"[Moduli spaces] have also appeared in theoretical physics like string theory: many computations of path integrals are reduced to integrals of Chern classes on such moduli spaces. "

Shing-Tung Yau

Phants and Surfaces

Yi Huang

University of Melbourne

December 4th, 2012

The moduli space $\mathcal{M}(S)$ of a surface S of genus g and with n punctures is:

$$\{\text{equivalence classes of Riemann surfaces homeomorphic to } S\}$$

The moduli space $\mathcal{M}(S)$ of a surface S of genus g and with n punctures is:

$$\{\text{equivalence classes of Riemann surfaces homeomorphic to } S\}$$

How can we study it?

The moduli space $\mathcal{M}(S)$ of a surface S of genus g and with n punctures is:

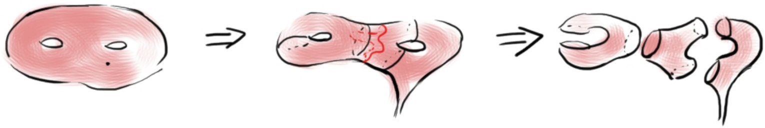
$$\{\text{equivalence classes of Riemann surfaces homeomorphic to } S\}$$

How can we study it?

- build every Riemann surface.
- identify biholomorphic ones.

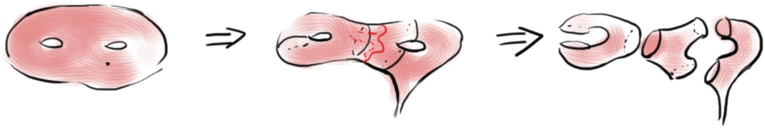
The anatomy of a Riemann surface:

- A Riemann surface $R \Rightarrow$ a unique hyperbolic surface R_h.



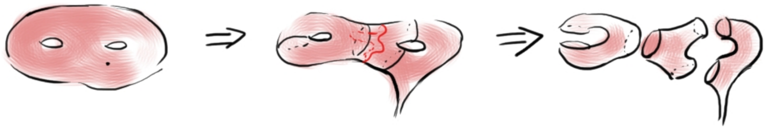
The anatomy of a Riemann surface:

- A Riemann surface $R \Rightarrow$ a unique hyperbolic surface R_h.
- Simple loops on hyperbolic surfaces $\Rightarrow$ unique geodesics.



The anatomy of a Riemann surface:

- A Riemann surface $R \Rightarrow$ a unique hyperbolic surface R_h.
- Simple loops on hyperbolic surfaces $\Rightarrow$ unique geodesics.
- Cut along these geodesics $\Rightarrow$ hyperbolic pairs of pants.



So, how can we generate every Riemann surface?

So, how can we generate every Riemann surface?

- There's a unique pair of pants for any three boundary lengths.

So, how can we generate every Riemann surface?

- There's a unique pair of pants for any three boundary lengths.
- Can vary these lengths, and

So, how can we generate every Riemann surface?

- There's a unique pair of pants for any three boundary lengths.
- Can vary these lengths, and
- vary the gluing.

Teichmüller space

Take geodesics $\{\gamma_1, \ldots, \gamma_{3g-3+n}\}$ that decompose R_h into pairs of pants. For each γ_i, we have:

- $\mathbb{R}^+$ possible lengths ℓ_i, and
- $\mathbb{R}$ possible twists τ_i (keep track of some winding number)

Teichmüller space

Take geodesics $\{\gamma_1, \ldots, \gamma_{3g-3+n}\}$ that decompose R_h into pairs of pants. For each γ_i, we have:

- $\mathbb{R}^+$ possible lengths ℓ_i, and
- $\mathbb{R}$ possible twists τ_i (keep track of some winding number)

We get the Teichmüller space

$$\mathcal{T}(S) = (\mathbb{R}_+ \times \mathbb{R})^{3g-3+n}.$$

The same hyperbolic surface appears infinitely many times in Teichmüller space, and the identification map

$$\mathcal{T}(S) \to \mathcal{M}(S)$$

comes from a group action.

The same hyperbolic surface appears infinitely many times in Teichmüller space, and the identification map

$$\mathcal{T}(S) \to \mathcal{M}(S)$$

comes from a group action.

The Weil-Petersson 2-form

$$\Omega_{WP} := \mathrm{d}\ell_1 \wedge \mathrm{d}\tau_1 + \ldots + \mathrm{d}\ell_{3g-3+n} \wedge \mathrm{d}\tau_{3g-3+n}$$

is invariant under this group action, and makes $\mathcal{M}(S)$ a symplectic manifold.

Tweaking the boundary

Instead of uniformizing to have cusps at the punctures, we can ask for geodesics with specified boundary lengths $\vec{L} = (L_1, \ldots, L_n)$.

Tweaking the boundary

Instead of uniformizing to have cusps at the punctures, we can ask for geodesics with specified boundary lengths $\vec{L} = (L_1, \ldots, L_n)$.

- Repeat previous steps
  $\Rightarrow$ a new 2-form $\Omega_{WP}(\vec{L})$ on $\mathcal{M}(S)$ for each $\vec{L}$.

Tweaking the boundary

Instead of uniformizing to have cusps at the punctures, we can ask for geodesics with specified boundary lengths $\vec{L} = (L_1, \ldots, L_n)$.

- Repeat previous steps
  $\Rightarrow$ a new 2-form $\Omega_{WP}(\vec{L})$ on $\mathcal{M}(S)$ for each $\vec{L}$.
- The volume of $\mathcal{M}(S)$ for $\Omega_{WP}(\vec{L})^{(3g-3+n)}$ is a rational polynomial in π^2 and L_i^2!

Tweaking the boundary

Instead of uniformizing to have cusps at the punctures, we can ask for geodesics with specified boundary lengths $\vec{L} = (L_1, \dots, L_n)$.

- Repeat previous steps
  $\Rightarrow$ a new 2-form $\Omega_{WP}(\vec{L})$ on $\mathcal{M}(S)$ for each $\vec{L}$.
- The volume of $\mathcal{M}(S)$ for $\Omega_{WP}(\vec{L})^{(3g-3+n)}$ is a rational polynomial in π^2 and L_i^2!
- Shove the coefficients in a generating function and exponentiate to get a solution to the KdV equations!

What happens if we specify cone-angles at the punctures instead?

What happens if we specify cone-angles at the punctures instead?

Up to cone-angle π, everything we've talked about still holds true.

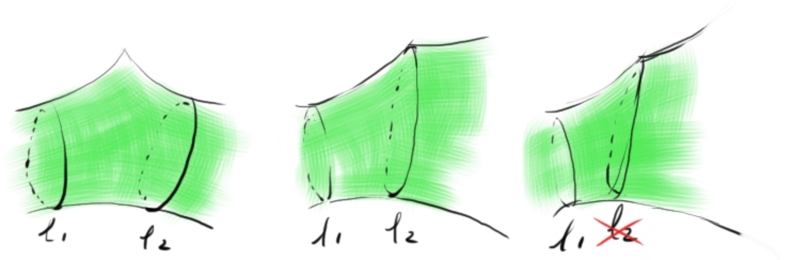
What happens between π and 2π?

What happens between π and 2π?

We lose (unique) geodesic representatives.

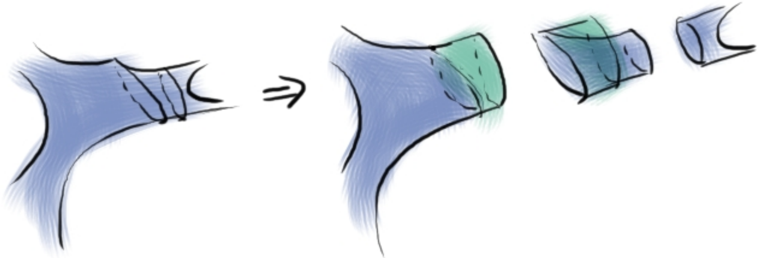
What happens between π and 2π?

We lose (unique) geodesic representatives.



To fix this: *push past* the cone-point.

When cutting surface along given broken geodesics, extend (and sometimes retract) our surface to obtain *phantom pants* or *phants* with geodesic boundary.



The upshot: we can now define length and twist parameters using these phants pieces!

The upshot: we can now define length and twist parameters using these phants pieces!

- Teichmüller spaces $\mathcal{T}(S)$ lives in the representation variety $\mathcal{R}(\pi_1(S), \mathbb{P}\mathrm{SL}_2(\mathbb{R}))$.

The upshot: we can now define length and twist parameters using these phants pieces!

- Teichmüller spaces $\mathcal{T}(S)$ lives in the representation variety $\mathcal{R}(\pi_1(S), \mathbb{PSL}_2(\mathbb{R}))$.
- $T_{[\phi]}\mathcal{R}(\pi_1(S), \mathbb{PSL}_2(\mathbb{R})) \cong \mathrm{H}^1(\pi_1(S), \mathfrak{sl}_2(\mathbb{R})_{Ad\circ\phi})$.

The upshot: we can now define length and twist parameters using these phants pieces!

- Teichmüller spaces $\mathcal{T}(S)$ lives in the representation variety $\mathcal{R}(\pi_1(S), \mathbb{PSL}_2(\mathbb{R}))$.
- $T_{[\phi]}\mathcal{R}(\pi_1(S), \mathbb{PSL}_2(\mathbb{R})) \cong \mathrm{H}^1(\pi_1(S), \mathfrak{sl}_2(\mathbb{R})_{Ad\circ\phi})$.
- Weil-Petersson form Ω_{WP} is a cup-product on $\mathrm{H}^1(\pi_1(S), \mathfrak{sl}_2(\mathbb{R}))$.

The upshot: we can now define length and twist parameters using these phants pieces!

- Teichmüller spaces $\mathcal{T}(S)$ lives in the representation variety $\mathcal{R}(\pi_1(S), \mathbb{P}\mathrm{SL}_2(\mathbb{R}))$.
- $T_{[\phi]}\mathcal{R}(\pi_1(S), \mathbb{P}\mathrm{SL}_2(\mathbb{R})) \cong \mathrm{H}^1(\pi_1(S), \mathfrak{sl}_2(\mathbb{R})_{Ad\circ\phi})$.
- Weil-Petersson form Ω_{WP} is a cup-product on $\mathrm{H}^1(\pi_1(S), \mathfrak{sl}_2(\mathbb{R}))$.

Our phants coordinates are very representation theoretic, so these coordinates willl produce the "correct" Weil-Petersson form.