Introduction

This small note is motivated by our desire to understand the behaviour of holomorphic differentials on (closed) Riemann surfaces. We begin by describing the construction of a genus g Riemann surface with a holomorphic differential on it (for any g). And use this as a basis for a heuristic analytic characterization of holomorphic differentials on Riemann surfaces.

Algebraic curves

It is well-known that any Riemann surface may be embedded in $\mathbb{C}P^n$ as a projective variety, that is: it can be written as the solution set to a homogeneous polynomial $F(z_0, \dots, z_n)$. In the special case that $n = 2$, we call the image of such a Riemann surface a *planar curve*.

A planar curve $X \subset \mathbb{C}P^2$ defined by a degree 3 homogeneous polynomial $F(x, y, z)$ may be thought of, without loss of generality, as a the solution set of a degree 3 polynomial $f(x, y) = F(x, y, 1)$ defined over $\mathbb{C}^2$. Then the $\mathbb{C}$-vector space of holomorphic differentials on X is given by:

$$\Omega^1(X) = \{\, p(x, y)\frac{dx}{\partial f/\partial y} \mid \deg(p) \leqslant d - 3 \,\}, \tag{1}$$

where d is the degree of f. You might be worried because $\partial f/\partial y$ can be 0, and this is resolved by considering the definition of the cotangent bundle in this context and interpreting this expression as a representative of an equivalence class of sections.

Riemann surfaces

We now turn to the problem of analytically construction non-trivial holomorphic differentials on Riemann surfaces. For the Riemann sphere this is known to be impossible – there are no such differentials, and for the torus

$$T = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}), \tag{2}$$

its space of holomorphic forms $\Omega^1(T)$ is one dimensional, and any such form pulls back, with respect to the universal covering map, to a multiple of dz over $\mathbb{C}$ – the universal cover of T.

Note that removing a closed disk from T results in an open Riemann surface homeomorphic to the once-punctured torus, and we may define a family of such tori:

$$T_\delta := (\mathbb{C} - \bigcup_{\substack{m+in \\ \in \mathbb{Z}+i\mathbb{Z}}} \overline{B}_\delta(m + ni))/(\mathbb{Z} + i\mathbb{Z}), \text{ for } \delta \geqslant 0. \tag{3}$$

The restriction of a holomorphic differential of T to being over T_δ for $\delta < \frac{1}{2}$ induces an injective morphism

$$\rho_\delta : \Omega^1(T) \to \Omega^1(T_\delta), \tag{4}$$

and it's fairly easy to see that ρ_δ is not surjective. In fact, we'll presently show that $\Omega^1(T_\delta)$ is infinite dimensional.

Weierstrass's elliptic function $\wp : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ is defined to be:

$$\wp(z) := \frac{1}{z^2} + \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(z+m+ni)^2} - \frac{1}{(m+ni)^2}. \tag{5}$$

Since Weierstrass's elliptic function is doubly periodic, that is:

$$\wp(z) = \wp(z+m+ni),$$

it can be regarded as a meromorphic function on the torus T:

$$\wp : T \to \mathbb{C} \cup \{\infty\}. \tag{6}$$

The poles of $\wp$ are position on $\mathbb{Z} + i\mathbb{Z}$, so the restriction of $\wp$ to T_δ is a holomorphic function. Then,

$$\{\wp^k \cdot \rho_\delta(\Omega^1(T))\}_{k \in \mathbb{Z}}$$

are all distinct subspaces of $\Omega^1(T_\delta)$. Henceforth, we'll intentionally confuse the elements of $\Omega^1(T)$ or $\Omega^1(T_\delta)$ with the pullbacks to their respective $(\mathbb{Z} + i\mathbb{Z})$-covers in $\mathbb{C}$. For example, we'll refer to the elements of $\wp \cdot \rho_\delta(\Omega^1(T))$ simply as $\alpha\wp(z)dz$. Let us now proceed to construct a genus 2 Riemann surface and then to analytically construct a holomorphic differential on it.

To begin with, we construct a function ζ that will be used for constructing the transition functions of the charts of our Riemann surface. Weierstrass's elliptic function $\wp$ has an order 2 pole at the integer points of $\mathbb{C}$, and its Laurent series expansion around 0 takes the form:

$$\wp(z) = \frac{1}{z^2} + c_2 z^2 + c_4 z^4 + c_6 z^6 + \ldots \tag{7}$$

Since $\wp(z)z^2$ is holomorphic around $z = 0$, the power series $\sum_{k>0} c_{2k} z^{2k+2}$ converges absolutely in closed ball around $z = 0$ and hence the following function

$$\zeta(z) := z \cdot \left(1 - \sum_{k=1}^{\infty} \frac{c_2 k}{2k+1} z^{2k+2}\right)^{-1}, \tag{8}$$

is well-defined in a sufficiently small neighborhood and satisfies that

$$\zeta^{-2}d\zeta = \wp(z)dz. \tag{9}$$

Since the derivative of ζ is non-zero about $z = 0$, the domain-restricted function

$$\zeta : \overline{B}_\epsilon(0) \to \mathbb{C}, \tag{10}$$

must be a holomorphic embedding for $\epsilon > 0$ sufficiently small. Notice that $\zeta(0) = 0$, so we may choose numbers $\epsilon' > \eta > 0$ such that:

$$\overline{B}_\eta(0) \subset \zeta(\overline{B}_\epsilon(0)) \subset \overline{B}_{\epsilon'}(0). \tag{11}$$

Now consider the open Riemann surfaces

$$S_1 := T_\eta \text{ and} \tag{12}$$

$$S_2 := (\mathbb{C} - \bigcup_{\substack{m+ni \\ \in \mathbb{Z}+i\mathbb{Z}}} (m+ni)\cdot\zeta^{-1}(\overline{B}_\eta))/(\mathbb{Z}+i\mathbb{Z}). \tag{13}$$

We can construct a closed genus 2 Riemann surface S by gluing S_1 and S_2 as follows:

$$S := S_1 \cup S_2/\sim, \text{where } z_1 \in S_1 \text{ is identified with } z_2 \in S_2 \text{ iff. } z_1 = \frac{-\eta^2}{\zeta(z_2)}. \tag{14}$$

Note: we're basically taking annular neighborhoods around the boundaries of S_1 and S_2 and identifying them.

The holomorphic differential on S_1 that takes the form $\eta^2 dz$ transforms to $\wp(z)dz$ on S_2 with respect to the transition function thanks to (9): this gives us a nontrivial holomorphic differential on all of S.

Fluff

Although we've only outlined the procedure to obtain a genus 2 Riemann surface, it's not too hard to extend these ideas to closed surfaces of arbitrarily high genus. In addition, note that there was some flexibility in our construction: we could have changed the modulus of either S_1 or S_2, and we could have varied η and introduced a twist in how we glued S_1 and S_2 by multiplying the identification condtion $\sim$ by a constant $e^{i\theta}$. By doing some analysis involving the ratios of integrals around certain cycles of S with respect to the differential produced along with it, we should be able to show that this procedure yields three complex-dimensional set of surfaces in $\mathcal{M}_2$. In fact, it's probably not too much of a stretch to put this together and conjecture that any genus 2 Riemann surface S can be expressed as the union of two punctured tori so that there exists a holomorphic form on S that takes the form dz on S_1 and $\wp(z)dz$ on S_2. Of course, if this were true, then we should probably expect that there's a reparametrization of S_1 and S_2 such that a linearly independent holomorphic form on S takes the form dz on S_2 and $\wp(z)dz$ on S_1. And if we're able to show that then we should probably be able to do it for any genus. So, basically, this should give us a characterization of a basis for $\Omega^1(X)$ in terms of torus components and Weierstrass's elliptic function?

Note: I've looked at the possible arrangements of dz and $\wp(z)dz$ on a genus g surface and the linear dependence between them, and it seems plausible that they'll generate dimension g vector space. Which is good news, I guess?