What do the following quadruples of numbers have in common?

(1,5,24,30), (1,6,14,21), (1,8,9,18), (1,9,10,10),
(2,3,10,15), (2,5,5,8), (3,3,6,6), (4,4,4,4)

Flipping numbers and curves

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Markoff triples

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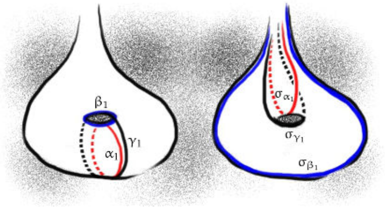

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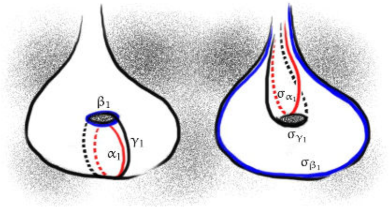

or as λ-lengths of an ideal triangulation $(\sigma_{\alpha_1}, \sigma_{\beta_2}, \sigma_{\gamma_1})$.

Representations

Given a 1-cusped torus $S_{1,1}$, $\pi_1(S_{1,1}) \cong \langle \xi, \eta \mid - \rangle$.

Any non-zero Markoff triple (x, y, z) arises as the traces of the following representation $\rho : \pi_1(S_{1,1}) \to \mathrm{SL}(2, \mathbb{C})$:

$$\rho(\xi) = \frac{1}{z}\begin{bmatrix} xz - y & xz^{-1} \\ xz & y \end{bmatrix}, \ \rho(\eta) = \frac{1}{z}\begin{bmatrix} yz - x & -yz^{-1} \\ -yz & x \end{bmatrix},$$
$$\rho(\xi\eta) = \begin{bmatrix} z & -z^{-1} \\ z & 0 \end{bmatrix}.$$

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Thus, Markoff triples arise as the characters of (type-preserving) representations.

Character varieties

- The traces of $\rho(\xi)$, $\rho(\eta)$ and $\rho(\xi\eta)$ of any $\rho : \pi_1(S_{1,1}) \to \mathrm{SL}(2, \mathbb{C})$ satisfies the Markoff triples relation.

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- The set of Markoff triples is the character variety for $S_{1,1}$.
- Any maximal dimensional component of the real character subvariety is the Teichmüller space $\mathcal{T}(S_{1,1})$:

$$\mathcal{T}(S_{1,1}) = \{ \text{ hyperbolic structures on } S_{1,1}, \text{ up to isotopy}\}$$

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- Flips can be thought of as extended mapping classes — (potentially non-orientable) homeomorphisms of $S_{1,1}$ up to isotopy.
- Flips and coordinate permutations generate the entire extended mapping class group of $S_{1,1}$.

Systolic geometry

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- Algorithm for working out the shortest geodesic (systole).
- The maximum of the systole function over Teichmüller space (and moduli space) is the $(3, 3, 3)$ cusped torus.
- The shortest geodesic for any 1-cusped hyperbolic torus is at most $2\operatorname{arccosh}(\frac{3}{2})$.

Geodesic growth rates

Let (x, y, z) and $(x' = yz - x, y, z)$ be flips of each other where $x \leq x'$. Generically, $yz \gg x$, thus:

$$\log(x') \approx \log(y) + \log(z).$$

$\Rightarrow$ *Fibonacci growth.*

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$\Rightarrow$ *Fibonacci growth.*

The length of a simple closed geodesics is roughly $2\log(\cdot)$ of its corresponding trace. Consider:

$$N_S(L) := \{\text{ simple closed geodesics on } S \text{ shorter than } L\},$$

Fibonacci growth $\Rightarrow N_S(L)$ is asymptotically $\alpha \cdot L^2$.

McShane identity

Rewriting the Markoff triples equation and the flipping relation:

$$1 = \frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy} \text{ and } 1 = \frac{x}{yz} + \frac{x'}{yz} \Rightarrow \frac{x'}{yz} = \frac{y}{xz} + \frac{z}{xy}.$$

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In the limit, we obtain *McShane identities*:

$$1 = \sum_{\gamma \in \mathrm{Sim}\pi_1(S)} \frac{2}{1 + \exp \ell_\gamma},$$

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Each term is the chance a geodesic shot out from the cusp on S won't hit γ before self-intersecting.

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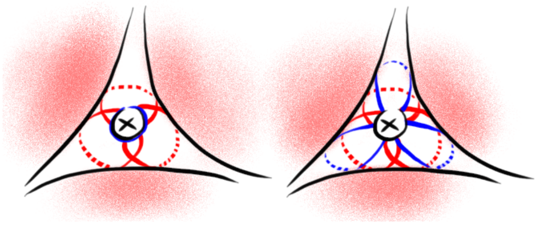

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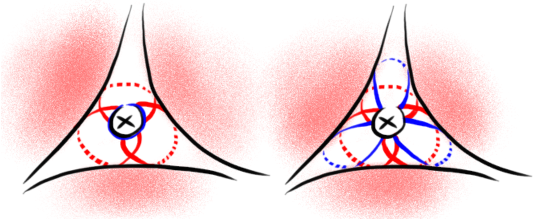

$\{Markoffquads\}$ = character variety for 3-cusped projective planes.

Real character subvariety → Teichmüller space.

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- A quad of such geodesics can be flipped to any other quad $\Rightarrow$ geodesic length generating algorithm.
- Flips can be thought of as extended mapping classes, and flips+permutations generate the (extended) mapping class group.

Systolic geometry

Theorem

The maximum of the systole function over the moduli space of 3-cusped projective plane is $2\operatorname{arcsinh}(2)$*, and uniquely attained by the* $(4, 4, 4, 4)$ *surface.*

Geodesic growth rates

Let (a, b, c, d) and (a', b, c, d) be flips of each other where $a \leq a'$, generically:

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Consider

$$N_S(L) := \{ \text{ 1-sided simple closed geodesics on } S \text{ shorter than } L \},$$

Fibonacci growth $\Rightarrow N_S(L)$ is between $O(L^{2.430})$ and $O(L^{2.477})$.

McShane identity

We similarly obtain the following sum refinement:

$$1 = \frac{a+b+c+d}{bcd} + \frac{a+b+c+d}{acd} + \frac{a+b+c+d}{abd} + \frac{a+b+c+d}{abc}$$

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Theorem

Given any 3-cusped projective plane S,

$$1 = \sum_{\gamma \in \mathrm{Sim}\pi_1^1(S)} \frac{2}{1+\exp \ell_\gamma},$$

where $\mathrm{Sim}\pi_1^1(S)$ is the set of 2-sided simple closed geodesics on S.

Future directions

Number theory: Integer Markoff triples flip to integer Markoff triples, and integer Markoff triples are central in number theory:

- rational approximation;
- Markoff's theorem for quadratic forms;
- the unicity conjecture.

Markoff quad equivalents?

Geometry: Are BQ-conditions trace-based characterisations of quasi-Fuchsian representations?

Are there geometric interpretations for more general Markoff-Hurwitz numbers?

In case anyone asks...

$$\rho : F_3 = \langle \alpha, \beta, \gamma \rangle \to \mathrm{SL}^{\pm}(2, \mathbb{C})$$

$$\alpha \mapsto \frac{1}{a+b+c+d} \begin{bmatrix} ab & b(a+c) \\ a(a+d) & a(a+c+d) \end{bmatrix},$$

$$\beta \mapsto \frac{1}{a+b+c+d} \begin{bmatrix} ab & -b(b+d) \\ -a(b+c) & b(b+c+d) \end{bmatrix},$$

$$\gamma \mapsto \frac{1}{a+b+c+d} \begin{bmatrix} ab + c(a+b+c+d) & b(a+c) \\ -a(b+c) & -ab \end{bmatrix}.$$