result is called a *McShane identity* [McS91]:

Given any 2-cusped hyperbolic torus S with cusps numbered 1 and 2, then

$$1 = \sum_{\{\gamma_1, \gamma_2\} \in \mathcal{P}} \frac{2}{1 + e^{\frac{1}{2}(\ell_{\gamma_1} + \ell_{\gamma_2})}},$$

where $\mathcal{P}$ is the collection of pairs of pants embedded in S which contain cusp 1; the pair of simple closed geodesics $\{\gamma_1, \gamma_2\}$ encode the other two boundaries of the specified pair of pants.

Note 3.8. We emphasise the fact that the individual summands on the right are dependant upon the geometry of S, but sum to something independant of the geometry of S.

The mapping class group $\mathrm{Mod}(R_{1,2})$ has a natural action on $\mathcal{P}$, where an arbitrary mapping class $[h]$ acts by taking $\{\gamma_1, \gamma_2\}$ to the geodesic representatives of $\{h \circ \gamma_1, h \circ \gamma_2\}$. In particular, $\mathcal{P}$ may be partitioned into two orbits:

1. $\mathcal{P}_1$: pairs of pants bordered by cusp 1, one interior simple closed curve γ and cusp 2;
2. $\mathcal{P}_2$: pairs of pants bordered by cusp 1 and two non-peripheral simple closed curves γ_1, γ_2.

Since $\mathcal{P}_1$ consists of embedded pairs of pants with cusps 1 and 2 as two of its boundaries, we remove the redundancy of writing down cusp 2 and simply encode elements of this set by the non-peripheral simple closed geodesic γ.

Let $V_{1,2}$ denote the Weil-Petersson volume of $\mathcal{M}(R_{1,2}, \mathbf{0})$, and let Ω_{WP} denote the Weil-Petersson volume form, then:

$$\begin{aligned} V_{1,2} = \int_{\mathcal{M}(R_{1,2},\mathbf{0})} \Omega_{WP} &= \int_{\mathcal{M}(R_{1,2},\mathbf{0})} \sum_{\{\gamma_1,\gamma_2\}\in\mathcal{P}} \frac{2\Omega_{WP}}{1+e^{\frac{1}{2}(\ell_{\gamma_1}+\ell_{\gamma_2})}} \\ &= \int_{\mathcal{M}(R_{1,2},\mathbf{0})} \sum_{\gamma\in\mathcal{P}_1} \frac{2\Omega_{WP}}{1+e^{\frac{1}{2}\ell_\gamma}} + \int_{\mathcal{M}(R_{1,2},\mathbf{0})} \sum_{\{\gamma_1,\gamma_2\}\in\mathcal{P}_2} \frac{2\Omega_{WP}}{1+e^{\frac{1}{2}(\ell_{\gamma_1}+\ell_{\gamma_2})}}. \end{aligned} \tag{3.12}$$

Recall that

$$\mathcal{M}(R_{1,2}, \mathbf{0}) = \mathcal{T}(R_{1,2}, \mathbf{0})/\mathrm{Mod}(R_{1,2}) \text{ and } \mathcal{M}(R_{1,2}, [\gamma], \mathbf{0}) = \mathcal{T}(R_{1,2}, \mathbf{0})/\mathrm{Stab}([\gamma]).$$

The orbit-stabiliser theorem says that we have a canonical bijection

$$\mathcal{P}_1 \cong \mathrm{Mod}(R_{g,m})/\mathrm{Stab}([\gamma]),$$

and so we may think of $\mathcal{M}(R_{1,2}, [\gamma])$ as a "$\mathcal{P}_1$-cover" of $\mathcal{M}(R_{1,2}, \mathbf{0})$. Hence, we may distribute the summands of the integrand of the left integral in (3.12) throughout $\mathcal{M}(R_{1,2}, [\gamma], \mathbf{0})$ to obtain:

$$\int_{\mathcal{M}(R_{1,2},\mathbf{0})} \sum_{\gamma\in\mathcal{P}_1} \frac{2\Omega_{WP}}{1+e^{\frac{1}{2}\ell_\gamma}} = \int_{\mathcal{M}(R_{1,2},[\gamma],\mathbf{0})} \frac{2\Omega_{WP}}{1+e^{\frac{1}{2}\ell_\gamma}}.$$

Similarly, the fact that

$$\mathcal{P}_2 \cong \mathrm{Mod}(R_{1,2})/\mathrm{Stab}\{[\gamma_1], [\gamma_2]\} \cong (\mathrm{Mod}(R_{1,2})/\mathrm{Stab}([\gamma_1], [\gamma_2]))/\mathbb{Z}_2$$

allows us to distribute the summands of the integrand of the right integral in (3.12) throughout $\mathcal{M}(R_{1,2}, ([\gamma_1], [\gamma_2]), \mathbf{0})$ to obtain:

$$\int_{\mathcal{M}(R_{1,2},\mathbf{0})} \sum_{\{\gamma_1,\gamma_2\}\in\mathcal{P}_1} \frac{2\Omega_{WP}}{1+e^{\frac{1}{2}(\ell_{\gamma_1}+\ell_{\gamma_2})}} = \int_{\mathcal{M}(R_{1,2},([\gamma_1],[\gamma_2]),\mathbf{0})} \frac{\Omega_{WP}}{1+e^{\frac{1}{2}(\ell_{\gamma_1}+\ell_{\gamma_2})}}.$$

Note 3.9. We've abused notation by using Ω_{WP} to refer to the Weil-Petersson volume on distinct spaces. This is only a minor notation abuse because the forms are related by pullback with respect to the covering maps from $\mathcal{M}(R_{1,2},[\gamma],\mathbf{0})$ or $\mathcal{M}(R_{1,2},([\gamma_1],[\gamma_2]),\mathbf{0})$ to $\mathcal{M}(R_{1,2},\mathbf{0})$.

The moduli space $\mathcal{M}(R_{1,2},[\gamma],\mathbf{0})$ is diffeomorphic to

$$\left\{(\ell,\tau+\ell\mathbb{Z},[S])\in\mathbb{R}_+\times\mathbb{S}^1\times\mathcal{M}(R_{1,1,0})\;\middle|\;\begin{array}{l}S\text{ is a connected component obtained from}\\ \text{cutting }\gamma\text{ on some surface in }\mathcal{M}(R_{1,2},\mathbf{0})\text{ with }\ell_\gamma=\ell\end{array}\right\}$$

In terms of Fenchel-Nielsen coordinates with respect to a Fenchel-Nielsen class containing $[\gamma]$, the distributed integral over $\mathcal{M}(R_{1,2},[\gamma],\mathbf{0})$ is given by:

$$\begin{aligned}\int_{\mathcal{M}(R_{1,2},[\gamma],\mathbf{0})}\frac{2\Omega_{WP}}{1+e^{\frac{1}{2}\ell_\gamma}}&=\int_0^\infty\int_0^\ell\frac{2}{1+e^{\frac{1}{2}\ell}}\cdot V_{1,1}(\ell)\,d\tau\,d\ell\\&=\int_0^\infty\frac{\ell(\frac{\pi^2}{6}+\frac{\ell^2}{24})}{1+e^{\frac{1}{2}\ell}}d\ell=\frac{17\pi^4}{180}.\end{aligned}$$

The moduli space $\mathcal{M}(R_{1,2},([\gamma_1],[\gamma_2]),\mathbf{0})$ is diffeomorphic to

$$\left\{\begin{array}{c}(\ell_1,\tau_1+\ell_1\mathbb{Z},\ell_2,\tau_2+\ell_2\mathbb{Z},[S])\in\\ \mathbb{R}_+\times\mathbb{S}^1\times\mathbb{R}_+\times\mathbb{S}^1\times\mathcal{M}(R_{0,2,1})\end{array}\;\middle|\;\begin{array}{l}S\text{ is a connected component obtained from cutting}\\ \gamma_1,\gamma_2\text{ on some surface in }\mathcal{M}(R_{1,2},\mathbf{0})\text{ with }\ell_{\gamma_i}=\ell_i.\end{array}\right\},$$

where the ℓ_i are the length parameters for $[\gamma_i]$ and the τ_i are twist parameters for $[\gamma_i]$. Using Fenchel-Nielsen coordinates arising from some Fenchel-Nielsen class containing $[\gamma_1]$ and $[\gamma_2]$,

$$\int_{\mathcal{M}(R_{1,2},([\gamma_1],[\gamma_2])}\frac{\Omega_{WP}}{1+e^{\frac{1}{2}(\ell_{\gamma_1}+\ell_{\gamma_2})}}=\int_0^\infty\int_0^\infty\frac{\ell_1\ell_2}{1+e^{\frac{1}{2}(\ell_1+\ell_2)}}d\ell_1\,d\ell_2=\frac{7\pi^4}{45}$$

Summing up these two volumes:

$$V_{1,2}=\frac{17\pi^4}{180}+\frac{7\pi^4}{45}=\frac{\pi^4}{4}.$$

3.3.1 The Structure of Mirzakhani's Integration Scheme

The starting point for Mirzakhani's volume calculation is a McShane identity. And to compute the Weil-Petersson volume of a general moduli space $\mathcal{M}(R_{g,m},\mathbf{L})$, Mirzakhani generalised McShane identities from cusped hyperbolic surfaces to bordered hyperbolic surfaces:

Theorem 3.16 (Mirzakhani's generalised McShane identity for bordered surfaces, [Mir07a]**).** *Given a hyperbolic surface* S*, with borders of length* $\mathbf{L}=(L_1,\dots,L_m)$*, then:*

$$L_1=\sum_{i=2}^m\sum_{\gamma\in\mathcal{P}_{1,i}}\log\left(\frac{e^{\frac{L_1+L_i+\ell_\gamma}{2}}+e^{\frac{L_1-L_i+\ell_\gamma}{2}}+e^{L_1}+e^{\ell_\gamma}}{e^{\frac{-L_1+L_i+\ell_\gamma}{2}}+e^{\frac{-L_1-L_i+\ell_\gamma}{2}}+e^{-L_1}+e^{\ell_\gamma}}\right) \tag{3.13}$$

$$+\sum_{\{\gamma_1,\gamma_2\}\in\mathcal{P}_2}2\log\left(\frac{e^{\frac{L_1}{2}}+e^{\frac{\ell_{\gamma_1}+\ell_{\gamma_2}}{2}}}{e^{\frac{-L_1}{2}}+e^{\frac{\ell_{\gamma_1}+\ell_{\gamma_2}}{2}}}\right), \tag{3.14}$$

where $\mathcal{P}_{1,i}$ is the collection of embedded pairs of pants on S which contain borders 1 and i, and $\mathcal{P}_2$ is the collection of embedded pairs of pants on S which contain of border 1 and two non-peripheral geodesics.

This McShane identity allows us to express $L_1 V_{g,m}(\mathbf{L})$ as integrals over moduli spaces of the form $\mathcal{M}(R_{g,m},[\gamma],\mathbf{L})$ for summands corresponding to those in (3.13) and $\mathcal{M}(R_{g,m},([\gamma_1],[\gamma_2]),\mathbf{L})$ for those in (3.14).

The fact that the moduli space $\mathcal{M}(R_{g,m},[\gamma],\mathbf{L})$ is diffeomorphic to

$$\left\{ (\ell, \tau+\ell\mathbb{Z}, S) \in \mathbb{R}_+ \times \mathbb{S}^1 \times \mathcal{M}(R_{g,m-1}) \;\middle|\; \begin{array}{l} S \text{ is a connected component obtained from} \\ \text{cutting } \gamma \text{ on some surface in } \mathcal{M}(R_{g,m},\mathbf{L}) \text{ with } \ell_\gamma = \ell \end{array} \right\}$$

means that the integrals corresponding to the summands in (3.13) reduce to integrals of the products of the Weil-Petersson volumes of lower dimensional moduli spaces, multiplied by some multivariate elementary functions. The integrals for the summands in (3.14) may be similarly reduced. We now explicitly state Mirzakhani's recursion [Mir07a] for completeness. The presentation is taken from [Do08].

Given $\mathbf{L} = (L_1,\ldots,L_m)$, let $\hat{\mathbf{L}}_k$ denote the vector $(L_1,\ldots,L_{k-1},L_{k+1},\ldots,L_m)$. Moreover, given an unordered subset of indices $I = \{i_1,i_2,\ldots,i_n\}$, let $\mathbf{L}_I$ denote the vector $(L_{i_1},L_{i_2},\ldots,L_{i_n})$ arranged in any order. Lastly, given two *disjoint* index sets I and J, we use the notation

$$I \sqcup_2^m J \text{ to mean that } I \cup J = \{2,\ldots,m\}.$$

Then the Weil-Petersson volume $V_{g,m}(\mathbf{L})$ of the moduli space $\mathcal{M}(R_{g,m},\mathbf{L})$ may be recursively calculated via:

$$\begin{aligned} 2\frac{\partial}{\partial L_1} L_1 V_{g,m}(\mathbf{L}) &= \int_0^\infty \int_0^\infty xy\; H(x+y,L_1) V_{g-1,m+1}(x,y,\hat{\mathbf{L}}_1)\; dxdy \\ &\quad + \sum_{\substack{g_1+g_2=g \\ I \sqcup_2^m J}} \int_0^\infty \int_0^\infty xy\; H(x+y,L_1) V_{g_1,|I|+1}(x,\mathbf{L}_I) V_{g_2,|J|+1}(y,\mathbf{L}_J)\; dxdy \\ &\quad + \sum_{k=2}^m \int_0^\infty x\,(H(x,L_1+L_k)+H(x,L_1-L_k))\, V_{g,m-1}(x,\hat{\mathbf{L}}_k)\; dx, \end{aligned}$$

where the function $H : \mathbb{R}^2 \to \mathbb{R}$ is given by:

$$H(x,y) := \frac{1}{1+e^{\frac{x+y}{2}}} + \frac{1}{1+e^{\frac{x-y}{2}}}.$$

And the base cases of the recursion are:

$$V_{0,1}(L_1) = 0,\;\; V_{0,2}(L_1,L_2) = 0,\;\; V_{0,3}(L_1,L_2,L_3) = 1 \text{ and } V_{1,1}(L_1) = \frac{\pi^2}{12} + \frac{L_1^2}{48}.$$

Note 3.10. In theory, Mirzakhani's integration scheme is more general than for Weil-Petersson volume calculations (and for deriving simple geodesic growth rates). Her ideas may be used to integrate any function expressible in the form:

$$\sum_{(\gamma_1,\ldots,\gamma_k)\in\mathcal{P}} F(\ell_{\gamma_1},\ldots,\ell_{\gamma_k}),$$

where $\mathcal{P} = \mathrm{Mod}(R)\cdot(\gamma_1,\ldots,\gamma_k)$ is some orbit space for a tuple of simple closed geodesics — or indeed, multicurves. We're aware of no examples (apart from Mirzakhani's volume calculations) of integrable candidate functions F where this sum converges to a geometrically meaningful function over the moduli space. Although, Norbury does implement this machinery to compute the integral of a function over the moduli space of 1-cusped Klein bottles [Nor08]. Unfortunately, it is actually reasonably simple to perform the integral in this particular example, and Mirzakhani's integration scheme is somewhat unnecessary.

Chapter 4

McShane Identities

A subsurface R' of a hyperbolic surface R may be identified by its boundary geodesics $\partial R'$ in R, and hence by the mixed class $[\partial R] = (\Gamma, A)$ representing $\partial R'$. This gives a canonical bijection between the set of subsurfaces of R of some fixed topological type R' and the $\mathrm{Mod}(R)$ orbit $\mathrm{Mod}(R) \cdot (\Gamma, A)$. In particular, the collection of subsurfaces of R of a given topological type R' inherits a $\mathrm{Mod}(R)$ action from this bijection.

A *McShane-type identity* for a cusped/bordered hyperbolic surface R with boundary lengths $\mathbf{L}$ is an expression of the form:

$$g(\mathbf{L}) = \sum_{P \in \mathcal{P}} \mathrm{Gap}(P),$$

where the right hand side is summed over geodesic bordered pair of pants $P \subset R$ and

$$\mathcal{P} = \bigcup \mathrm{Mod}(R) \cdot P_i$$

is a disjoint union of $\mathrm{Mod}(R)$-orbits of embedded pairs of pants P_i in R. The sum is independent of the underlying geometry of $[R] \in \mathcal{M}(R, \mathbf{L})$, whilst the individual summands are highly dependant. Since pairs of pants P_i are geometrically determined by its boundary lengths $\ell_{\alpha_i}, \ell_{\beta_i}, \ell_{\gamma_i}$, the function Gap is usually given as a real function $f : \mathbb{R}_+^3 = \mathcal{M}(R_{0,3,0})$ [1], although one or two of the inputs of the Gap function may be taken from the m-tuple $\mathbf{L}$ of R boundary lengths.

McShane's (original) identities are derived from measure theoretically partitioning a length 1 horocycle η around cusp 1 on a cusped surface S. The horocycle η parametrises all the possible directions in which one may shoot out geodesic from cusp 1, and the points on η are partitioned based on the behaviour of these launched geodesics.

To begin with, the *Birman-Series (geodesic sparsity) theorem* [BS85] tells us that geodesic paths launched from cusp 1 almost always self-intersect.

[1] Or $\mathbb{R}_+^2$ for $\mathcal{R}_{0,2,1}$ and $\mathbb{R}_+$ for $\mathcal{R}_{0,1,2}$.

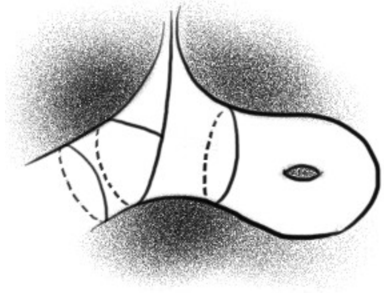


FIGURE 4.1: A lasso and the unique pair of pants containing it.

Definition 4.1. *We call the segment of a self-intersecting geodesic up to its first point of self-intersection a* lasso *(consider Figure 4.1).*

Each lasso shot out from cusp 1 lies on a unique embedded pair of pants $P \subset S$ containing cusp 1. Conversely, the length 1 horocycle η at cusp 1 of the pair of pants $P \subset S$ contains four "windows" of horocyclic segments from which every launched geodesic self-intersects, and have lassos which lie on P.

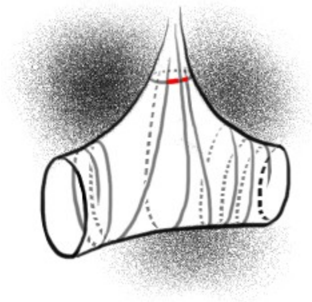


FIGURE 4.2: A pair of horocyclic regions corresponding to spiralling geodesics.

These four horocyclic regions come in pairs — one pair at the front of P and one at the back. Moreover, each pair of horocyclic regions is separated by the unique ideal geodesic on P with both end points up cusp 1. And the other two boundary points of these two horocyclic segments are given by ideal geodesics on P which spiral about the two other boundaries of P. The length of these four horocyclic subsegments add to $\frac{2}{1+e^{\frac{1}{2}(\ell_{\gamma_1}+\ell_{\gamma_2})}}$ (Lemma 4.24).

Definition 4.2. *The horocyclic regions on η from which the projected geodesics have lassos which lie on an embedded subsurface P is called the* gap region *for P.*

We partition the length of η as a sum indexed by embedded pairs of pants P in S containing cusp 1 to obtain the following identity:

Theorem 4.3 (McShane [McS98])**.** *Given a hyperbolic cusped surface S, let $\mathcal{P}$ denote the collection of embedded pairs of pants $P \subset S$ that border cusp 1, then:*

$$1 = \sum_{P \in \mathcal{P}} \frac{2}{1 + e^{\frac{1}{2}(\ell_{\gamma_1}+\ell_{\gamma_2})}}.$$

One powerful application of McShane-type identities is the computing of Weil-Petersson volumes for the moduli spaces $\mathcal{M}(R_{g,m,n}, \mathbf{L})$ of bordered hyperbolic surfaces with boundary length $\mathbf{L}$: recall from Subsection 3.3.1 that Mirzakhani (Note 3.10, [Mir07a]) used McShane-type identities to distribute

$$g(\mathbf{L})V_{g,m}(\mathbf{L}) = \int_{\mathcal{M}(R,\mathbf{L})} \sum_{P\in\mathcal{P}} g(\mathbf{L})\ \Omega_{WP}.$$

as an integral over the moduli spaces $\mathcal{M}(R, [\partial P_i], \mathbf{L})$, where $[\partial P_i]$ denotes the curve class corresponding to the boundary of P_i:

$$g(\mathbf{L})V_{g,m}(\mathbf{L}) = \sum_i \int_{\mathcal{M}(R,[\partial P_i],\mathbf{L})} \mathrm{Gap}(f_* P_i)\ \Omega_{WP}.$$

4.1 Generalisations

The *first method* of deriving a new types of McShane's identities is to extend beyond the world of cusped surfaces. For example: we've already seen Mirzakani's generalisation of McShane's original identities for cusped hyperbolic surfaces to include bordered hyperbolic surfaces (Theorem 3.16). In [TWZ06], Tan-Wong-Zhang independantly derive this identity, and generalise it to further include hyperbolic surfaces with small cone-points. Various other generalisations of McShane identities exist, here is a non-exhaustive list:

- bordered hyperbolic surfaces [Mir07a, TWZ06]
- closed hyperbolic surfaces [LT11]
- cone-pointed hyperbolic surfaces with small ($\leqslant \pi$) cone angles [TWZ06],
- non-orientable cusped and bordered hyperbolic surfaces [Nor08],
- Markoff triples (including quasi-Fuchsian representations of the 1-cusped torus group) [Bow96, Bow98]
- quasi-Fuchsian representations [AMS06],
- closed hyperbolic surfaces with one marked point [Hua12],
- Markoff quads (includes quasi-Fuchsian representations of the 3-cusped projective plane group) [HN13].

In recent work [Hu13, HTZ14], Hu-Tan-Zhang (implicitly) construct a mapping class group equivariant map

$$\mathcal{V} : \mathcal{T}(R_{1,1,0}, L) \to \mathcal{T}(R_{1,0,1})$$

from the Teichmüller space of 1-bordered tori with length L boundary to the Teichmüller space of 1-cusped tori. The individual summands of the McShane identity for the cusped case

McShane identity pullback (as functions on $\mathcal{T}(R_{1,0,1})$) and give a new McShane identity for 1-bordered tori. For completeness, we state the map $\mathcal{V}$, as it isn't explicitly given in [Hu13, HTZ14]. We state $\mathcal{V}$ in terms as a function from the λ-length coordinates given in (3.11) to the λ-length coordinates given in (3.9). Define the function:

$$\begin{aligned}\nu(x,y,z) = &\left(\frac{y}{xz}+\frac{1}{3}\left(\frac{1}{2}-\frac{y}{xz}\right)\left(\frac{4\sinh^2(\frac{L}{4})}{x^2+4\sinh^2(\frac{L}{4})}+\frac{4\sinh^2(\frac{L}{4})}{z^2+4\sinh^2(\frac{L}{4})}\right)\right)\\ &\times\left(\frac{z}{xy}+\frac{1}{3}\left(\frac{1}{2}-\frac{z}{xy}\right)\left(\frac{4\sinh^2(\frac{L}{4})}{x^2+4\sinh^2(\frac{L}{4})}+\frac{4\sinh^2(\frac{L}{4})}{y^2+4\sinh^2(\frac{L}{4})}\right)\right).\end{aligned}$$

Then, the mapping class group equivariant map $\mathcal{V}:\mathcal{T}(R_{1,1,0},L)\to\mathcal{T}(R_{1,0,1})$ is given by:

$$\mathcal{V}(\lambda_1,\lambda_2,\lambda_2) = \left(\nu(\lambda_1,\lambda_2,\lambda_3)^{-\frac{1}{2}},\nu(\lambda_2,\lambda_3,\lambda_1)^{-\frac{1}{2}},\nu(\lambda_3,\lambda_1,\lambda_2)^{-\frac{1}{2}}\right). \tag{4.1}$$

The *second method* for obtaining McShane-type identities is to sum over $\mathrm{Mod}(S)$-orbits of subsurfaces of S other than pairs of pants. For example: decomposing a pair of pants P containing cusp 1 along the unique ideal geodesic on P with both ends going up cusp 1, gives two pairs of half-pants P_1 and P_2. And geodesics which self-intersect on P, when launched within the gap region on P_i, will remain on P_i (Figure 4.2). This observation allows us to split each summand in the cusped McShane identity into two smaller numbers, thus yielding a McShane identity with finer summands:

Proposition 4.4. *Given a cusped hyperbolic surface S, let η denote the length 1 horocycle around cusp 1. Set $\mathcal{P}$ to denote the collection of embedded pairs of half-pants in S which border cusp 1. For a pair of half-pants $P\in\mathcal{P}$, let $\ell_\gamma(P)$ denote the length of the closed geodesic boundary of P and let $\ell_{\gamma_\infty}(P)$ denote the length of the arch of P truncated at η. Then,*

$$1=\sum_{P\in\mathcal{P}}2e^{\frac{-1}{2}(\ell_\gamma+\ell_{\gamma_\infty})}.$$

Proof. The proof that the lasso of (almost) every geodesic launched from cusp 1 lies on a unique pair of half-pants is covered during the proof of Theorem 4.5. We presently resign ourselves simply to computing the combined length of the gap regions on these pairs of half-pants.

Let h denote the length of the horocyclic subsegment of η on P. By equation (3.7),

$$h\exp(\tfrac{\ell_{\gamma_\infty}}{2})=2\cosh(\tfrac{\ell_\gamma}{2})\Rightarrow h=2\exp(\tfrac{-\ell_{\gamma_\infty}}{2})\cosh(\tfrac{\ell_\gamma}{2}).$$

Moreover, by equation (4.21) in the proof of Lemma 4.24, the horocyclic region corresponding to launched geodesics which self-intersect on P is given by:

$$h-\sqrt{h^2-\tfrac{4}{e^{\ell_{\gamma_\infty}}}}=2e^{\frac{-1}{2}\ell_{\gamma_\infty}}\left(\cosh(\tfrac{\ell_\gamma}{2})-\sinh(\tfrac{\ell_\gamma}{2})\right)=2e^{\frac{-1}{2}(\ell_\gamma+\ell_{\gamma_\infty})}, \tag{4.2}$$

and the result follows. □

Note 4.1. Similar refinements exist for surfaces with geodesic boundaries and cone-points with small cone angles.

4.2 McShane Identity for Crowned Surfaces

We now generalise the refined McShane identity (Proposition 4.4) to crowned hyperbolic surfaces. To simplify the statement of our next theorem, we regard cusps as length 0 boundary geodesics.

Theorem 4.5. *Given a crowned surface S with m closed boundary geodesics $(\beta_1, \ldots, \beta_m)$ of lengths $(L_1, \ldots, L_m) \in \mathbb{R}^m_{\geqslant 0}$ and k arches $(\alpha_1, \ldots, \alpha_k)$. We partially decorate S with length 1 horocycles at the cusps of S and length 1 horocyclic segment at the tines of S; let*

- *$\mathcal{S}_i$ be the collection of embedded ideal triangles with the α_i opposite to tine 1 (left figure in Figure 4.3), each ideal triangle is denoted by the two bi-infinite geodesics $\{\sigma, \tau\}$ adjacent to tine 1;*
- *$\mathcal{P}_j$ be the collection of embedded half-pants with its tine based at tine 1 and with boundary j as a boundary component (center figure in Figure 4.3), each such pair of half-pants is denoted by its bi-infinite geodesic boundary μ;*
- *$\mathcal{P}$ be the collection of all[2] embedded half-pants with its tine based at tine 1 (center and right figures in Figure 4.3), each pair of half-pants is denoted by $\{\gamma, \gamma_\infty\}$, where γ is the closed geodesic boundary and γ_∞ is the bi-infinite geodesic boundary of this pair of half-pants.*

Then,

$$1 = \sum_{i=1}^{k} \sum_{\{\sigma,\tau\} \in \mathcal{S}_i} e^{\frac{1}{2}(\ell_{\alpha_i} - \ell_\sigma - \ell_\tau)} + \sum_{j=1}^{m} \sum_{\mu \in \mathcal{P}_j} 2e^{\frac{-1}{2}\ell_\mu} \sinh\left(\tfrac{L_j}{2}\right)$$
$$+ \sum_{\{\gamma,\gamma_\infty\} \in \mathcal{P}} 2e^{\frac{-1}{2}(\ell_\gamma + \ell_{\gamma_\infty})},$$

where for any ideal geodesic β, the positive number ℓ_β denotes the length of β truncated at the length 1 horocycles at the cusps and tines of S.

Note 4.2. When the surface S has only cusps — no crowns and no closed boundary geodesics, the sets $\mathcal{S}_i$ and $\mathcal{P}_j$ are empty and Theorem 4.5 reduces to Proposition 4.4. Hence, combined with (4.2) and (4.22), Theorem 4.5 allows us to recover McShane's original identities (Theorem 4.3).

Proof. By gluing S with an isometric but orientation-reversed copy of itself along its boundaries, we obtain the double dS of S. By construction, dS is a cusped hyperbolic surface and by the Birman-Series theorem (Theorem 4.10), simple geodesics occupy a set of area 0 on dS and hence on S. Therefore, almost all of the geodesics launched from tine 1 may be classified as:

[2]Including those in $\mathcal{P}_j$.

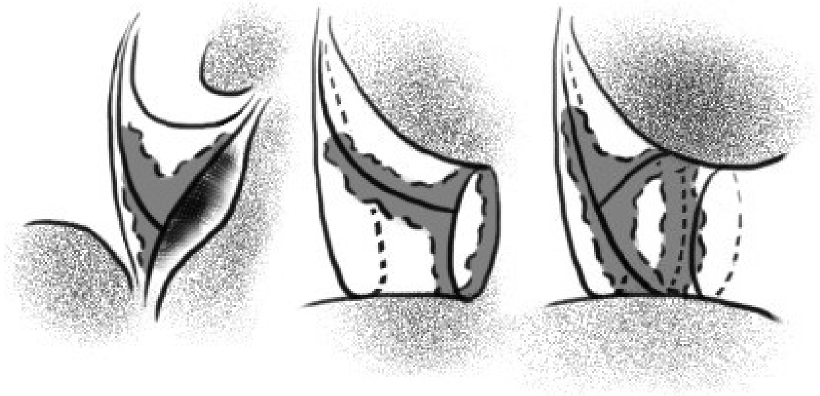


FIGURE 4.3: Fattening up geodesics to subsurfaces.

- those which hit a (boundary) arch without self-intersecting,
- those which hit a closed geodesic boundary without self-intersecting
- and those which intersect.

Lemma 4.6. *A geodesic launched from tine* 1 *which*

- *hits arch* α_i*, lies on a unique embedded ideal triangle* T *in* S*, and* $T \in \mathcal{S}_i$*;*
- *hits a closed geodesic boundary* β_j*, lies on a unique embedded pair of half-pants* P*, and* $P \in \mathcal{P}_j$*;*
- *self-intersects, forms a lasso. And this lasso lies on a unique embedded pair of half-pants* $P \in \mathcal{P}$*.*

Proof. The basic idea in all three cases is to take the subsegment of a launched geodesic up to either its first self-intersection or boundary-intersection and to *fatten* this up to a subsurface — as depicted in Figure 4.3. We now analyse each of these cases separately:

Given a geodesic δ launched from tine 1 that hits arch α_i, consider the two bi-infinite piecewise geodesics obtained by travelling along δ until we hit α_i and then following α_i either to the left or to the right. These two broken bi-infinite geodesics are uniquely homotopy equivalent to respective bi-infinite geodesics σ and τ, and the geodesics σ, τ, α_i bound an embedded ideal triangle $T \in \mathcal{S}_i$. The fact that hyperbolic 2-gons are impossible means that δ cannot hit either σ or τ and hence cannot leave T before hitting α_i and terminating. Therefore, δ lies on T. For the uniqueness of T, if an ideal triangle T' contains δ, then the fact that δ terminates at α_i means that α_i is one of the boundaries of T'. The other two sides of T' are then homotopy equivalent to the two bi-infinite broken geodesics we formed at the start. By the uniqueness of geodesic representatives (or the impossibility of 2-gons), the other two sides of T' must be σ and τ, and hence $T' = T$.

The proof for the second claim is similar. Given a geodesic δ launched from tine 1 that hits β_j, consider the bi-infinite piecewise geodesic obtained by travelling along δ from tine 1 to the boundary, then following β_j around once (in either direction), then back up δ. This bi-infinite broken geodesic is uniquely homotopy equivalent to a bi-infinite geodesic μ with both ends up tine 1, and the geodesics μ and β_j bound a pair of half-pants $P \in \mathcal{P}_j$. The fact that hyperbolic 2-gons are impossible means that δ cannot hit μ, and hence cannot leave P before hitting β_j (and terminating). Therefore, δ lies on P. For uniqueness, if a pair of half-pants P' contains δ, then the fact that δ terminates at β_j means that P' must contain β_j as a boundary component. The bi-infinite boundary of P' must be homotopy equivalent to the bi-infinite piecewise geodesic we formed at the start (of this paragraph), and by the uniqueness of homotopic geodesic representatives, the bi-infinite geodesic boundary of P' must be μ, and hence $P' = P$.

Finally, given a geodesic that self-intersects, consider its lasso δ. A lasso is composed of two parts: a broken geodesic *loop* attached to the end of a geodesic *spoke*. Let γ be the unique simple closed geodesic homotopy equivalent to the loop of δ, and let γ_∞ be the unique bi-infinite geodesic homotopy equivalent to the bi-infinite piecewise geodesic obtained by travelling along δ to its end and then back to tine 1 along the spoke of δ. These two geodesics γ and γ_∞ bound an embedded pair of half-pants $P \in \mathcal{P}$. Due to the impossibility of hyperbolic 2-gons, δ cannot exit P via γ_∞. If δ leaves P via γ, then the geodesics γ and δ either bound a hyperbolic 2-gon or a hyperbolic geodesic triangle with internal angles summing to greater than π (Figure 4.4). Since both of these things are impossible, the lasso of δ must lie within

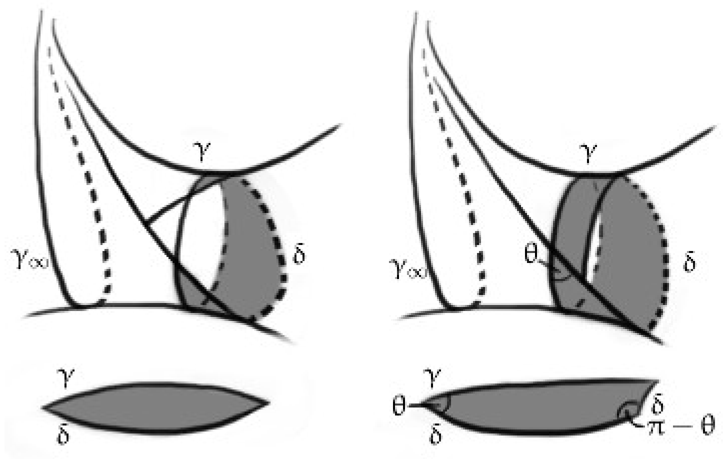


FIGURE 4.4: Figures which violate Gauss-Bonnet.

P. The uniqueness of P follows the same arguments as before: the boundaries of any pair of half-pants containing δ are homotopy equivalent to the two broken geodesics described at the beginning of this paragraph. The uniqueness of geodesic representatives then ensures that P is unique. $\square$

Returning to the proof of Theorem 4.5, we need only to determine the length of the horocyclic region of geodesics launched from tine 1 which hit α_i, β_j or self-intersect. For an embedded ideal triangle $T \in \mathcal{S}_i$ bordered by $\{\sigma, \tau, \alpha_i\}$, every geodesic on T launched from tine 1 hits α_i (apart from σ and τ themselves). By Proposition 1.3, the length of the horocyclic segment at tine 1 is given by

$$\frac{\lambda_{\alpha_i}}{\lambda_\sigma \lambda_\tau} = \exp \tfrac{1}{2}(\ell_{\alpha_i} - \ell_\sigma - \ell_\tau).$$

For an embedded pair of half-pants $P \in \mathcal{P}$, the combined length of the two horocyclic regions constituting the gap region of P is

$$2\exp\tfrac{-1}{2}(\ell_\gamma + \ell_{\gamma_\infty}).$$

In addition, for an embedded pair of half-pants $P \in \mathcal{P}_j$ with closed boundary β_j and boundary arch μ, the length of the horocyclic region (corresponding to geodesics which hit β_j before self-intersecting) is given by the total length of the horocyclic segment on P minus the length of the region of self-intersecting geodesics:

$$2e^{\frac{-1}{2}\ell_\mu}\cosh(\tfrac{L_j}{2}) - 2e^{\frac{-1}{2}(L_j+\ell_\mu)} = 2e^{\frac{-1}{2}\ell_\mu}\sinh(\tfrac{L_j}{2}).$$

The desired McShane identity follows from this measure theoretic partitioning of the length of the horocycle segment at tine 1 described in Lemma 4.6. □

Note 4.3. Our proof was independent of whether we partitioned a horocyclic segment at a tine or a horocycle at a cusp. Thus, we may instead replace "tine 1" with "cusp 1" in the definitions of $\mathcal{S}_i$, $\mathcal{P}_j$ and $\mathcal{P}$, and obtain a McShane identity (of the same form) for decomposing cusp horocycles on crowned surfaces with cusps.

4.2.1 A curious example

We now consider this McShane identity for hyperbolic crowned surfaces S homeomorphic to $R_{0,0,0}^{(1,1)}$ — a $(1,1)$-crowned annulus. Let's first try to understand what the collections $\mathcal{S}_i, \mathcal{P}_j, \mathcal{P}$ are for such a surface. Since S has no closed geodesic boundaries, there are no sets of the form $\mathcal{P}_j$. Moreover, there's a unique pair of embedded half-pants $P = \{\gamma, \alpha_1\}$ with its tine at tine 1, hence $\mathcal{P} = \{P\}$. On S, simple geodesics can't shoot out from tine 1 and come back and hit α_1 without somehow forming a hyperbolic 2-gon. Thus $\mathcal{S}_1$ is empty. On the other hand, there's a $\mathbb{Z}$ family of ideal triangles $\triangle_i \in \mathcal{S}_2$ containing α_2. Fix an arbitrary pair (σ_0, σ_1) of disjoint non-peripheral ideal geodesics. Then, $\triangle_0 := (\alpha_1, \alpha_2, \sigma_0, \sigma_1)$ gives an ideal triangulation of S. Define σ_2 to be the diagonally opposite geodesic to σ_0 with respect to $\triangle_0$ and note that

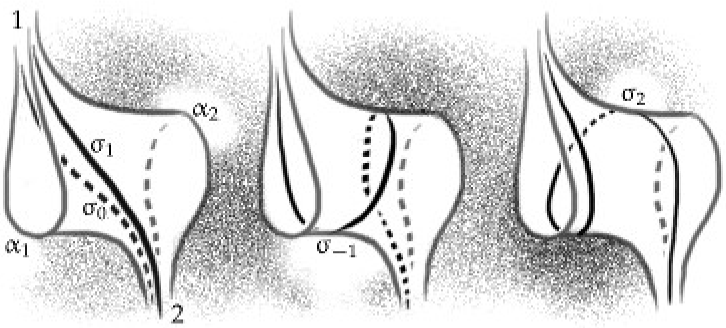


FIGURE 4.5: An example of $\triangle_0 := (\alpha_1, \alpha_2, \sigma_0, \sigma_1)$ and ideal geodesics σ_{-1} and σ_2.

$\triangle_1 := (\alpha_1, \alpha_2, \sigma_1, \sigma_2)$ is a new triangulation of S. We produce a sequence of ideal triangulations $\{\triangle_i\}_{i\in\mathbb{Z}_{\geqslant 0}}$ and ideal geodesics $\{\sigma_i\}$ by iteratively taking σ_{i+2} to be the opposite diagonal to σ_i

with respect to $\triangle_i$ and setting

$$\triangle_{i+1} := (\alpha_1, \alpha_2, \sigma_{i+1}, \sigma_{i+2}) \text{ for } i \in \mathbb{Z}_{\geqslant 0}.$$

We similarly define a sequence of ideal triangulations $\triangle$ and ideal geoesics $\{\sigma_i\}$ in the negative direction by taking σ_{i-1} to be the opposite diagonal to σ_{i+1} with respect to $\triangle_i$ and setting

$$\triangle_{i-1} := (\alpha_1, \alpha_2, \sigma_{i-1}, \sigma_i) \text{ for } i \in \mathbb{Z}_{\leqslant 0}.$$

This sequence $\{\triangle_i\}_{i\in\mathbb{Z}}$ of ideal triangulations produces a sequence of ideal triangles $\{\sigma_i, \sigma_{i+1}\} \in \mathcal{S}_2$ bounded by the geodesics σ_i, σ_{1+1} and α_2. And every element of $\mathcal{S}_2$ arises in this way. The statement of our McShane identity for $(1,1)$-crowned annuli is therefore:

$$1 = 2e^{\frac{-1}{2}(\ell_\gamma + \ell_{\alpha_1})} + \sum_{i\in\mathbb{Z}} e^{\frac{1}{2}(\ell_{\alpha_2} - \ell_{\sigma_1} - \ell_{\sigma_2})}. \tag{4.3}$$

Proposition 4.7. *Let λ_σ denote the λ-length of an ideal geodesic on the length 1 horocycle-decorated hyperbolic $(1,1)$-crowned annulus S, then:*

$$\tanh(\tfrac{\ell_\gamma}{2}) = \sum_{i\in\mathbb{Z}} \frac{\lambda_{\alpha_1}^2}{\lambda_{\alpha_1}^2 + \lambda_{\sigma_i}^2 + \lambda_{\sigma_{i+1}}^2}.$$

Proof. To see this, first observe that by equation (3.7),

$$\lambda_{\alpha_1} = \cosh(\tfrac{\ell_\gamma}{2}) = \lambda_{\alpha_2}. \tag{4.4}$$

Now, the length 1 horocyclic segment at tine 1 is partitioned by $\triangle_i$ into three subsegments. Summing up their lengths and replacing λ_{α_2} with λ_{α_1} using (4.4), we obtain:

$$1 = \frac{\lambda_{\alpha_1}}{\lambda_{\sigma_i}\lambda_{\sigma_{i+1}}} + \frac{\lambda_{\sigma_i}}{\lambda_{\alpha_1}\lambda_{\sigma_{i+1}}} + \frac{\lambda_{\sigma_{i+1}}}{\lambda_{\alpha_1}\lambda_{\sigma_i}}, \text{ or equivalently, } \lambda_{\alpha_1}\lambda_{\sigma_i}\lambda_{\sigma_{i+1}} = \lambda_{\alpha_1}^2 + \lambda_{\sigma_i}^2 + \lambda_{\sigma_{i+1}}^2.$$

Thus, we see that the summands of the McShane identity in (4.3) are:

$$e^{\frac{1}{2}(\ell_{\alpha_2} - \ell_{\sigma_1} - \ell_{\sigma_2})} = \frac{\lambda_{\alpha_1}}{\lambda_{\sigma_i}\lambda_{\sigma_{i+1}}} = \frac{\lambda_{\alpha_1}^2}{\lambda_{\alpha_1}^2 + \lambda_{\sigma_i}^2 + \lambda_{\sigma_{i+1}}^2}$$

To obtain the $\tanh(\frac{\ell_\gamma}{2})$ term, we subtract $2e^{\frac{-1}{2}(\ell_\gamma + \ell_{\alpha_1})}$ from 1:

$$1 - 2e^{\frac{-1}{2}(\ell_\gamma + \ell_{\alpha_1})} = 1 - \frac{1}{e^{\frac{\ell_\gamma}{2}}\cosh(\frac{\ell_\gamma}{2})} = \frac{e^{\ell_\gamma} - 1}{e^{\ell_\gamma} + 1} = \tanh(\tfrac{\ell_\gamma}{2}).$$

□

Proposition 4.7 is closely related to Theorem 1 of [Nor08], which states that on any hyperbolic 1-cusped Klein bottle K:

$$\tanh(\tfrac{\ell_\gamma}{2}) = \sum_{i\in\mathbb{Z}} \frac{1}{1 + \sinh^2(\frac{\ell_{\gamma_i}}{2}) + \sinh^2(\frac{\ell_{\gamma_{i+1}}}{2})},$$

where γ is the unique 2-sided simple closed geodesic on K and $\{\gamma_i\}_{i\in\mathbb{Z}}$ is the collection of all 1-sided simple closed geodesics on K ordered in such a way that for any $i \in \mathbb{Z}$, the geodesics γ_i and γ_{i+1} are disjoint.

Given a 1-cusped Klein bottle K, there is a unique ideal geodesic α on K disjoint from γ. Cutting K along α results in a hyperbolic $(1,1)$-crowned annulus S with unlabelled tines. Conversely, there's a unique way to glue a $(1,1)$-crowned annulus S to form a 1-cusped Klein bottle K. Thus, the moduli space $\mathcal{M}(S)$ of $(1,1)$-crowned annuli is a double cover of the moduli space $\mathcal{M}(K)$. This map $\mathcal{M}(S) \to \mathcal{M}(K)$ is a 2-fold cover, with the quotient group given by the $\mathbb{Z}/2\mathbb{Z}$-action exchanging the tine labels. We intentionally conflate geodesics on a $(1,1)$-crowned annulus S with their corresponding geodesics on K.

We decorate S with length 1 horocyclic segments at both of its tines, and correspondingly decorate K with length $2 = 1+1$ horocycles at its cusp. Recall (equation (4.4)) that the λ-lengths $\lambda_{\alpha_1} = \lambda_{\alpha_2}$ are equal to $\cosh(\frac{\ell_\gamma}{2})$. Further observe that both the ordering of the $\{\sigma_i\}$ and the ordering of the $\{\gamma_i\}$ are unique up to an affine isomorphism of $\mathbb{Z}$. In particular, we may relabel the indices of $\{\sigma_i\}_{i\in\mathbb{Z}}$ with an affine isomorphism so that any 1-sided simple geodesic γ_i on K is assigned the unique ideal geodesic σ_i on K disjoint from both α and γ_i.

Lemma 4.8. *Given the above correspondence between $\{\gamma_i\}$ and $\{\sigma_i\}$,*

$$\sinh\left(\frac{\ell_{\gamma_i}}{2}\right) = \frac{\lambda_{\sigma_i}}{\lambda_{\alpha_1}}.$$

Proof. Fix a pair of disjoint 1-sided simple closed geodesics γ_i and γ_{i+1} on K. Cutting K along γ_i and γ_{i+1} results in a pair of pants, hence we may regard a 1-cusped Klein bottle as a pair of pants with its two non-cuspidal boundaries glued to themselves form cross-caps (Figure 4.6). The natural reflection isometry for pairs of pants extends therefore to an involution on K, the

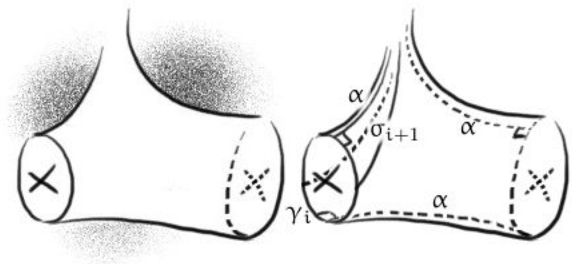


FIGURE 4.6: A 1-cusped Klein bottle and some geodesics, the crosses denote cross-caps.

fixed points of which precisely correspond to α — the unique ideal geodesic on K disjoint from γ. Since this involution set-wise preserves the geodesic γ_i, it must also set-wise preserve the ideal geodesic σ_i. As can be seen from Figure 4.6, cutting K along α, γ_i and σ_{i+1} results in two isomorphic pentagons and two isomorphic triangles. These are right-angle triangles with one ideal vertex; their shortest side is of length $\frac{1}{2}\ell_{\gamma_{i+1}}$ and their respective hypotenuses are precisely half of σ_{i+1}. Using the same computations as we used to calculate (3.10) from

Figure 3.2, we see that:

$$H\lambda_{\sigma_{i+1}} = \sinh(\tfrac{\ell_{\gamma_i}}{2}),$$

where H is the length of the subsegment of the length 2 horocycle on K that is contained on one of these right-angled triangles. Since $(\sigma_i, \sigma_{i+1}, \alpha)$ gives an ideal triangulation of K, we can use Proposition 1.3 to assert that $H = \frac{\lambda_{\sigma_i}}{\lambda_\alpha \lambda_{\sigma_{i+1}}}$. Substituting this into the above formula and noting that $\lambda_\alpha = \lambda_{\alpha_1}$ yields the desired result. □

Note 4.4. An immediate consequence of the above result is that the summands in Norbury's McShane identity [Nor08, Theorem 1] are equal to those in ours:

$$\frac{1}{1+\sinh^2(\frac{\ell_{\gamma_i}}{2})+\sinh^2(\frac{\ell_{\gamma_{i+1}}}{2})} = \frac{\lambda^2_{\alpha_1}}{\lambda^2_{\alpha_1}+\lambda^2_{\sigma_i}+\lambda^2_{\sigma_{i+1}}}.$$

Note 4.5. This equivalence of McShane identities of differing hyperbolic surfaces is not a general phenomenon. Although, there are generic theoretical statements that one may use to relate our McShane identities for a crowned surface S and a hyperbolic supersurface $S' \supset S$. For example, if S' is obtained from S by capping off some boundary arc α_i, where $j \neq 1$, with a 1-cusped monogon, then the cusp/tine 1 McShane identity on S' may be though of as a refinement of the cusp/tine 1 McShane identity on S obtained by partitioning each summand over the $\mathcal{S}_i$ into infinitely many terms. A similar statement may be made when capping off α_1 with a 1-cusped monogon, although one needs to first subtract $\frac{2}{2+\lambda_{\alpha_1}}$ from the McShane identity on S' and renormalise the sum to 1 before we can say that the resulting summands are a refinement of the McShane identity on S.

4.2.2 Weil-Petersson Volume Integration

Certain moduli spaces of crowned surfaces seem to naturally arise as domains of integration in Mirzakhani's volume recursion calculation. We again use the computation of $V_{1,2}$ in Section 3.3 as an example to illustrate this phenomenon.

First observe that the collection

$$\mathcal{A} := \{ \text{ ideal geodesics with both ends up cusp 1 } \}$$

is in natural bijection with the collection

$$\mathcal{P} := \{ \text{ pairs of pants in } R_{1,0,2} \text{ which contain cusp 1} \}.$$

This bijection is give by assigning $\sigma \in \mathcal{A}$ to the unique pair of pants $P \in \mathcal{P}$ containing σ. Moreover, the partitioning of $\mathcal{P}$ into

$$\mathcal{P}_1 := \{ \text{ pairs of pants bounded by cusps 1 and 2 } \} \text{ and}$$
$$\mathcal{P}_2 := \{ \text{ pairs of pants bounded by two non-peripheral simple closed curves } \}$$

corresponds to the partition of $\mathcal{A}$ into:

- $\mathcal{A}_1$: ideal geodesics which cut $R_{1,0,2}$ into two surfaces of types $R^{(1)}_{1,0,0} \cup R^{(1)}_{0,0,1}$;
- $\mathcal{A}_2$: ideal geodesics which cut $R_{1,0,2}$ into a surface of type $R^{(1,1)}_{0,0,1}$.

Thus, the McShane identity for a hyperbolic surface S homeomorphic to $R_{1,2}$ via a label-preserving map $h : R_{1,2} \to S$ may be expressed in the form:

$$1 = \sum_{\sigma_1 \in \mathcal{A}_1} f_1([S - h_\# \sigma_1]) + \sum_{\sigma_2 \in \mathcal{A}_2} f_2([S - h_\# \sigma_2])$$

where $[S - h_\# \sigma_1]$ is a point in $\mathcal{M}(R^{(1)}_{1,0,0}) \times \mathcal{M}(R^{(1)}_{0,1,0})$ obtained by cutting S along the geodesic representative $h_\# \sigma_1$ of the homotopy class $h_*[\sigma_1]$. Likewise, $[S - h_\# \sigma_2]$ is a point in $\mathcal{M}(R^{(1,1)}_{0,1,0})$. Next observe that the moduli spaces $\mathcal{M}(R_{1,2,0}, [\gamma])$ and $\mathcal{M}(R_{1,2,0}, ([\gamma_1], [\gamma_2]))$ satisfy

$$\mathcal{M}(R_{1,2,0}, [\gamma]) \cong \mathcal{M}(R^{(1)}_{1,0,0}) \times \mathcal{M}(R^{(1)}_{0,1,0}) \cong \mathcal{M}(R^{(1)}_{1,0,0}) \text{ and}$$
$$\mathcal{M}(R_{1,2,0}, ([\gamma_1], [\gamma_2])) \cong \mathcal{M}(R^{(1,1)}_{0,1,0}).$$

as symplectic manifolds, and their respective Weil-Petersson volume forms Ω_{WP} agree. Thus,

$$V_{1,2} = \int_{\mathcal{M}(R^{(1)}_{1,0,0})} f_1 \, \Omega_{WP}(R^{(1)}_{1,0,0}) + \int_{\mathcal{M}(R^{(1,1)}_{0,1,0})} f_2 \, \Omega_{WP}(R^{(1,1)}_{0,1,0}).$$

More generally, for a cusped hyperbolic surface $R_{g,0,n}$ the collection $\mathcal{A}$ of bi-infinite geodesics with both ends up cusp 1 can be partitioned into $\mathcal{A}_1, \ldots, \mathcal{A}_N$, where each set $\mathcal{A}_i$ consists of ideal geodesics so that for any two $\sigma, \sigma' \in \mathcal{A}_i$, the (potentially disconnected) surfaces $R_{g,0,n} - \sigma$ and $R_{g,0,n} - \sigma'$ are (label-preservingly) homeomorphic. Fix a collection $\sigma_1, \ldots, \sigma_N$ of respective elements of $\mathcal{A}_1, \ldots, \mathcal{A}_N$.

Since $\mathcal{A}$ is in natural bijection with the collection $\mathcal{P}$ of pairs of pants which contain cusp 1, the McShane identity for a genus g hyperbolic surface S with n cusps homeomorphic to $R_{g,0,n}$ via $h : R_{g,0,n} \to S$ may be expressed in the form

$$1 = \sum_{i=1}^{N} \sum_{\sigma \in \mathcal{A}_i} f_i([S - h_\# \sigma]),$$

where $[S - \sigma]$ is a point in $\mathcal{M}(R_{g,0,n} - \sigma_i)$. Then,

Lemma 4.9. *The Weil-Petersson volume $V_{g,n}$ of $\mathcal{M}(R_{g,0,n})$ is given by:*

$$V_{g,n} = \sum_{i=1}^{N} \int_{\mathcal{M}(R_{g,0,n} - \sigma_i)} f_i \, \Omega_{WP}(R_{g,0,n} - \sigma_i).$$

Note 4.6. Lemma 4.9 is a paraphrasing of line 9 of the proof of [Mir07a, Theorem 7.1] in terms of integrals over moduli spaces $\mathcal{M}(R_{g,0,n} - \sigma_i)$ of crowned surfaces. Note that if an ideal geodesic σ_i separates $R_{g,0,n}$ into R_a and R_b, we identify the moduli space $\mathcal{M}(R_{g,0,n} - \sigma_i)$ with the Cartesian product $\mathcal{M}(R_a) \times \mathcal{M}(R_b)$. Its Weil-Petersson volume form $\Omega_{WP}(R_{g,0,n})$ is the wedge product

$$\Omega_{WP}(R_a) \wedge \Omega_{WP}(R_b) = \Omega_{WP}(R_b) \wedge \Omega_{WP}(R_a).$$

4.3 Closed Surfaces with One Marked Point

A marked point p on a closed hyperbolic surface S may be regarded as a cone-point with cone-angle 2π. Since it is impossible to have a hyperbolic pair of pants with one boundary that's a cone-point with cone-angle 2π, any McShane-type identity for hyperbolic surfaces with one marked point must necessarily consist of sums over surfaces which aren't pairs of pants. We derived one such identity in [Hua12], and now briefly outline the main points of this work.

We first show that almost every geodesic launched from a marked point on a hyperbolic surface self-intersects. Since a marked point on a closed hyperbolic surface S is the same thing as a 2π cone-point, we first extend the Birman-Series theorem for hyperbolic surfaces with small cone-angles [TWZ06] to hyperbolic surfaces with arbitrarily large cone-angles. The desired result on the sparsity of simple geodesics follows as a corollary of this general theorem:

Theorem 4.10. *Given any complete finite-volume hyperbolic surface S with a finite collection C of cone points, fix an integer k. Then the points constituting all complete hyperbolic geodesics possibly broken at C with at most k intersections is nowhere dense on S and has Hausdorff dimension 1.*

The proof of the Birman-Series theorem may be broken into three steps:

1. Take a geodesic polygonal fundamental domain R with the restricted covering map $\pi : R \to S$. Show that the number of isotopy classes of n-segmented geodesic arcs on S with endpoints on $\pi(\partial R)$ grows polynomially in n.
2. Show that, with respect to n, an exponentially decreasing width neighborhood of any representative of such an isotopy class will cover all other representatives of the same isotopy class.
3. By increasing n, we prove that the area covered by such geodesic arcs is bounded by a polynomial divided by an exponential and must tend to 0, and use this to obtain the desired result.

Readers wishing to see the details of this proof are encouraged to consult [Hua12]. The above extension of the Birman-Series theorem differs from [BS85] and [TWZ06] in that we allow for large cone-angles and broken geodesics.

Corollary 4.11. *Given a complete finite-volumed hyperbolic surface S and any countable collection of points $C \subset S$, the set of points which lie on geodesics possibly broken at points in C has zero Lebesgue measure.*

Proof. Broken geodesics which meet infinitely many points in C may be decomposed into finite arcs which join points in C. Since C is countable, this collection of finite arcs is also countable and hence occupies no area on S. Now, order C as $\{p_i\}_{i\in\mathbb{N}}$ (or a finite sequence if C is finite). Broken geodesics on S which meet points in C only finitely many times eventually arise as a subset of the set of points

$$\{x \in S \mid x \text{ lies on a simple geodesic possibly broken at the first } k \text{ points of } C\}.$$

By Theorem 4.10, these sets (and hence their union) are measure 0. □

The above result shows that the area within a radius ϵ ball around a marked point p covered by the initial segment of a simple (broken) geodesic launched from p is 0. Since the fraction of the total area occupied by these simple geodesics is the same as the fraction of all the directions around p occupied by these simple geodesics, the above result tells us that almost every direction from p shoots out a geodesic that's self-intersecting.

Using the same fattening construction as in the proof of Theorem 4.5, the lasso of (almost) any geodesic launched from p may be fattened to induces a unique *immersed* pair of half-pants on S. These lasso-induced immersed pairs of half-pants can take several forms: they may be embedded (leftmost in Figure 4.7), topologically a 1-holed torus (second from the left in Figure 4.7) or topologically a pair of pants (third from the left in Figure 4.7).

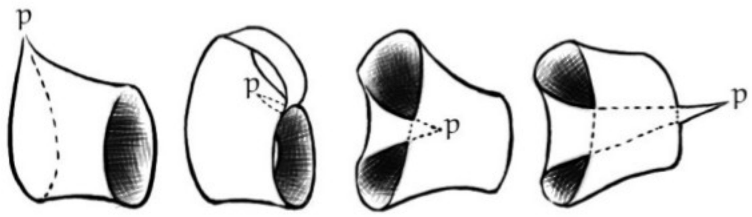


FIGURE 4.7: The left three are examples of lasso-induced immersed pairs of half-pants, the rightmost is not.

Recall that each pair of half-pants P contains a small *gap region* (Definition 4.2) of directions at p, characterised as the maximal set of directions where the loop of any geodesic shot out within this gap region lies within P. We call the size of this gap region (measured as an angle $\leqslant 2\pi$) the *gap-angle*. The gap-angle for an embedded pair of half-pants (based as p) is:

$$2\arcsin\left(\frac{\cosh(\frac{\ell_\gamma}{2})}{\cosh(\frac{\ell_{\gamma_p}}{2})}\right) - 2\arcsin\left(\frac{\sinh(\frac{\ell_\gamma}{2})}{\sinh(\frac{\ell_{\gamma_p}}{2})}\right).$$

For strictly immersed pairs of half-pants, the gap-angle needs to be smaller to avoid overcounting: geodesics launched within the corresponding region might self-intersect "prematurely" and induce a different pair of half-pants. The following figure gives an example of this phenomenon:

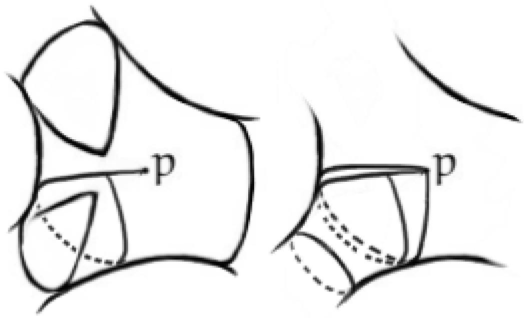


FIGURE 4.8: An example of how a lasso might lie on two immersed pairs of half-pants.

Theorem 4.12. *Given a closed hyperbolic surface* S *with marked point* p*, let* $\mathcal{HP}$ *denote the collection of half-pants lasso-induced at* p*. We define the real function* Gap : $\mathcal{HP} \to [0, \pi]$ *to output the gap-angle of the directions from* p *that shoot out geodesics whose lassos lie in* P*. Then,*

$$\sum_{P \in \mathcal{HP}} \mathrm{Gap}(P) = 2\pi, \tag{4.5}$$

where the Gap *function is*

$$2\arcsin\left(\frac{\cosh(\frac{\ell_\gamma}{2})}{\cosh(\frac{\ell_{\gamma_P}}{2})}\right) - 2\arcsin\left(\frac{\sinh(\frac{\ell_\gamma}{2})}{\sinh(\frac{\ell_{\gamma_P}}{2})}\right)$$

for embedded *pairs of half-pants, but not for* strictly immersed *pairs of half-pants.*

We close this subsection by stating the Gap function in terms of explicit length parameters on the input pairs of (strictly) immersed lasso-induced half-pants P.

When P is topologically a thrice-holed torus, we need two pieces of geometric information from P to define its gap-angle. First we must know the position of $p \in P$, which we specify using two parameters τ and δ: we know that P is the isometric immersion of a unique pair of half-pants $\tilde{P}$. There are two preimages for p in $\tilde{P}$ and there is a unique way to reach the preimage of p on the interior of $\tilde{P}$ by launching orthogonally from the cuff of $\tilde{P}$ as per the black dotted line in the left figure in Figure 4.9. We set $\tau \in [0, \ell_\gamma)$ to parametrise the position of the launching point on the cuff, so that the point on the cuff which orthogonally projects to p is set to 0; the parameter δ then denotes the distance between the interior preimage of p and the cuff of $\tilde{P}$.

The second piece of information we require counts (with sign) how many times the tip of the zipper of P wraps around itself. Specifically, consider the unique shortest geodesic β between the boundary/zipper preimage of p in $\tilde{P}$ and the cuff of $\tilde{P}$ (as shown in red). We define n to be the number of times $\iota(\beta)$ intersects itself, signed to be positive if β shoots out from p in the same direction that τ is increasing, and negative in the direction that τ is decreasing. We refer to Figure 4.9, the parameter n is -1 in this case because it goes against the orientation on the cuff in which τ is increasing. Note that specifying these parameters does not specify the whole geometry of P.

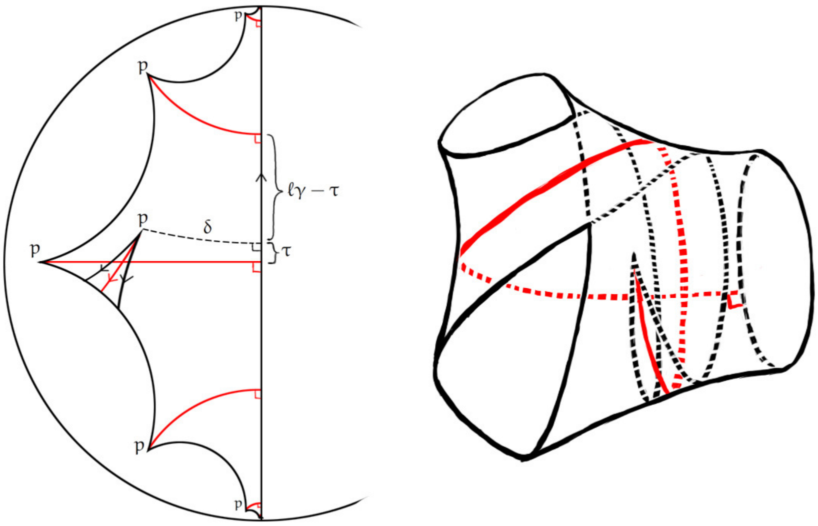


FIGURE 4.9: An example of a $n = -1$ immersed pair of pants.

Given this setup, if P is topologically a thrice-holed sphere and $n = 0$, then the gap-angle of P is:

$$\begin{aligned}
\operatorname{Gap}(P) &= \operatorname{Gap}(\ell_\gamma, \ell_{\gamma_p}, \tau, \delta, n = 0) \\
&= \max\left\{\Theta(\delta, \tau, \operatorname{arccosh}\left(\frac{\sinh(\frac{\ell_\gamma}{2})}{\sinh(\frac{\ell_{\gamma_p}}{2})}\right)) - \arcsin\left(\frac{\sinh(\frac{\ell_\gamma}{2})}{\sinh(\frac{\ell_{\gamma_p}}{2})}\right), 0\right\} \\
&+ \max\left\{\Theta(\delta, \ell_\gamma - \tau, \operatorname{arccosh}\left(\frac{\sinh(\frac{\ell_\gamma}{2})}{\sinh(\frac{\ell_{\gamma_p}}{2})}\right)) - \arcsin\left(\frac{\sinh(\frac{\ell_\gamma}{2})}{\sinh(\frac{\ell_{\gamma_p}}{2})}\right), 0\right\},
\end{aligned} \tag{4.6}$$

where $\Theta(x, y, z)$ is defined by:

$$\Theta(x, y, z) = \frac{1}{2}\arccos\left(\frac{2(\cosh(x)\cosh(y)\sinh(z) - \sinh(x)\cosh(z))^2}{(\cosh(x)\cosh(y)\cosh(z) - \sinh(x)\sinh(z))^2 - 1)} - 1\right).$$

And if $n \neq 0$, then the gap-angle of P is:

$$\begin{aligned}
\operatorname{Gap}(P) = \operatorname{Gap}(\ell_\gamma, \ell_{\gamma_p}, \tau, \delta, n) &= \Theta(\delta, |n\ell_\gamma - \tau|, \operatorname{arccosh}\left(\frac{\sinh(\frac{\ell_\gamma}{2})}{\sinh(\frac{\ell_{\gamma_p}}{2})}\right)) \\
- \max\left\{\arcsin\left(\frac{\sinh(\frac{\ell_\gamma}{2})}{\sinh(\frac{\ell_{\gamma_p}}{2})}\right), \Theta(\delta, |n\ell_\gamma - \tau| - \ell_\gamma, \operatorname{arccosh}\left(\frac{\sinh(\frac{\ell_\gamma}{2})}{\sinh(\frac{\ell_{\gamma_p}}{2})}\right))\right\}&.
\end{aligned} \tag{4.7}$$

Now for the case when P is topologically a one-holed torus, the parameters τ and δ are similarly defined. The gap-angle is:

$$\begin{aligned}
&\mathrm{Gap}(P) = \mathrm{Gap}(\ell_\gamma, \ell_{\gamma_P}, \tau, \delta) \\
&= 2\arcsin\left(\frac{\cosh(\frac{\ell_\gamma}{2})}{\cosh(\frac{\ell_{\gamma_P}}{2})}\right) - \Theta(\delta, \ell_\gamma \left\lceil \frac{\Psi - \tau}{\ell_\gamma} \right\rceil + \tau, \operatorname{arccosh}\left(\frac{\sinh(\frac{\ell_\gamma}{2})}{\sinh(\frac{\ell_{\gamma_P}}{2})}\right)) \\
&- \Theta(\delta, \ell_\gamma \left\lceil \frac{\Psi - (\ell_\gamma - \tau)}{\ell_\gamma} \right\rceil + \ell_\gamma - \tau, \operatorname{arccosh}\left(\frac{\sinh(\frac{\ell_\gamma}{2})}{\sinh(\frac{\ell_{\gamma_P}}{2})}\right)),
\end{aligned} \tag{4.8}$$

where Ψ is given by:

$$\Psi = \frac{1}{2}\log\left(\frac{\cosh^2(\delta)}{\sinh^2(\frac{\ell_\gamma}{2})} - \frac{\cosh^2(\delta)}{\sinh^2(\frac{\ell_{\gamma_P}}{2})}\right).$$

Note 4.7. The index set $\mathcal{HP}$ used in the above result from [Hua12] does not denote the collection of all pairs of immersed half-pants on S (based at p), but instead refers only to the collection of immersed half-pants induced by the lasso of some geodesic launched from p. This is not a mapping class group invariant set and hence does not adhere to the form of a McShane identity as per our description at the start of this chapter. This minor issue is easily remedied by adding on additional pairs of immersed half-pants P' into this summation, and setting their gap-angles $\mathrm{Gap}(P')$ to be 0.

4.4 Markoff Triples and Markoff Quads

In [Bow98], Bowditch used a trace-based proof to derive a generalisation of the McShane's original identity [McS91]:

Theorem 4.13 (Bowditch)**.** *A quasi-Fuchsian representation of the fundamental group* $\pi_1(S_{1,1})$ *of a 1-cusped torus* $S_{1,1}$ *gives a quotient hyperbolic 3-manifold* $X = \mathbb{H}^3/\pi_1(S_{1,1})$*. Then,*

$$\frac{1}{2} = \sum_{\gamma \in \mathcal{C}_{1,1}} \frac{1}{1 + e^{\ell_\gamma}},$$

where the sum is taken over the collection $\mathcal{C}_{1,1}$ *of simple geodesics in* X *and* ℓ_γ *is the complexified geodesic length of* γ*.*

We briefly outline a few of the key ideas behind Bowditch's proof.

4.4.1 Markoff Triples

Consider the abstract simplicial complex $\mathrm{Cur}_{1,1}$ for the universal set $\mathcal{C}_{1,1}$ given by assigning to each

- simple closed geodesic $\alpha \in \mathcal{C}_{1,1}$, a 0-cell $\{\alpha\}$;

- pair of once-intersecting simple closed geodesics α, β, a 1-cell $\{\alpha, \beta\}$;
- triple of pairwise once-intersecting simple closed geodesics α, β, γ, a 2-cell $\{\alpha, \beta, \gamma\}$.

The geometric realisation of $\mathrm{Cur}_{1,1}$, minus its 0-skeleton may be identified with the *Farey triangulation* [Bow98] — an ideal triangulation of the hyperbolic plane $\mathbb{H}$. Regarding $\mathbb{H}$ as the upperhalf-plane in $\mathbb{C} \cup \{\infty\}$, the 0-skeleton of $\mathrm{Cur}_{1,1}$ then corresponds to the extended rational numbers $\mathbb{Q} \cup \{\infty\} \subset \mathbb{R} \cup \{\infty\} = \partial\mathbb{H}$. Any 1-cell $\{\alpha_1, \beta_1\}$ borders two 2-cells $\{\alpha_1, \beta_1, \gamma_1\}$ and $\{\alpha_1, \beta_1, \gamma_2\}$, and we may think of the 1-cells of $\mathrm{Cur}_{1,1}$ as *flips* going from $\{\alpha_1, \beta_1, \gamma_1\}$ to $\{\alpha_1, \beta_1, \gamma_2\}$, or vice versa. The connectedness of $\mathrm{Cur}_{1,1}$ means that a fixed initial 2-cell $\{\alpha_1, \beta_1, \gamma_1\}$ may be flipped to any other 2-cell.

The *simple length spectrum* on a hyperbolic surface is the multiset consisting of the lengths of all of its simple closed geodesics. We've already seen that flipping allows us to generate all the 2-cells of $\mathrm{Cur}_{1,1}$ from a fixed initial 2-cell $\{\alpha_1, \beta_1, \gamma_1\}$. Our previous topological statement about generating unordered triples of pairwise once-intersecting geodesics may be promoted to a geometric statement about generating ordered triples of (elementary functions of) geodesic lengths in the simple length spectrum

$$(a_1, b_1, c_1) := \left(2\cosh(\tfrac{\ell_{\alpha_1}}{2}), 2\cosh(\tfrac{\ell_{\beta_1}}{2}), 2\cosh(\tfrac{\ell_{\gamma_1}}{2})\right).$$

Specifically, let $\rho : \pi_1(S_{1,1}) \to \mathrm{SL}(2, \mathbb{R})$ be a lift of the monodromy representation $\pm\rho : \pi_1(S_{1,1}) \to \mathrm{PSL}(2, \mathbb{R})$ given by the hyperbolic structure of $S_{1,1}$. We choose the signs of the lift ρ so that

$$a_1 := \mathrm{Tr}\rho(\alpha_1) = 2\cosh(\tfrac{\ell_{\alpha_1}}{2}) \text{ and } b_1 := \mathrm{Tr}\rho(\beta_1) = 2\cosh(\tfrac{\ell_{\beta_1}}{2}),$$

where we've interpreted the simple closed geodesics α and β as conjugacy classes of simple homotopy classes in $\pi_1(S_{1,1})$. There are precisely two simple closed geodesics γ_1, γ_2 that each intersect α_1 once and β_1 once, and the Fricke relation [Gol09] tells us that $c_1 := \mathrm{Tr}\rho(\gamma_1)$ and $c_2 := \mathrm{Tr}\rho(\gamma_2)$ satisfy:

$$a_1 b_1 c_i = a_1^2 + b_1^2 + c_i^2, \text{ or equivalently: } 1 = \frac{a_1}{b_1 c_i} + \frac{b_1}{a_1 c_i} + \frac{c_i}{a_1 b_1}. \tag{4.9}$$

Bowditch calls any (nowhere 0) complex solution (a_1, b_1, c_i), a *Markoff triple*.[3]

Since c_1, c_2 are the roots of the polynomial

$$p(x) = x^2 - a_1 b_1 x + a_1^2 + b_1^2 = (x - c_1)(x - c_2),$$

we obtain the following relations by comparing the coefficients of the powers of x:

$$c_1 + c_2 = a_1 b_1 \text{ and } c_1 c_2 = a_1^2 + b_1^2. \tag{4.10}$$

[3]Integer solutions of this Diophantine equation are *Markoff numbers* multiplied by 3.

The left identity of (4.10), and the symmetry of the Markoff equation (4.9) tells us that given an initial Markoff triple (a_1, b_1, c_1), we may generate three new solutions:

$$\begin{aligned}(a_2, b_1, c_1) &:= (b_1c_1 - a_1, b_1, c_1),\\ (a_1, b_2, c_1) &:= (a_1, a_1c_1 - b_1, c_1),\\ (a_1, b_1, c_2) &:= (a_1, b_1, a_1b_1 - c_1).\end{aligned}$$

In number theoretic circles, this type of technique is called *root flipping* or *Vieta jumping*. We may store these flipped triples by assigning them to their respective 2-cells in $\mathrm{Cur}_{1,1}$. Hence, the connectedness of $\mathrm{Cur}_{1,1}$ means that this algorithm allows us to generate the entire simple geodesic spectrum. Note also that the data of these triples may be equivalently stored by assigning to each 0-cell $\{\alpha\}$ the trace $\mathrm{Tr}\rho(\alpha)$. This is an example of a *Markoff map*: a complex function $\phi : \mathrm{Cur}^0_{1,1} \to \mathbb{C}$ on the set of 0-cells of the curve complex, satisfying

$$\phi(\{\alpha\})^2 + \phi(\{\beta\})^2 + \phi(\{\gamma\})^2 = \phi(\{\alpha\})\phi(\{\beta\})\phi(\{\gamma\})$$

for every 2-cell $\{\alpha, \beta, \gamma\}$. The fact that everything is phrased in terms of traces and handled algebraically allowed Bowditch to generalise McShane's identity to the set of all Markoff maps ϕ satisfying the *BQ-conditions*:

1. $|\phi|$ is less than 2 for at most finitely many 0-cells in $\mathrm{Cur}^0_{1,1}$, and
2. the image of ϕ avoids $[-2, 2] \subset \mathbb{C}$.

For Markoff maps induced by taking the trace of a (type-preserving) representation $\rho : \pi_1(S_{1,1}) \to \mathrm{SL}(2, \mathbb{C})$, the BQ-conditions may be interpreted as saying that ρ has finitely many conjugacy classes of simple elements with trace with absolute value less than 2, and that ρ has no elliptic or (non-peripheral) parabolic elements. These two conditions are satisfied by all quasi-Fuchsian representations, and hence the set of Markoff maps arising from quasi-Fuchsian representations is a subclass of the set of all Markoff maps satisfying BQ-conditions. Indeed, Bowditch conjectured that every BQ-satisfying Markoff map arises from quasi-Fuchsian representation — a conjecture that is still open.

Bowditch works with the dual complex $\Omega := \mathrm{Cur}^*_{1,1}$ of $\mathrm{Cur}_{1,1}$, and assigns orientations to the 1-cells $\{\alpha, \beta\}^*$ in $\Omega^1 := (\mathrm{Cur}^*_{1,1})^1$ of the dual curve complex such that $\{\alpha, \beta\}$ is oriented to point from β to α if $|\phi(\{\beta\})| \geqslant |\phi(\{\alpha\})|$. This orientation gives us discrete dynamics on the 1-skeleton of Ω, and local analysis shows that there is a unique dual 0-cell $\{\alpha, \beta, \gamma\}^*$ in Ω^0 such that the three dual 1-cells $\{\alpha, \beta\}^*, \{\alpha, \gamma\}^*, \{\beta, \gamma\}^*$ connected to $\{\alpha, \beta, \gamma\}^*$ are each pointing into $\{\alpha, \beta, \gamma\}^*$. Bowditch uses this fact to derive the following result:

Theorem 4.14 (Bowditch, [Bow98])**.** *The unique maximum of the systole*[4] *function over the moduli space of 1-cusped hyperbolic tori is* $2\operatorname{arccosh}(\frac{3}{2})$.

Note 4.8. This result can be obtained with basic hyperbolic trigonometry, although Bowditch's proof also holds for hyperbolic 3-folds corresponding to type-preserving quasi-Fuchsian representations of $\pi_1(S_{1,1})$.

[4] Recall that a *systole* of a surface is the length of its shortest essential curve.

We can also use these dynamics to study the growth rate of the length of simple closed geodesics on $S_{1,1}$. Solving for c_i in (4.9), we see that

$$c_i = \frac{a_1 b_1}{2}\left(1 \pm \sqrt{1 - \frac{4}{a_1^2} - \frac{4}{b_1^2}}\right).$$

Hence, for a_1, b_1 sufficiently large,

$$\max\{\log|c_1|, \log|c_2|\} \approx \log|a_1| + \log|b_1|.$$

Thus suggests that for a Markoff map, we might be able to asymptotically approximate the growth of the function $\log|\phi|$ by a function $F_{\{\alpha,\beta\}} : \mathrm{Cur}^0_{1,1} \to \mathbb{R}^+$ such that:

- the 2-tuple $\{\alpha, \beta\}$ is a 1-cell in $\mathrm{Cur}_{1,1}$,
- $F_{\{\alpha,\beta\}}(\{\alpha\}) = F_{\{\alpha,\beta\}}(\{\beta\}) = 1$, and
- $F_{\{\alpha,\beta\}}(\{\delta'\}) = F_{\{\alpha,\beta\}}(\{\alpha'\}) + F_{\{\alpha,\beta\}}(\{\beta'\})$, whenever a dual 0-cell $\{\alpha', \beta', \gamma'\}^*$ is closer to $\{\alpha, \beta\}^*$ than a dual 0-cell $\{\alpha', \beta', \delta'\}^*$.

Any function $f : \mathrm{Cur}^0_{1,1} \to \mathbb{R}^+$ satisfying the bound

$$\frac{1}{\kappa} F_{\{\alpha,\beta\}} \leqslant f \leqslant \kappa F_{\{\alpha,\beta\}} \tag{4.11}$$

for cofinitely many 0-cells in $\mathrm{Cur}^0_{1,1}$ is said to obey *Fibonacci growth rates*[5].

Theorem 4.15 (Bowditch, [Bow98])**.** *Given a Markoff map ϕ satisfying the BQ-conditions, the function $\log_+|\phi|$ defined by:*

$$\log_+|\phi(\{\xi\})| := \max\{\log|\phi(\{\xi\})|, 0\} \textit{ for } \xi \in \mathrm{Cur}^0_{1,1}$$

has Fibonacci growth rates.

We now use this result (and its proof in [Bow98]) to prove the following:

Proposition 4.16. *Define the function*

$$\mathcal{S}_\phi(L) := \mathrm{Card}\left\{\{\xi\} \in \mathrm{Cur}^0_{1,1} \mid |\phi(\{\xi\})| \leqslant L\right\},$$

then there exist positive real numbers $C_1, C_2 \in \mathbb{R}^+$ such that

$$C_1 L^2 < \mathcal{S}_\phi(L) < C_2 L^2.$$

Proof. Define the function

$$\mathcal{S}_{F_{\{\alpha,\beta\}}} := \mathrm{Card}\left\{\{\xi\} \in \mathrm{Cur}^0_{1,1} \mid F_{\{\alpha,\beta\}}(\{\xi\}) \leqslant L\right\}.$$

[5]Bowditch shows that the definition of Fibonacci growth rate is independent of the edge used to define it $\{\alpha, \beta\}$.

We first show that there exist constants $C_1, C_2 \in \mathbb{R}^+$ such that

$$C_1 L^2 < \mathcal{S}_\phi(L) < C_2 L^2 \text{ for all } L \text{ sufficiently large}$$

if and only if

$$C_1' L^2 < \mathcal{S}_{F_{\{\alpha,\beta\}}}(L) < C_2' L^2 \text{ for all } L \text{ sufficiently large.}$$

By definition, the statement that $\mathcal{S}_{F_{\{\alpha,\beta\}}}(L) > C_1' L^2$ means that there is a set $U_L \subset \mathrm{Cur}^0_{1,1}$ of size $C_1' L^2$ such that for all $\{\xi\} \in U_L$,

$$F_{\{\alpha,\beta\}}(\{\xi\}) < L$$

and hence by (4.11), $\log_+ |\phi(\{\xi\})| < \kappa L$. Since ϕ satisfies the BQ-conditions, $|\phi| < 1$ for finitely many 0-cells and thus we may assume for large enough L that $\log_+ |\phi| \equiv \log |\phi|$. Thus,

$$|\phi(\{\xi\})| < e^{\kappa L} < 2\cosh(\kappa L).$$

By definition, this means that $\mathcal{S}_\phi(L') > C_1' L^2 = C_1 L'^2$ for $C_1 := \frac{C_1'}{\kappa^2}$ and when $L' := \kappa L$ is large enough.

Similarly, the statement that $\mathcal{S}_{F_{\{\alpha,\beta\}}}(L) < C_2' L^2$ means that there is a set $V_L \subset \mathrm{Cur}^0_{1,1}$ of size $C_2' L^2$ such that for all $\{\xi\} \notin V_L$,

$$F_{\{\alpha,\beta\}}(\{\xi\}) > L$$

and hence by (4.11), $\log_+ |\phi(\{\xi\})| > \frac{L}{\kappa}$. Thus for L large enough,

$$|\phi(\{\xi\})| > e^{\frac{L}{\kappa}} > 2\cosh(\tfrac{L}{2\kappa}).$$

By definition, this means that $\mathcal{S}_\phi(L') < C_2' L^2 < C_2 L'^2$ for $C_2 = 4\kappa^2 C_2'$ and when $L' := \frac{L}{2\kappa}$ is large enough. To obtain the converse implication, we employ essentially the same arguments with the roles of $\mathcal{S}_{F_{\{\alpha,\beta\}}}$ and $\mathcal{S}_\phi$ reversed.

Finally, to complete the proof of this proposition, from page 19 of [Bow98] we know that:

$$\mathcal{S}_{F_{\{\alpha,\beta\}}}(L) = 2\sum_{k=1}^{\lfloor L \rfloor} \varphi(k),$$

where φ denotes the Euler totient function. And hence is equal to $\frac{6}{\pi^2} L^2 + O(L \log L)$ by a standard result in number theory [HW79, page 268] [6]. □

We now say a few words on how the algebraic structure of Markoff maps lead to a McShane identity. Observe that the right identity of (4.10) is equivalent to $\frac{c_1}{a_1 b_1} = \frac{a_1}{b_1 c_2} + \frac{b_1}{a_1 c_2}$, and allows

[6] Slightly better bounds exist for the sum of the totient functions, for example: [Wal63].

us to decompose each of the three summands in the right identity of (4.9) into two smaller summands:

$$\begin{aligned} 1 &= \frac{a_1}{b_1c_1} + \frac{b_1}{a_1c_1} + \frac{c_1}{a_1b_1} \\ &= \frac{b_1}{a_2c_1} + \frac{c_1}{a_2b_1} + \frac{a_1}{b_2c_1} + \frac{c_1}{a_1b_2} + \frac{a_1}{b_1c_2} + \frac{b_1}{a_1c_2}. \end{aligned}$$

Since these resulting summands take the same form as our initial summands, they may each be successively split into finer summands. In the limit, this finite sum converges to the McShane identity for $S_{1,1}$. In particular, up to changing the roles of a, b, c, the limiting summands (equation (4.21)) take the form

$$\lim_{n\to\infty} \frac{b_{i_n}}{a_i c_{i_n}} = 1 - \sqrt{1 - 4a_i^{-2}} = \frac{4}{a_i^2 + \sqrt{a_i^4 - 4a_i^2}} = \frac{2}{1 + e^{\ell_\alpha}},$$

where the indices $i_n \to \infty$ as $n \to \infty$. This is of course only a sketch of the idea behind Bowditch's trace-based proof of the McShane identities — a proof that necessitates some analysis on the 1-cells of the dual complex of $\mathrm{Cur}_{1,1}$ to avoid invoking the Birman-Series theorem.

4.4.1.1 A Geometric Interpretation

We now consider one geometric interpretation for the summands constituting the McShane identity in the Fuchsian case.

Simple closed geodesics on $S_{1,1}$ are in natural bijection with the collection of (simple) ideal geodesics on $S_{1,1}$ by assigning to a simple closed geodesic γ the unique ideal geodesic σ_γ disjoint from γ. Thus, given a triple of closed geodesics $\{\alpha_1, \beta_1, \gamma_1\}$ which pairwise intersect once, we obtain a corresponding triple of ideal geodesics $\{\sigma_{\alpha_1}, \sigma_{\beta_1}, \sigma_{\gamma_1}\}$ — an unordered ideal triangulation of $S_{1,1}$ (Figure 4.10).

Let $\lambda_{\alpha_1}, \lambda_{\beta_1}, \lambda_{\gamma_1}$ be the corresponding λ-lengths of $\sigma_{\alpha_1}, \sigma_{\beta_1}, \sigma_{\gamma_1}$ truncated at the length 2 horocycle on $S_{1,1}$. Then the decomposition of the length 2 horocycle on $S_{1,1}$ into the six horocyclic segments cut by $\{\sigma_{\alpha_1}, \sigma_{\beta_1}, \sigma_{\gamma_1}\}$ tells us that:

$$2 = 2\left(\frac{\lambda_{\alpha_1}}{\lambda_{\beta_1}\lambda_{\gamma_1}} + \frac{\lambda_{\beta_1}}{\lambda_{\alpha_1}\lambda_{\beta_1}} + \frac{\lambda_{\gamma_1}}{\lambda_{\alpha_1}\lambda_{\beta_1}}\right).$$

By Proposition 3.7, the trace a_1 of α_1 is equal to the λ-length λ_{α_1} for σ_{α_1} truncated at the length 2 horocycle on $S_{1,1}$. Thus, the above identity is equivalent to the right side of (4.9). Moreover, the right side of (4.10) gives us the identity

$$\frac{c_1}{a_1b_1} = \frac{a_1}{b_1c_1} + \frac{b_1}{a_1c_2}.$$

Replacing an ideal geodesic in an ideal triangulation $\triangle$ of $S_{1,1}$ with the "opposite" diagonal ideal geodesic cuts one of the horocyclic segments induced by $\triangle$ into two smaller segments

(Note 1.1). Thus, the algebraic splitting of summands forming the heart of Bowditch's proof may be geometrically interpreted as carving the length 2 horocycle η on $S_{1,1}$ into shorter horocyclic segments by cutting η at its intersection points with ideal geodesics.

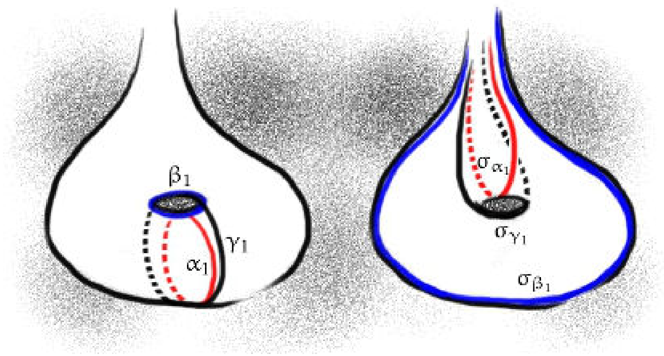


FIGURE 4.10: Corresponding triples of closed geodesics and ideal geodesics.

4.4.2 Markoff Quads

Motivated by Bowditch's work, Norbury and I have found a very similar phenomenon for a 3-cusped projective planes S [HN13], regarded as a lift of a representation

$$\rho : \pi_1(S) \to \mathrm{PSL}_2^{\pm}(\mathbb{R})$$

into $\mathrm{SL}_2^{\pm}(\mathbb{R})$ so as to satisfy certain trace positivity conditions. In analogy with the 1-cusped torus case, given three pairwise once-intersecting simple closed geodesics $\alpha_1, \beta_1, \gamma_1$, there are precisely two simple closed geodesics δ_1, δ_2 which each pairwise once-intersect the first three geodesics. The traces a_1, b_1, c_1, d_i for these four pairwise once-intersecting simple closed geodesics $\alpha_1, \beta_1, \gamma_1, \delta_i$ satisfy:

$$\frac{d_i}{a_1 + b_1 + c_1 + d_i} = \frac{a_1 + b_1 + c_1 + d_i}{a_1 b_1 c_1}. \tag{4.12}$$

We call 4-tuples of complex numbers which satisfy (4.12) *Markoff quads*. Since d_1, d_2 are the roots of the polynomial

$$p(x) = x^2 + (2a_1 + 2b_1 + 2c_1 - a_1 b_1 c_1)x + (a_1 + b_1 + c_1)^2,$$

we obtain the following relations

$$a_1 b_1 c_1 = (a_1 + b_1 + c_1 + d_1) + (a_1 + b_1 + c_1 + d_2) \text{ and} \tag{4.13}$$

$$d_1 d_2 = (a_1 + b_1 + c_1)^2. \tag{4.14}$$

These formulae allows us to define equivalent root flips for Markoff quads given respectively by:

$$\begin{aligned}(a_1, b_1, c_1, d_1) &\mapsto (a_1, b_1, c_1, a_1b_1c_1 - 2a_1 - 2b_1 - 2c_1 - d_1) \text{ and}\\ &\mapsto (a_1, b_1, c_1, \tfrac{1}{d_1}(a_1 + b_1 + c_1)^2).\end{aligned} \tag{4.15}$$

Much like the 1-cusped torus case, given an initial quad we can generate and store infinitely many Markoff quads in a curve complex $\mathrm{Cur}(S)$ [7]. Where $\mathrm{Cur}(S)$ is an abstract simplicial complex for the universal set $\mathcal{C}^1(S)$ of 1-sided simple closed geodesics in S:

- 0-cells $\{\alpha\}$ correspond to 1-sided simple closed geodesics $\alpha \in \mathrm{Cur}(S)$;
- 1-cells $\{\alpha, \beta\}$ correspond to pairs of once-intersecting 1-sided simple closed geodesics $\alpha, \beta \in \mathrm{Cur}(S)$;
- 2-cells $\{\alpha, \beta, \gamma\}$ correspond to triples of pairwise once-intersecting 1-sided simple closed geodesics $\alpha, \beta, \gamma \in \mathrm{Cur}(S)$;
- 3-cells $\{\alpha, \beta, \gamma, \delta\}$ correspond to 4-tuples of pairwise once-intersecting 1-sided simple closed geodesics $\alpha, \beta, \gamma, \delta \in \mathrm{Cur}(S)$.

Specifically, we may store these Markoff quads either as 4-tuples assigned to the 3-cells $\mathrm{Cur}(S)^3$ of $\mathrm{Cur}(S)$ or as numbers assigned to the 0-cells $\mathrm{Cur}(S)^0$ of $\mathrm{Cur}(S)$. We once again refer to this latter option as an example of a *Markoff map*: a function $\phi : \mathrm{Cur}(S)^0 \to \mathbb{C}$ so that

$$(\phi(\{\alpha\}) + \phi(\{\beta\}) + \phi(\{\gamma\}) + \phi(\{\delta\}))^2 = \phi(\{\alpha\})\phi(\{\beta\})\phi(\{\gamma\})\phi(\{\delta\}) \tag{4.16}$$

for every 3-cell $\{\alpha, \beta, \gamma, \delta\}$. We say that these Markoff maps satisfy *BQ-conditions* if:

1. for any $k \in \mathbb{R}_+$, the product $|\phi(\{\alpha\})\phi(\{\beta\})|$ is less than k for at most finitely many 1-cells $\{\alpha, \beta\}$ in $\mathrm{Cur}(S)^1$,
2. and $\phi(\{\alpha\})\phi(\{\beta\})$ avoids $[0, 4]$ for every 1-cell $\{\alpha, \beta\}$ in $\mathrm{Cur}(S)^1$.

Markoff maps arising from taking the trace of a quasi-Fuchsian representation of $\pi_1(S)$ again satisfy these BQ-conditions, and it is tempting to wonder if every Markoff map satisfying BQ-conditions arises from taking the trace of a quasi-Fuchsian representation.

As with Bowditch's work, Norbury and I work in the dual complex — defining discrete dynamics on the 1-skeleton of the dual complex and using root flipping to derive the following result:

Theorem 4.17. *The maximum of the systole function over the moduli space of 3-cusped hyperbolic projective planes is* $2\mathrm{arcsinh}(2)$.

[7] This curve complex appears in [Sch82].

In addition, solving for d_i in equation (4.12) yields:

$$d_i = \frac{a_1 b_1 c_1}{4}\left(1 \pm \sqrt{1 - \frac{4}{a_1 b_1} - \frac{4}{a_1 c_1} - \frac{4}{b_1 c_1}}\right)^2.$$

Hence, for a_1, b_1, c_1 sufficiently large,

$$\max\{\log|d_1|, \log|d_2|\} \approx \log|a_1| + \log|b_1| + \log|c_1|.$$

Based on this observation, we generalise the definition of Fibonacci growth rates to $\mathrm{Cur}(S)^0$ and show that $\log_+|\phi|$ satisfies Fibonacci growth rates. We then use this result to obtain the following growth rate bounds for 1-sided simple closed geodesics on S:

Theorem 4.18. *Define the function*

$$\mathcal{S}_\phi(L) := \mathrm{Card}\left\{\{\xi\} \in \mathrm{Cur}(S)^0 \mid |\phi(\{\xi\}) \leqslant L\right\},$$

then there exist constants $C_1, C_2 \in \mathbb{R}^+$ *such that*

$$C_1 L^2 < \mathcal{S}_\phi(L) < C_2 L^3.$$

Note 4.9. Having used this root-flipping-based algorithm to generate some elements of the 1-sided simple length spectrum for a particular 3-cusped projective plane, Norbury and I've seen that the growth rate of the 1-sided simple length spectrum doesn't seem to asymptotically behave like a polynomial in L. This is in sharp contrast with what occurs with oriented hyperbolic surfaces S', as McShane-Rivin and Mirzakhani [MR95, Riv01, Mir08] have shown that the growth rate is always a polynomial in L of degree equal to the (real) dimension of the moduli space $\mathcal{M}(S', \mathbf{L})$ of S'.

We further use this technology to generalise Norbury's McShane identity for 3-cusped projective planes [Nor08] to Markoff maps satisfying BQ-conditions. In particular, the analogous sum to 1 for 3-cusped hyperbolic projective planes S corresponding to equation (4.9) is:

$$\begin{aligned} 1 &= \tfrac{a_1}{a_1+b_1+c_1+d_i} + \tfrac{b_1}{a_1+b_1+c_1+d_i} + \tfrac{c_1}{a_1+b_1+c_1+d_i} + \tfrac{d_1}{a_1+b_1+c_1+d_i} \\ &= \tfrac{a_1+b_1+c_1+d_i}{b_1 c_1 d_i} + \tfrac{a_1+b_1+c_1+d_i}{a_1 c_1 d_i} + \tfrac{a_1+b_1+c_1+d_i}{a_1 b_1 d_i} + \tfrac{a_1+b_1+c_1+d_i}{a_1 b_1 c_1}. \end{aligned} \tag{4.17}$$

Which, combined with equation (4.13) tells us that:

$$\tfrac{a_1+b_1+c_1+d_1}{a_1 b_1 c_1} = \tfrac{a_1+b_1+c_1+d_2}{b_1 c_1 d_2} + \tfrac{a_1+b_1+c_1+d_2}{a_1 c_1 d_2} + \tfrac{a_1+b_1+c_1+d_2}{a_1 b_1 d_2}.$$

By symmetry, such splitting formulas hold for each of the summands in (4.17) and iteratively splitting each resulting summand results in an even finer partition of 1. Each resulting summand of this finer partition take the same form and may be further split into three summands. . . et cetera. Splitting *ad infinitum*, we obtain the following McShane identity:

Theorem 4.19. *Let* X *be the hyperbolic manifold corresponding to a quasi-Fuchsian representation of* $\pi_1(S)$*, then*

$$1 = \sum_{\epsilon \in \mathcal{C}^2(S)} \frac{2}{1 + e^{\frac{1}{2}\ell_\epsilon}},$$

where the sum is taken over the collection $\mathcal{C}^2(S)$ *of 2-sided simple closed geodesics on* X*.*

Note 4.10. The more immediate presentation for this McShane identity obtained from taking the partition limit is a sum over the 1-cells $\{\alpha, \beta\}$ in $\mathrm{Cur}(S)^1$. We use the fact that given a pair of once-intersecting 1-sided simple closed geodesics α, β, there is a unique 2-sided simple closed geodesic ϵ disjoint from α and β to translate the summands of this limiting partition into the above identity.

Note 4.11. The McShane identity that we obtain in [HN13] is actually (potentially) more general than Theorem 4.19. Norbury and I proved a McShane identity for Markoff maps on $\mathrm{Cur}(S)$ satisfying BQ-conditions. As previously stated, quasi-Fuchsian representations are examples of Markoff maps, but it is not known if all Markoff maps arise from quasi-Fuchsian representations. In either case, something interesting occurs: either we obtain a McShane identity that holds for more general objects than quasi-Fuchsian representations or we have an algebraic characterisation of quasi-Fuchsian representations.

As a final application, we use the root-flipping algorithm to determine the structure of the moduli space $\mathcal{M}(S)$ of 3-cusped hyperbolic projective planes. First, we showed that the collection of positive real Markoff quads is a model for the Teichmüller space of the 3-cusped projective plane S:

$$\mathcal{T}(S) := \left\{ (X, f) \;\middle|\; \begin{array}{l} f : S \to X \text{ is a homeomorphism} \\ \text{between cusped hyperbolic surfaces} \end{array} \right\} / \sim_T$$

where $(X_1, f_1) \sim_T (X_2, f_2)$ if and only if $f_2 \circ f_1^{-1}$ is isotopic equivalent to a hyperbolic isometry. We refer to equivalence classes $[X, f]$ of such surfaces as *marked surfaces*.

Proposition 4.20. *Given a 4-tuple* $\{\alpha, \beta, \gamma, \delta\} \in \mathrm{Cur}(S)^3$*, the map (unique up to ordering* $\alpha, \beta, \gamma, \delta$*)*

$$\begin{aligned} \mathcal{T}(S) &\to \{(a, b, c, d) \in \mathbb{R}_+^4 \mid (a + b + c + d)^2 = abcd\} \\ [X, f] &\mapsto (2\sinh\frac{1}{2}\ell_\alpha, 2\sinh\frac{1}{2}\ell_\beta, 2\sinh\frac{1}{2}\ell_\gamma, 2\sinh\frac{1}{2}\ell_\delta) \end{aligned}$$

is a real-analytic diffeomorphism, where $\ell_\alpha([X, f])$ *denotes the length of the geodesic representative* $f_\# \alpha$ *of* $f_*(\alpha)$ *on* X*.*

Note 4.12. This is a Fricke-Klein-type embedding theorem for the Teichmüller space of 3-cusped projective planes.

Recall that the moduli space

$$\mathcal{M}(S) := \{X \mid X \text{ is a cusped hyperbolic surface label-preserving homeomorphic to } S\}/\sim_M,$$

where $X_1 \sim_M X_2$ if and only if they're isometric surfaces, is the quotient of the Teichmüller space $\mathcal{T}(S)$ by the mapping class group $\mathrm{Mod}(S)$ consisting of isotopy classes $[h]$ of homeomorphisms h of S.

To study the mapping class group, we first show that the four root flips at each of the coordinates of (a_1, b_1, c_1, d_1) as given by (4.15) are realised by homeomorphisms f_1, f_2, f_3, f_4 of S preserving three of the geodesics in $\{\alpha_1, \beta_1, \gamma_1, \delta_1\}$ and taking the fourth to its "flip" — the unique distinct 1-sided simple closed geodesic that pairwise once-intersects the other three. The mapping classes $[f_1], [f_2], [f_3], [f_4]$ freely generate a finite-index normal subgroup F of $\mathrm{Mod}(S)$, and we show that the quotient space $\mathcal{T}(S)/F$ is an open octahedron with four (open) triangles of order 2 orbifold points glued onto a collection of four non-adjacent sides. Using the fact that this is a finite cover of the moduli space:

Theorem 4.21. *The moduli space $\mathcal{M}(S)$ of 3-cusped projective planes is homeomorphic to an open 3-ball with an open hemisphere of order 2 orbifold points glued on, and a line of orbifold points running straight through the center of this 3-ball — joining two antipodal points of the glued on orbifold hemisphere. The orbifold points on this line are of order 2, except for the very center of the 3-ball, which is order 4.*

4.4.2.1 Geometric Interpretation

As with the punctured torus case, the summands in equation (4.17) correspond to the lengths of horocyclic segments constituting (any) one of the three length 1 horocycles on the 3-cusped projective plane S.

Given a 4-tuple of pairwise once-intersecting simple closed geodesics $\{\alpha, \beta, \gamma, \delta\}$ on S, there is a unique 6-tuple of ideal geodesics $\{\sigma_{\alpha\beta}, \sigma_{\alpha\gamma}, \sigma_{\alpha\delta}, \sigma_{\beta\gamma}, \sigma_{\beta\delta}, \sigma_{\gamma\delta}\}$ (Figure 4.11) where $\sigma_{\xi\eta}$ is the unique ideal geodesic which does not intersect the distinct simple closed geodesics $\xi, \eta \in \{\alpha, \beta, \gamma, \delta\}$ and has its ends up different cusps. This 6-tuple of ideal geodesics gives an ideal triangulation of S, and let $\lambda_{\alpha\beta}, \lambda_{\alpha\gamma}, \lambda_{\alpha\delta}, \lambda_{\beta\gamma}, \lambda_{\beta\delta}, \lambda_{\gamma\delta}$ denote the corresponding λ-lengths of this ideal triangulation truncated at the three length 1 horocycles of S.

Proposition 4.22. *The traces*

$$a = 2\sinh(\tfrac{1}{2}\ell_\alpha), b = 2\sinh(\tfrac{1}{2}\ell_\beta), c = 2\sinh(\tfrac{1}{2}\ell_\gamma), d = 2\sinh(\tfrac{1}{2}\ell_\delta)$$

of $\alpha, \beta, \gamma, \delta$ may be expressed in terms of $\lambda_{\alpha\beta}, \lambda_{\alpha\gamma}, \lambda_{\alpha,\delta}, \lambda_{\beta\gamma}, \lambda_{\beta\delta}, \lambda_{\gamma\delta}$ as:

$$a = \frac{\lambda_{\alpha\beta}\lambda_{\alpha\gamma}}{\lambda_{\beta\gamma}} = \frac{\lambda_{\alpha\beta}\lambda_{\alpha\delta}}{\lambda_{\beta\delta}} = \frac{\lambda_{\alpha\gamma}\lambda_{\alpha\delta}}{\lambda_{\gamma\delta}}, \; b = \frac{\lambda_{\alpha\beta}\lambda_{\beta\gamma}}{\lambda_{\alpha\gamma}} = \frac{\lambda_{\alpha\beta}\lambda_{\beta\delta}}{\lambda_{\alpha\delta}} = \frac{\lambda_{\beta\gamma}\lambda_{\beta\delta}}{\lambda_{\gamma\delta}}$$
$$c = \frac{\lambda_{\alpha\gamma}\lambda_{\beta\gamma}}{\lambda_{\alpha\beta}} = \frac{\lambda_{\alpha\gamma}\lambda_{\gamma\delta}}{\lambda_{\alpha\delta}} = \frac{\lambda_{\beta\gamma}\lambda_{\gamma\delta}}{\lambda_{\beta\delta}}, \; d = \frac{\lambda_{\alpha\delta}\lambda_{\beta\delta}}{\lambda_{\alpha\beta}} = \frac{\lambda_{\alpha\delta}\lambda_{\gamma\delta}}{\lambda_{\alpha\gamma}} = \frac{\lambda_{\beta\delta}\lambda_{\gamma\delta}}{\lambda_{\beta\gamma}}.$$

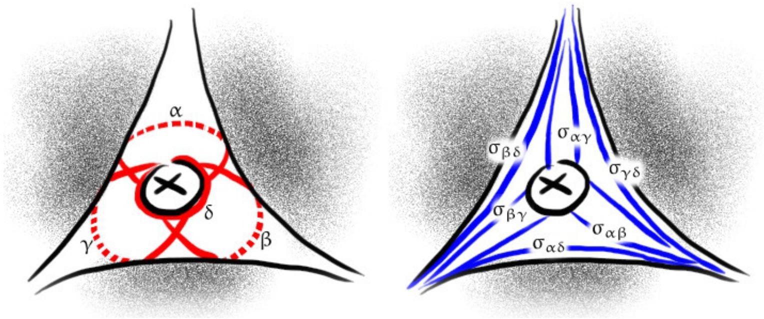


FIGURE 4.11: A 4-tuple of simple closed geodesics $\alpha, \beta, \gamma, \delta$ and a corresponding ideal triangulation $\sigma_{\alpha\beta}, \sigma_{\alpha\gamma}, \sigma_{\alpha\delta}, \sigma_{\beta\gamma}, \sigma_{\beta\delta}, \sigma_{\gamma\delta}$.

Proof. First note that because $\alpha, \beta, \gamma, \delta$ are 1-sided geodesics, their respective traces are $2\sinh(\frac{1}{2}\cdot)$ of their lengths. We now derive the above identity for d, the others follow from symmetry.

Observe that there is an embedded 1-crowned hyperbolic Möbius strip $M \subset S$ that contains δ, as shown in Figure 4.12.

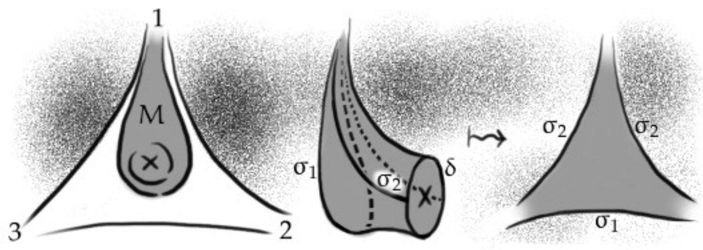


FIGURE 4.12: A 1-cusped Möbius band M embedded in a 3-cusped projective plane.

Denote the λ-length of the boundary arch σ_1 of M by λ_1 and the λ-length of the unique (non-peripheral) ideal geodesic σ_2 in M by λ_2. Cutting along σ_2 decomposes M into one ideal triangle (Figure 4.12), and the horocyclic segment at the tine of M is of length:

$$h = \frac{2}{\lambda_1} + \frac{\lambda_1}{\lambda_2^2}.$$

Cutting M along δ results in a pair of half-pants with its closed geodesic boundary begin of length $2\ell_\delta$. By equation (3.7):

$$2\cosh(\ell_\delta) = h\lambda_1 = 2 + \frac{\lambda_1^2}{\lambda_2^2}, \text{ and hence } d = 2\sinh(\tfrac{\ell_\delta}{2}) = \frac{\lambda_1}{\lambda_2}.$$

Having expressed d in terms of the λ-lengths of ideal geodesics on S, we can now use the Ptolemy relation to compute λ_1 and λ_2 in terms of $\lambda_{\alpha\beta}, \lambda_{\alpha\gamma}, \lambda_{\alpha,\delta}, \lambda_{\beta\gamma}, \lambda_{\beta\delta}$ and $\lambda_{\gamma\delta}$. Going through the calculations using Figure 4.13 and the ideal Ptolemy relation, we obtain that:

$$d = \frac{\lambda_1}{\lambda_2} = \frac{\lambda_{\alpha\beta}\lambda_{\alpha\gamma}\lambda_{\alpha\delta} + \lambda_{\alpha\beta}\lambda_{\beta\gamma}\lambda_{\beta\delta} + \lambda_{\alpha\gamma}\lambda_{\beta\gamma}\lambda_{\gamma\delta} + \lambda_{\alpha\delta}\lambda_{\beta\delta}\lambda_{\gamma\delta}}{\lambda_{\alpha\beta}\lambda_{\alpha\gamma}\lambda_{\beta\gamma}}. \tag{4.18}$$

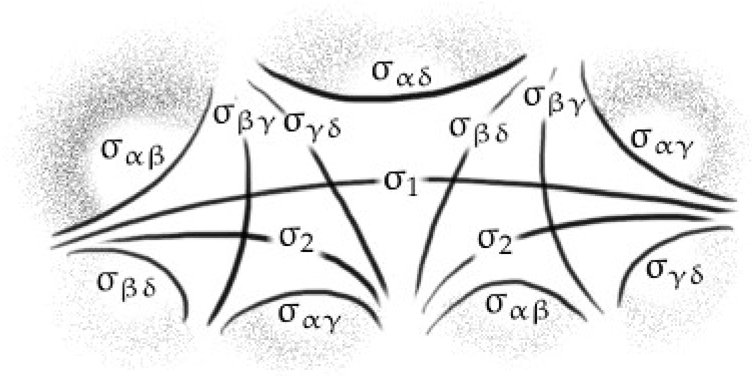


FIGURE 4.13: A figure for computing the trace of δ via ideal Ptolemy relations.

Note 4.13. The homogeneity of (4.18) means that this identity for d (and the corresponding identities for a, b, c) is independent of the horocycles we choose for defining these λ-lengths.

Finally, invoking the fact the three decorating horocycles on S are of length 1, Proposition 1.3 gives us that:

$$\lambda_{\alpha\beta}\lambda_{\alpha\gamma}\lambda_{\alpha\delta} + \lambda_{\alpha\beta}\lambda_{\beta\gamma}\lambda_{\beta\delta} + \lambda_{\alpha\gamma}\lambda_{\beta\gamma}\lambda_{\gamma\delta} + \lambda_{\alpha\delta}\lambda_{\beta\delta}\lambda_{\gamma\delta} = \begin{cases} \lambda_{\alpha\beta}\lambda_{\alpha\gamma}\lambda_{\beta\delta}\lambda_{\gamma\delta} \\ \lambda_{\alpha\beta}\lambda_{\alpha\delta}\lambda_{\beta\gamma}\lambda_{\gamma\delta} \\ \lambda_{\alpha\gamma}\lambda_{\alpha\delta}\lambda_{\beta\gamma}\lambda_{\beta\delta}. \end{cases}$$

Substituting each of these three identities into (4.18) then yields the result. □

The ideal triangulation $\{\sigma_{\alpha\beta}, \sigma_{\alpha\gamma}, \sigma_{\alpha\delta}, \sigma_{\beta\gamma}, \sigma_{\beta\delta}, \sigma_{\gamma\delta}\}$ cuts up the length 1 horocycle at cusp 1 into four horocyclic segments. In terms of λ-lengths, this partition is given by:

$$1 = \frac{\lambda_{\alpha\delta}}{\lambda_{\beta\delta}\lambda_{\gamma\delta}} + \frac{\lambda_{\beta\gamma}}{\lambda_{\alpha\gamma}\lambda_{\gamma\delta}} + \frac{\lambda_{\beta\gamma}}{\lambda_{\alpha\beta}\lambda_{\beta\delta}} + \frac{\lambda_{\alpha\delta}}{\lambda_{\alpha\beta}\lambda_{\alpha\gamma}}. \tag{4.19}$$

Utilising Proposition 4.22, we see that

$$\lambda_{\alpha\beta} = \sqrt{ab},\ \lambda_{\alpha\gamma} = \sqrt{ac},\ \lambda_{\alpha\delta} = \sqrt{ad},\ \lambda_{\beta\gamma} = \sqrt{bc},\ \lambda_{\beta\delta} = \sqrt{bd},\ \lambda_{\gamma\delta} = \sqrt{cd}.$$

Substituting this into equation (4.19) yields:

$$1 = \sqrt{\frac{a}{bcd}} + \sqrt{\frac{b}{acd}} + \sqrt{\frac{c}{abd}} + \sqrt{\frac{d}{abc}}.$$

Since $\sqrt{abcd} = a + b + c + d$, the above line becomes:

$$1 = \frac{a}{a+b+c+d} + \frac{b}{a+b+c+d} + \frac{c}{a+b+c+d} + \frac{d}{a+b+c+d}.$$

Which is precisely the top line of (4.17). For cusps 2 and 3, the corresponding decompositions for (4.19) are respectively given by:

$$\begin{aligned} 1 =& \frac{\lambda_{\alpha\gamma}}{\lambda_{\beta\gamma}\lambda_{\gamma\delta}} + \frac{\lambda_{\beta\delta}}{\lambda_{\alpha\delta}\lambda_{\gamma\delta}} + \frac{\lambda_{\alpha\gamma}}{\lambda_{\alpha\beta}\lambda_{\alpha\delta}} + \frac{\lambda_{\beta\delta}}{\lambda_{\alpha\beta}\lambda_{\beta\gamma}}, \\ 1 =& \frac{\lambda_{\alpha\beta}}{\lambda_{\beta\gamma}\lambda_{\beta\delta}} + \frac{\lambda_{\alpha\beta}}{\lambda_{\alpha\gamma}\lambda_{\alpha\delta}} + \frac{\lambda_{\gamma\delta}}{\lambda_{\alpha\delta}\lambda_{\beta\delta}} + \frac{\lambda_{\gamma\delta}}{\lambda_{\alpha\gamma}\lambda_{\beta\gamma}}. \end{aligned}$$

Substituting these λ-lengths for traces once again yields the top line of (4.17).

4.5 Derivation by Ptolemy Relation

This speculative final section describes one potential way of generalising Bowditch's proof of McShane's original 1-cusped torus identity to arbitrary cusped hyperbolic surfaces. It is based on interpreting Markoff triples as λ-lengths instead of traces. The resulting strategy gives something slightly different to purely trace based proofs (as can be seen from Norbury and my proof for 3-cusped projective plane McShane identities), although it seems possible to tweak[8] this strategy sufficiently so as to recover aspects of a purely trace based proof.

We've already seen that Markoff triples may be thought of as λ-lengths instead of the traces of simple closed geodesics using the correspondence between triples of once-intersecting simple closed geodesics and ideal triangulations on a 1-cusped hyperbolic torus (Proposition 3.7). Phrased in this language, Bowditch's proof of the McShane identity may be broken into the following steps:

1. Construct a(n abstract) simplicial complex — the arc complex $\mathcal{A}(R)$ for a cusped hyperbolic surface R.
2. Define a *Markoff map* $\phi : \mathcal{A}(R)^0 \to \mathbb{R}_+$ on the 0-cells $\mathcal{A}(R)^0$ of $\mathcal{A}(R)$, such that the top dimensional cells of $\mathcal{A}(R)$ assign integral algebraic relations on the ϕ-values associated to the vertices of such a top dimensional cell.
3. Use the horocyclic length partition-based interpretation of these algebraic relations via the ideal Ptolemy relation to describe how to refine a given partition.
4. Show that the limiting summands of a sequence of iteratively refined partitions correspond to the summands of the cuspidal McShane identity.

Step 1: Consider the arc complex $\mathcal{A}(R)$ on R. Specifically, define the abstract simplicial complex where we assign

- a 0-cell $\{\sigma_1\}$ to each (simple) bi-infinite ideal geodesic σ_1 on R;
- a 1-cell $\{\sigma_1, \sigma_2\}$ to each pair of disjoint bi-infinite ideal geodesics σ_1, σ_2 on R;

[8] We do not say anything about how to do this.

- a 2-cell $\{\sigma_1, \sigma_2, \sigma_3\}$ to each 3-tuple of disjoint bi-infinite ideal geodesics $\sigma_1, \sigma_2, \sigma_3$ on R;

  $\vdots$
- a $3|\chi(R)|$-cell $\{\sigma_1, \ldots, \sigma_{3|\chi(R)|}\}$ to each $3|\chi(R)|$-tuple of disjoint bi-infinite ideal geodesics on R. Since this is a maximal collection, each such cell as corresponds naturally to an ideal triangulation of R.

Note 4.14. The geometric realisation of $\mathcal{A}(R)$ may be constructed using Penner's canonical triangulation-based cellularisation of the decorated Teichmüller space $\hat{\mathcal{T}}(R)$.

Step 2: The λ-length of the ideal geodesics on R give us a prototype of what we call a *Markoff map* on $\mathcal{A}(R)$. Specifically, we may define a function on the 0-cells of the arc complex $\mathcal{A}(R)$

$$\phi_\lambda : \mathcal{A}(R)^0 \to \mathbb{R}_+$$

that assigns to each $\{\sigma\} \in \mathcal{A}(R)^0$ its λ-length $\phi_\lambda(\{\sigma\}) := \lambda_\sigma$ of σ truncated at the length 1 horocycles on R. Note that we've opted to use the length 1 horocycles instead of length 2 (as per the Markoff triples); for all intents and purposes it makes no difference.

Although defined on the 0-cells of the arc complex $\mathcal{A}(R)$, the data of the function ϕ_λ may be alternatively stored by assigning to each top dimensional cell $\{\sigma_1, \ldots, \sigma_{3|\chi(R)|}\}$ a collection of $3|\chi(R)|$ positive real numbers

$$\phi_\lambda(\{\sigma_1\}) := \lambda_{\sigma_1},\ \phi_\lambda(\{\sigma_2\}) := \lambda_{\sigma_2},\ \ldots,\ \phi_\lambda(\{\sigma_{3|\chi(R)|}\}) := \lambda_{\sigma_{3|\chi(R)|}}.$$

For the 1-cusped torus case the 1-skeleton of the dual of the arc complex $\mathcal{A}(S_{1,1})$ is a tree, and hence there's a sensible and canonical (up to a finite number of choices) way of compatibly ordering the corresponding 3-tuples of numbers. It is perhaps better to think of ϕ_λ as a point $\phi_\lambda \in \mathbb{R}_+^{\mathcal{A}(S)^0}$, and to regard each $\phi(\{\sigma\}) := \lambda_\sigma$ as the σ-th coordinate of ϕ_λ.

As previously demonstrated (and also described in [Pen87]), for each top dimensional cell $\{\sigma_1, \ldots, \sigma_{3|\chi(R)|}\}$ the length 1 horocycle conditions for the n cusps of S give us n integral polynomial relations:

$$0 = P_i(\lambda_{\sigma_1}, \ldots, \lambda_{\sigma_{3|\chi(R)|}}), \text{ where } i = 1, \ldots, n, \tag{4.20}$$

for this collection of $3|\chi(R)|$ "coordinates" to satisfy.

Now, any function $\phi : \mathcal{A}(R^0) \to \mathbb{R}_+$ may be thought of as a point in $\mathbb{R}_+^{\mathcal{A}(S)^0}$. We say that such a point ϕ is a *Fuchsian*[9] *Markoff map* for R if for every top dimensional cell $\{\sigma_1, \ldots, \sigma_{3|\chi(R)|}\}$, the corresponding coordinates of the point $\phi \in \mathbb{R}_+^{\mathcal{A}(S)^0}$ satisfy the equations constituting (4.20). It is unsurprising that marked surfaces $[S, f]$ induce Markoff maps by mapping a 0-cell $\{\sigma\}$ to the λ-length of the ideal bi-infinite geodesic representative $f_\#\sigma$ of the curve $f_*\sigma$ in S. In fact, Theorem 3.1 of [Pen87] tells us that every Fuchsian Markoff map arises in such a manner:

[9] This nomenclature optimistically anticipates future work with general complex Markoff maps.

Proposition 4.23. *The space of Fuchsian Markoff maps* $\phi : \mathcal{A}(R)^0 \to \mathbb{R}_+$ *for* R *is a model for the Teichmüller space* $\mathcal{T}(R)$.

Step 3: To go from one top dimensional cell $\triangle$ to an adjacent top dimensional cell $\triangle'$, we replace a constituent geodesic $\sigma \in \triangle$ by the opposite diagonal σ' in the unique ideal quadrilateral in $R - (\triangle - \{\sigma\})$. Cyclically labelling the λ-lengths of the sides of the ideal quadrilateral containing σ and σ' by $\lambda_a, \lambda_b, \lambda_c, \lambda_d$ and the λ-lengths of σ and σ' respectively by λ_e and λ_f, the ideal Ptolemy relation (Proposition 1.4) tells us that:

$$\lambda_e \lambda_f = \lambda_a \lambda_c + \lambda_b \lambda_d.$$

This is algebraically equivalent to either of the following statements:

$$\frac{\lambda_e}{\lambda_a \lambda_b} = \frac{\lambda_c}{\lambda_b \lambda_f} + \frac{\lambda_d}{\lambda_a \lambda_f} \text{ or } \frac{\lambda_e}{\lambda_c \lambda_d} = \frac{\lambda_a}{\lambda_d \lambda_f} + \frac{\lambda_b}{\lambda_c \lambda_f},$$

which arise from cutting one horocyclic segment into two horocyclic segments (Note 1.1).

Since the cusp 1 horocycle condition for the ideal triangulation $\triangle$ is a partition of 1 into terms of the form $\frac{\lambda_{\sigma_i}}{\lambda_{\sigma_j} \lambda_{\sigma_k}}$, the above identities tell us how we may break this partition of 1 into finer and finer summands. In particular, we may visualise this refinement process by starting at the dual vertex $\triangle^*$ of the dual complex $\mathcal{A}(S)^*$ of the arc complex and gradually branching out to other vertices along the 1-skeleton of the dual complex.

Step 4: To see that this process eventually produces the correct summands for the McShane identity, we conclude this chapter by explicitly describing a sequence of ideal geodesics which cut away slivers of these horocyclic segments to give the desired limiting summand of the McShane identity.

Lemma 4.24. *The total length of the four horocyclic segments constituting the gap region at the cusp 1 of a pair of pants* P *bordered by cusp 1 and two closed geodesic boundaries* γ_1 *and* γ_2 *is*

$$\frac{2}{1 + \exp \frac{1}{2}(\ell_{\gamma_1} + \ell_{\gamma_2})}.$$

Proof. Given a cusped hyperbolic surface S, recall that the summand for an embedded pair of pants $P \subset S$ containing cusp 1 corresponds to the length of the four horocyclic segments constituting the gap region for P (Figure 4.2). Two of these segments lie on the pair of half-pants $P_1 \subset P$, and the reflection symmetry of P_1 means that they're of the same length. Denote one of these two segments lying on the pair of half-pants P_i by η_i, we now calculate an upperbound for the length of η_1.

Consider an embedded $(1,1)$-crowned annulus in S that contains P_1, and triangulate it with $\triangle_0 := (\alpha_1, \alpha_2, \sigma_0, \sigma_1)$. We define σ_2 as the diagonally opposite geodesic to σ_0 with respect to $\triangle_0$ and note that $\triangle_1 := (\alpha_1, \alpha_2, \sigma_1, \sigma_2)$ is a new geodesic ideal triangulation of our $(1,1)$-crowned annulus. This is the same construction as used in Subsection 4.2.1, and we produce from it a sequence of triangulations $\{\triangle_i\}$ and ideal geodesic arcs $\{\sigma_i\}$ in this way by repeatedly taking σ_{i+2} to be the opposite diagonal to σ_i with respect to $\triangle_i$ and setting

$$\triangle_{i+1} := (\alpha_1, \alpha_2, \sigma_{i+1}, \sigma_{i+2}).$$

These geodesics cut the horocycle η_1 into shorter and shorter segments of lengths $\frac{\lambda_{\sigma_i}}{\lambda_{\alpha_1}\lambda_{\sigma_{i+1}}}$. Let x denote the limiting width as i tends to infinity, and let h denote the length of the horocyclic segment lying in P_1 of the original length 1 horocycle. Then,

$$h = \frac{\lambda_{\sigma_{i-1}}}{\lambda_{\alpha_1}\lambda_{\sigma_i}} + \frac{\lambda_{\sigma_{i+1}}}{\lambda_{\alpha_1}\lambda_{\sigma_i}}.$$

Taking the limit, this gives us the quadratic equation: $0 = x^2 - hx + \lambda_{\alpha_1}^{-2}$. Solving for the smaller root for $x \leqslant \frac{h}{2}$, we obtain that:

$$x = \frac{h}{2} - \sqrt{\frac{h^2}{4} - \frac{1}{\lambda_{\alpha_1}^2}}. \tag{4.21}$$

By construction, $x \geqslant \ell(\eta_1)$ because any geodesic launched within η_1 must self-intersect (before leaving P_1) and hence cannot be one of the $\{\sigma_i\}$. By symmetry, $y \geqslant \ell(\eta_2)$ for

$$y = \frac{1-h}{2} - \sqrt{\frac{(1-h)^2}{4} - \frac{1}{\lambda_{\alpha_1}^2}},$$

constructed similarly for P_2. Finally, using the fact that:

$$\begin{aligned} h\lambda_{\alpha_1} &= 2\cosh(\tfrac{\ell_{\gamma_1}}{2}), \\ (1-h)\lambda_{\alpha_1} &= 2\cosh(\tfrac{\ell_{\gamma_2}}{2}), \\ \text{and } \lambda_{\alpha_1} &= 2\cosh(\tfrac{\ell_{\gamma_1}}{2}) + 2\cosh(\tfrac{\ell_{\gamma_2}}{2}), \end{aligned}$$

where γ_1 and γ_2 are the respective closed geodesic boundaries of P_1 and P_2, we obtain the following comparison:

$$\begin{aligned} \frac{2}{1+e^{\frac{1}{2}(\ell_{\gamma_1}+\ell_{\gamma_2})}} =& 2\ell(\eta_1) + 2\ell(\eta_2) \\ \leqslant& 2x + 2y = 1 - \sqrt{h^2 - 4\lambda_{\alpha_1}^{-2}} - \sqrt{(1-h)^2 - 4\lambda_{\alpha_1}^{-2}} \\ =& 1 - \frac{2\sinh(\frac{\ell_{\gamma_1}}{2}) + 2\sinh(\frac{\ell_{\gamma_2}}{2})}{2\cosh(\frac{\ell_{\gamma_1}}{2}) + 2\cosh(\frac{\ell_{\gamma_2}}{2})} = \frac{2}{1+e^{\frac{1}{2}(\ell_{\gamma_1}+\ell_{\gamma_2})}}. \end{aligned} \tag{4.22}$$

Therefore, $x = \ell(\eta_1)$ and $y = \ell(\eta_2)$. $\square$

Note 4.15. A small Porism of the above proof is that on a (1-tined) pair of half-pants with decorating horocyclic segment of length h and ideal boundary arch of λ-length λ, the gap region is of length:

$$h - \sqrt{h^2 - 4\lambda^{-2}}.$$

Note 4.16. Similar λ-length and Ptolemy relation based computations may be done for bordered surfaces to give an interpretation of the summands for Mirzakhani's bordered surface McShane identities as areas of regions spiralling around a geodesic border within a certain hypercycle around that border. Although we have not been able to extend this technique to derive Tan-Wong-Zhang's small cone-angle McShane identity [TWZ06].

Ultimately, we would like to be able to replicate Bowditch's strategy of studying the growth rates of these λ-lengths (via some Fibonacci-growth-like bound) and hence avoid having to invoke the Birman-Series theorem to derive a McShane identity. This seems plausible, but rather finnicky — perhaps it'll constitute future work.

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