A McShane-type identity for closed surfaces

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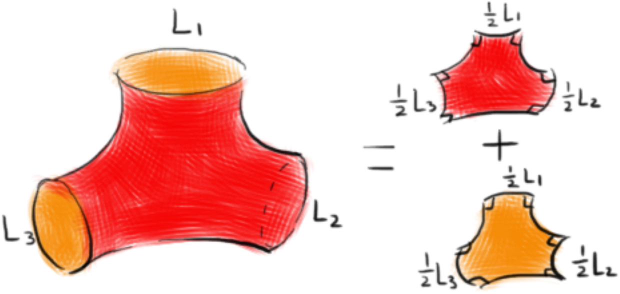
Consider the following questions:

1. What does a geodesic launched from a point on a (constant curvature) sphere look like?
2. What does a geodesic launched from a point on a (flat) torus look like?
3. And what does a geodesic launched from a point on a hyperbolic surface look like?

The simplest hyperbolic surface is a *pair of pants* with geodesic and/or cusp boundaries.



- Its geometry is completely determined by its boundary lengths.
- It has a reflection isometry.

There are 4 types of behaviours for geodesics emanating from the *tip* of the cusp.

- $1\times$*zipper* geodesic.
- $4\times$spiralling geodesics.
- $2\times$intervals of geodesics which exit the pair of pants.
- $4\times$intervals worth of self-intersecting geodesic - call this the *spiral region*.

The chance of choosing a direction within this spiral region is:

$$\frac{2}{1+\exp\frac{1}{2}(\ell_{\gamma_1}+\ell_{\gamma_2})}$$

ℓ_{γ_1} and ℓ_{γ_2} are the lengths of the (non-cusp) boundaries γ_1, γ_2 of our pair of pants.

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1. Almost every geodesic launched from a cusp self-intersects.
2. Such a geodesic is launched within the spiral region of a unique pair of pants in our surface.
3. The chance of launching a self-intersecting geodesic from a chosen cusp can be written as a sum over the set of embedded pairs of pants containing this cusp.

And so we obtain something called a *McShane identity* [McShane]:

$$1 = \sum_{\{\gamma_1, \gamma_2\}} \frac{2}{1 + \exp \frac{1}{2}(\ell_{\gamma_1} + \ell_{\gamma_2})}$$

where γ_1 and γ_2 are internal geodesics which, along with the chosen cusp, bound a pair of pants embedded in our surface.

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4. Non-orientable hyperbolic surfaces. [Norbury]

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Strategy for closed hyperbolic surfaces

What happens if, instead of a cusp, we launched from a normal point on a closed hyperbolic surface?

1. Does *almost every* geodesic launched from a selected point self-intersect?
2. Is such a geodesic always launched within the spiral region of a *unique* pair of pants?
3. If so, then we have a new McShane-type identity!

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We might (in theory) be able to do something quite similar with the moduli spaces of hyperbolic surfaces with large cone-angles.

Dinner time?