On De-Klein-ing Lie's

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A long long time ago. . .

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- Projective geometry: "lines are infinitely large circles."

How can we formalise this idea that lines are just BIG circles?

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3. The points at ∞ are the ones where the first coordinate is 0:

$$[0 : x : y]$$

To be precise, the projective n-space $\mathbb{P}^n$ is given by:

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and we have the n-space $\mathbb{R}^n$ sitting inside of it as:

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Example 2: $\mathbb{P}^2 = \mathbb{R}^2 \cup \{[0 : x, y]\} = \mathbb{R}^2 \cup \mathbb{P}^1 = \mathbb{R}^2 \cup \mathbb{R}^1 \cup \mathbb{R}^0$.

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Yes: scaling by a commutes with multiplication by a matrix M.

$$M \begin{bmatrix} ax_0 \\ ax_1 \\ \cdot \\ \cdot \\ \cdot \\ ax_n \end{bmatrix} = M\, a \cdot \begin{bmatrix} x_0 \\ x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = a \cdot M \begin{bmatrix} x_0 \\ x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}.$$

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The above $1 - 2 - 3$ structure is Klein's *Erlangen program*.

The "points" in this geometry can be thought of as lines in $\mathbb{R}^{n+1}$ going through the origin.

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Spoiler: they're the *same* geometry!

Plücker's line geometry

A line in $\mathbb{R}^3$ is given by two points on the line:

$$u = (u_1, u_2, u_3),\ v = (v_1, v_2, v_3)$$
$$\Rightarrow 3 + 3 = 6 \text{ dimensions.}$$

Each one of these points is free to move along the line,

$$\Rightarrow\ 6 - 1 - 1 = 4 \text{ dimensions worth of lines.}$$

Instead of giving a line in $\mathbb{R}^3$ in terms of u and v, just knowing the following two vectors suffice:

$$d := v - u, \;\; m := u \times v.$$

Changing the position of u or v,

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Let's see *why*. . .

So, we should really think of the pair d and m as one point in $\mathbb{P}^5$:

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The space of lines in $\mathbb{R}^3$ is:

$$\mathcal{L} = \{\ [d : m] \in \mathbb{P}^5 :\ d_1 m_1 + d_2 m_2 + d_3 m_3 = 0\}.$$

Intersecting lines

Fun fact $1A$:

Given this way of specifying lines, two lines in $\mathbb{R}^3$ given by

$$[d_1, d_2, d_3, m_1, m_2, m_3] \text{ and } [d_1', d_2', d_3', m_1', m_2', m_3']$$

intersect if and only if

$$d_1 m_1' + d_1' m_1 + d_2 m_2' + d_2' m_2 + d_3 m_3' + d_3' m_3 = 0$$

Erlangen

We've got a space $\mathcal{L}$ (part 1 of Erlangen).

Each point in $\mathcal{L}$ is a line in $\mathbb{R}^3$ (part 3 of Erlangen).

Which matrices act on $\mathcal{L}$ (part 2 of Erlangen)?

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Easy/cheap answer: the ones that send lines to lines in $\mathbb{R}^3$
$\Rightarrow$ the 6×6 matrices that preserve

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Fun fact $2A$:

Any 6×6 matrix that preserves

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So, these matrices (i.e.: symmetries of the line geometry) preserve intersections between lines!

Onto Lie's. . . *stuff*

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So we see that the spheres in $\mathbb{R}^3$ are given by:

$$\{(B, C, D, E, R) \in \mathbb{R}^5 : \; E = B^2 + C^2 + D^2 - R^2\}$$

Let's Projectivise!

We can projectivise our current set of spheres in $\mathbb{R}^3$ by projectivising:

$$(B, C, D, E, R) \mapsto [a : aB : aC : aD : aE : aR] = [a : b : c : d : e : r].$$

Therefore, our set of spheres now becomes:

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Projectivising adds stuff at infinity, so what have we just added?

Tangent spheres

Fun fact $1B$:

Given this way of specifying spheres, two spheres in $\mathbb{R}^3$ given by

$$[a : b : c : d : e : r] \text{ and } [a' : b' : c' : d' : e' : r']$$

are (internally) tangent to eachother if and only if

$$ae' + a'e - 2b'b - 2c'c - 2d'd + 2r'r = 0$$

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So, these matrices (i.e.: symmetries of the sphere geometry) preserve (internal) tangency between spheres!

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- Their underlying points are solution sets to polynomials on $\mathbb{P}^5$:

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- Their defining equations are almost the same thing:

$$d_1 \mapsto b + ic, m_1 \mapsto b - ic, d_2 \mapsto d - r, m_2 \mapsto d + r$$
$$d_3 \mapsto a, d_4 \mapsto -e$$

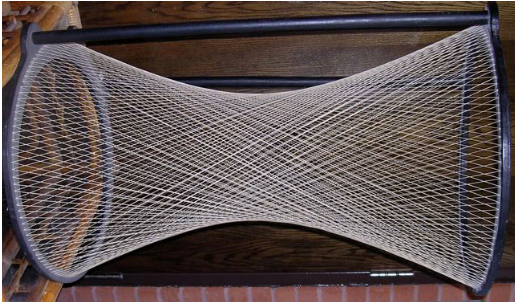
Pictures Please?

Geometry: points, curves, surfaces, spaces. . .

We have points, what are lines?

Answer: ruled surfaces - surfaces in $\mathbb{R}^3$ made out of straight lines.

A hyperboloid of 1-sheet can be made from a family of straight lines in two ways,



so that each line in family 1 intersects every other line in family 2.

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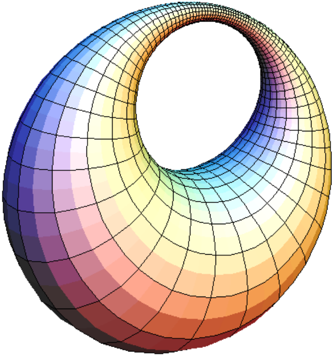
- each sphere in family 1 needs to tangentally touch spheres in family 2.

A Dupin's cylide! (a special type of torus)



Google "metamathological" for the gory details.