A McShane-type identity for closed surfaces

Yi Huang

University of Melbourne

September 25th, 2012

Consider the following questions:

1. What does a geodesic launched from a point on a (constant curvature) sphere look like?
2. What does a geodesic launched from a point on a (flat) torus look like?
3. And what does a geodesic launched from a point on a hyperbolic surface look like?

The simplest hyperbolic surface is a *pair of pants* with geodesic boundaries; a cusp is thought of as a length 0 geodesic. We'll start off dealing with pants with at least one cusp. Note that:

- Its geometry is completely determined by its boundary lengths.
- It has a reflection isometry.

There are 4 types of behaviours for geodesics emanating from the *tip* of the cusp.

1. One *zipper* geodesic.
2. Four spiralling geodesics.
3. Four intervals worth of self-intersecting geodesic
   - call this the *spiral region*.
4. Geodesics which exit the pair of pants.

The chance of choosing a direction within one this spiral region is:

$$\frac{2}{1 + \exp \frac{1}{2}(\ell_{\gamma_1} + \ell_{\gamma_2})}$$

where ℓ_{γ_1} and ℓ_{γ_2} are the lengths of the (non-cusp) boundaries of our pair of pants.

Strategy for cusped hyperbolic surfaces

Knowing about geodesics on a pair of pants helps us to understand general cusped hyperbolic surfaces:

1. Almost every geodesic launched from a cusp self-intersects.

Strategy for cusped hyperbolic surfaces

Knowing about geodesics on a pair of pants helps us to understand general cusped hyperbolic surfaces:

1. Almost every geodesic launched from a cusp self-intersects.
2. Such a geodesic is launched within the spiral region of a unique pair of pants in our surface.

Strategy for cusped hyperbolic surfaces

Knowing about geodesics on a pair of pants helps us to understand general cusped hyperbolic surfaces:

1. Almost every geodesic launched from a cusp self-intersects.
2. Such a geodesic is launched within the spiral region of a unique pair of pants in our surface.
3. The chance of launching a self-intersecting geodesic from a chosen cusp can be written as a sum over the set of embedded pairs of pants containing this cusp.

And so we obtain what's usually called a *McShane identity*:

$$1 = \sum_{\{\gamma_1, \gamma_2\}} \frac{2}{1 + \exp \frac{1}{2}(\ell_{\gamma_1} + \ell_{\gamma_2})},$$

where γ_1 and γ_2 are internal geodesics which, along with the chosen cusp, bound a pair of pants embedded in our surface.

There are generalisations of this identity for:

1. The mapping tori of cusped hyperbolic surfaces with respect to a pseudo-Anosov map.

There are generalisations of this identity for:

1. The mapping tori of cusped hyperbolic surfaces with respect to a pseudo-Anosov map.
2. Surfaces with geodesic boundary.

There are generalisations of this identity for:

1. The mapping tori of cusped hyperbolic surfaces with respect to a pseudo-Anosov map.
2. Surfaces with geodesic boundary.
3. Surfaces with cone-points with angle between 0 and π.

There are generalisations of this identity for:

1. The mapping tori of cusped hyperbolic surfaces with respect to a pseudo-Anosov map.
2. Surfaces with geodesic boundary.
3. Surfaces with cone-points with angle between 0 and π.
4. Non-orientable hyperbolic surfaces.

Strategy for closed hyperbolic surfaces

What happens if, instead of a cusp, we launched from a normal point on a closed hyperbolic surface?

Strategy for closed hyperbolic surfaces

What happens if, instead of a cusp, we launched from a normal point on a closed hyperbolic surface?

1. Does (almost) every geodesic launched from a selected point self-intersect?

Strategy for closed hyperbolic surfaces

What happens if, instead of a cusp, we launched from a normal point on a closed hyperbolic surface?

1. Does (almost) every geodesic launched from a selected point self-intersect?
2. Is such a geodesic always launched within the spiral region of a *unique* pair of pants?

Strategy for closed hyperbolic surfaces

What happens if, instead of a cusp, we launched from a normal point on a closed hyperbolic surface?

1. Does (almost) every geodesic launched from a selected point self-intersect?
2. Is such a geodesic always launched within the spiral region of a *unique* pair of pants?
3. If so, then the chance of launching a self-intersecting geodesic from a chosen cusp can be written as a sum over the set of embedded pairs of pants containing this cusp.

How point 2 fails:

- There are no pairs of pants with our chosen point as a boundary component!

How point 2 fails:

- There are no pairs of pants with our chosen point as a boundary component!
- *To fix this, we use isometrically immersed pairs of half-pants instead.*

How point 2 fails:

- There are no pairs of pants with our chosen point as a boundary component!
- *To fix this, we use isometrically immersed pairs of half-pants instead.*
- A launched geodesic may intersect itself prematurely, hence its direction may be in the spiral regions of more than one pair of half-pants.

How point 2 fails:

- There are no pairs of pants with our chosen point as a boundary component!
- *To fix this, we use isometrically immersed pairs of half-pants instead.*
- A launched geodesic may intersect itself prematurely, hence its direction may be in the spiral regions of more than one pair of half-pants.
- *We do some trigonometry to discount these overcounted geodesics.*

How point 2 fails:

- There are no pairs of pants with our chosen point as a boundary component!
- *To fix this, we use isometrically immersed pairs of half-pants instead.*
- A launched geodesic may intersect itself prematurely, hence its direction may be in the spiral regions of more than one pair of half-pants.
- *We do some trigonometry to discount these overcounted geodesics.*

So, what do these summands look like?

So, what's the upshot?

- McShane identities $\Rightarrow$ Recursively calculate Weil-Petersson volumes of moduli spaces
- Symplectic reduction $\Rightarrow$ Recursion structure in the intersection number of these moduli spaces
- $\Rightarrow$ A proof of Witten's conjecture!

And we should be able to do something quite similar with the moduli spaces of hyperbolic surfaces with large cone-angles.