Notes for "*The Geometry of Markoff Numbers*" by Caroline Series.

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The point of this note is to explain the story [Coh55, Coh71] between real quadratic forms and simple closed geodesics on 1-cusped hyperbolic tori. Specifically, I want to gain some understanding for *Markoff's theorem for quadratic forms*:

Theorem 1 (Markoff)**.** *Given an arbitrary real quadratic form*

$$f(x,y) = ax^2 + bxy + cy^2 \textit{ with } \text{discriminant } \Delta(f) = b^2 - 4ac,$$

there are integer pairs $(x,y) \in \mathbb{Z} \times \mathbb{Z}$ *such that*

$$0 < \frac{|f(x,y)|}{\sqrt{\Delta(f)}} \leqslant \frac{1}{3} \textit{ unless if } f \textit{ is a } \text{Markoff form.}$$

There are countably many Markoff forms and they take the following form:

$$px^2 + (3p - 2a)xy + (b - 3a)y^2,$$

where $0 < a < \frac{p}{2}, a \equiv \pm r \mod p, bp - a^2 = 1$ *and* (p,q,r) *is a* Markoff triple. *That is, a triple of positive integers satisfying:*

$$p^2 + q^2 + r^2 = 3pqr.$$

I use Series's paper [Ser85] as a basis for these notes. In fact, she also uses this technology to explain a classification of irrational numbers based on how they may be approximated by truncations of their continued fraction representation.

1. Language

The 1-cusped torus S_0 is the unique 1-cusped torus with order 6 isometry group. It can be specified by the following (lift of the) monodromy representation (from $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{Isom}^+(\mathbb{H})$ into $\mathrm{SL}_2(\mathbb{R})$):

$$\rho_0 : \pi_1(S_0) \cong \langle \alpha, \beta \rangle \to \mathrm{SL}_2(\mathbb{Z}) \leqslant \mathrm{SL}_2(\mathbb{R})$$
$$\alpha \mapsto \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \beta \mapsto \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

Since S_0 is a quotient manifold of $\mathbb{H}$ by $\rho_0(\pi_1(S_0))$, we say that a geodesic on $\mathbb{H}$ *projects* to a geodesic on S_0.

2. Strategy

There are three key steps to proving this result:

1. Show that (all but countably many of) the compactly-supported simple geodesics on the order 6-symmetric 1-cusped torus S_0 are precisely the geodesics which don't meet the length 4 horocycle on S_0. That is to say: all self-intersecting geodesics meet the length 4 horocycle.
2. Use matrices M corresponding to simple closed geodesics (via a particular monodromy representation of S_0) to construct the Markoff forms Q_M and use the characterisation established by the previous step to obtain the following inequality:

$$Q_M(1,0) < \tfrac{1}{3}\sqrt{\Delta(Q_M)}.$$

3. Reformulate these Markoff forms in terms of Markoff triples instead of monodromy representation matrices.

We don't go through the third step because Series doesn't cover it in the paper and I have neither learned nor derived it. I'm going to outline everything in dot points — it helps me think.

3. Proof Outine

Let's start with Markoff's "classification" of irrational numbers. We assume for now a few geometric facts that we later encounter. E.g.: simple closed geodesics on S_0 do not meet the length 4 horocycle on S_0.

- Irrational numbers θ can be approximated by a sequence of rationals $\frac{p_n}{q_n}$. This is called a *good approximation* if there is a constant c such that:

$$\left|\theta - \frac{p_n}{q_n}\right| < \frac{c}{q_n^2}.$$

- These fractions $\{\frac{p_n}{q_n}\}$ are called the *convergents*. They are the n-the step truncations of the continued fraction for $\theta = [n_0, n_1, n_2, \ldots]$.
- Define $\nu(\theta)$ to be the infimum of all of the possible c:

$$\nu(\theta) := \inf\left\{c : \left|\theta - \frac{p}{q}\right| < \frac{c}{q^2} \text{ for infinitely many integers } q\right\}.$$

- Markoff [1879] showed that there's a discrete set of values ν_i decreasing to $\frac{1}{3}$ so that if $\nu(\theta) > \frac{1}{3}$, then $\nu(\theta) = \nu_i$ for some i. These $\{\nu_i\}$ are called the *Markoff spectrum* and the corresponding irrational numbers θ are called *Markoff irrationalities*.
- $\nu(\theta) \leqslant \frac{1}{\sqrt{5}}$, and equality is attained if and only if $\theta = [n_0, \ldots, n_k, 1, 1, 1, \ldots]$. Although Series doesn't (to my knowledge) explain how one obtains this result, it's probably got something to do with the three geodesics of shortest length [sic] on S_0.
- Series tesselates the universal cover $\mathbb{H}$ of S_0 with ideal triangles and paraphrases geodesics in terms of sequences specifying which ideal triangles they hit. She defines *periodic* and *characteristic* sequences.

- For any $\theta \in \mathbb{R}$, consider the "vertical" geodesic $\sigma(\theta)$ in the hyperbolic plane $\mathbb{H}$ joining the ideal points θ and ∞. This projects to a bi-infinite geodesic on S_0. We partition the set of irrational numbers θs into three classes:

  1. $\sigma(\theta)$ projects to a simple bi-infinite geodesic with one end going up into the cusp and the other end spiralling into a simple closed geodesic ($\sigma(\theta)$ is *eventually-characteristic* and *eventually-periodic*);
  2. $\sigma(\theta)$ projects to a simple bi-infinite geodesic with one end going up into the cusp and the other end spiralling into a leaf of some geodesic lamination that can be approximated by a sequence of simple closed geodesics ($\sigma(\theta)$ is *eventually-characteristic* and *never-periodic*);
  3. $\sigma(\theta)$ projects to a non-simple bi-infinite geodesic with at least one end going up into the cusp ($\sigma(\theta)$ is *never-characteristic*).

- Note that we ignore *rationalities* ($\theta \in \mathbb{Q}$), for which $\sigma(\theta)$ project to bi-infinite geodesics that (may self-intersect finitely many times and) have both ends up the cusp.
- The collection of *irrationalities* ($\theta \notin \mathbb{Q}$) falling within case 1 correspond to Markoff irrationalities. To see this, use the following fact:
- An arbitrary matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2,\mathbb{R})$ takes the following lift of the length 4 horocycle on S_0

  $$h_{\frac{3}{2}} := \left\{ z \in \mathbb{C} \mid \Im(z) = \tfrac{3}{2} \right\}$$

  to a horocycle in $\mathbb{H}$ tangent to $\frac{a}{c} \in \mathbb{R}$ and with Euclidean radius $\frac{1}{2\times\frac{3}{2}c^2} = \frac{1}{3c^2}$.
- Case 1 bi-infinite geodesics $\sigma(\theta)$ start within the cuspidal region bounded by the length 4 horocycle, but then spiral arbitrarily close one simple closed geodesic. Thus, they eventually get out of this cuspidal region for good.
- The following three conditions are equivalent:

  1. $\sigma(\theta)$ meets finitely many horocycles in $\rho_0(\pi_1(S_0)) \cdot h_{\frac{3}{2}}$;
  2. $|\theta - \frac{p}{q}| < \frac{1}{3q^2}$ for only finitely many $\frac{p}{q}$;
  3. $\nu(\theta) > \frac{1}{3}$.

  And this, combined with the previous two dot points, shows that Case 1 (when $\sigma(\theta)$ spirals to simple closed geodesics) corresponds to Markoff irrationalitie.
- For Case 2, the bi-infinite geodesics $\sigma(\theta)$ corresponds to a geodesic lamination and leaves the length 4 horocycle but then gets aribtrarily close (back) to the length 4 horocycle. Case 2 irrationalities θ satisfy $\nu(\theta) = \frac{1}{3}$, since for any $c > \frac{1}{3}$ we could find some a collection of $\frac{p}{q}$ so that

  $$\left|\theta - \frac{p}{q}\right| < \frac{c}{q^2} \text{ for infinitely many integers } q.$$

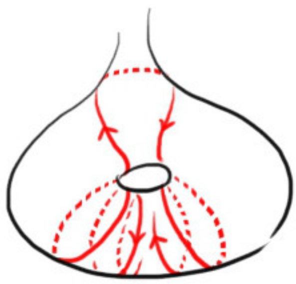

Figure 1: An example of a simple compactly-supported geodesic which meets the length 4 horocycle.

- For Case 3, we need to show that self-intersecting geodesics necessarily enter into the cuspidal region bounded by the length 4 horocycle. Thus concluding that $\nu(\theta) < \frac{1}{3}$ for all Case 3 geodesics. This is easy to see this using fundamental domains on $\mathbb{H}$, but it would be nice to have a more intrinsic way of proving this.

We turn now to Step 1 of the proof of Markoff's theorem for quadratic forms:

- We begin with the following easy fact, which I proved differently using a little trigonometry:

  Lemma 2. *No simple closed geodesic touches the length 4 horocycle on S_0.*

  It's fairly easy to show using trigonometry that this fact holds for any 1-cusped hyperbolic torus.

- The above lemma means that (compactly-supported simple) geodesics which may be arbitrarily closely approximated by simple closed geodesics, also can't touch the length 4 horocycle. These simple geodesics all arise as leaves of geodesic laminations. In fact, the only type of leaf of a compactly-supported geodesic lamination is a simple bi-infinite geodesic whose two ends spiral into the same simple closed geodesic (figure 1) on S_0 so that they're spiralling in the "opposite" direction[1].

- Moreover, the longer simple closed geodesics on S_0 get closer to the length 4 horocycle on S_0. Hence, geodesics which may be arbitrarily closely approximated by a sequence of length-increasing simple closed geodesics, get arbitrarily close to the length 4 horocycle.

- A much deeper result that I haven't been able to prove using just trigonometry[2] is the "converse" to the above result:

  Lemma 3. *Self-intersecting geodesics necessarily meet the length 4 horocycle.*

[1] I'm using Series's language here, and I assert that her figure 9 is incorrect.

[2] I have verified this fact geometrically for once self-intersecting closed geodesics and I wonder if it's possible to prove this by proving it for the collection of all self-intersecting closed geodesics.

The rough idea of the proof that Series gives[3] is to use a change of marking (in the Teichmüller theory sense of the word) to position a self-intersecting geodesic in such a way that it has a subsegment that meets our ideal triangulation (or rather, ideal quadrilateral tesselation) of $\mathbb{H}$ in one of two ways — both of which lead to hitting the length 4 horocycle on S_0.

- Haas shows that the above two lemmas hold for any 1-cusped torus, not just S_0 [Haa87, Haa88].

Step 2 is to use these simple closed geodesics to reproduce Markoff forms, albeit not yet as stated in Markoff's theorem.

- Given a matrix $M := \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \rho_0(\pi_1(S_0))$ corresponding to a geodesic in $\mathbb{H}$ that projects to a simple closed geodesic on our 1-cusped torus, the ideal fixed points of M are given by:

  $$\xi_M = \frac{(a-d) + \sqrt{(a-d)^2 - 4bc}}{2c} \text{ and } \xi'_M = \frac{(a-d) - \sqrt{(a-d)^2 - 4bc}}{2c}.$$

  We know from Step 1 that every Markoff irrationality arises from ξ_M or ξ'_M for some M.

- Using the universal cover, we know that $|\xi_M - \xi'_M| < 2 \times \frac{3}{2} = 3$. And it's easy to algebraically check that

  $$|\xi_M - \xi'_M| = \frac{\sqrt{(\mathrm{Tr}M)^2 - 4}}{Q_M(1,0)} = \frac{\sqrt{\Delta(Q_M)}}{Q_M(1,0)},$$

  for the quadratic form $Q_M(x,y) = cx^2 + (d-a)xy - by^2$. Therefore,

  $$Q_M(1,0) > \tfrac{1}{3}\sqrt{(\mathrm{Tr}M)^2 - 4} = \tfrac{1}{3}\sqrt{\Delta(Q_M)}.$$

- For any $g \in SL_2(\mathbb{R})$,

  $$Q_M(x,y) = Q_{gMg^{-1}}(g \cdot (x,y)). \tag{1}$$

  And for any coprime pair $(x,y) \in \mathbb{Z} \times \mathbb{Z}$, the matrix

  $$\left[\begin{smallmatrix} a & b \\ -y & x \end{smallmatrix}\right], \text{ where } ax + by = 1,$$

  is a matrix in $SL_2(\mathbb{Z})$ taking (x,y) to $(1,0)$. Therefore,

  $$\min_{(x,y)\in\mathbb{Z}^2} \frac{Q_M(x,y)}{\sqrt{(\mathrm{Tr}M)^2 - 4}} = \min_{(x,y)\in\mathbb{Z}^2} \frac{\gcd(x,y)^2 \times Q_{gMg^{-1}}(1,0)}{\sqrt{(\mathrm{Tr}M)^2 - 4}} > \frac{1}{3}.$$

  Which is precisely the statement that these *Markoff forms* Q_M must satisfy (as per Markoff's theorem).

[3] I think that it might be her own proof, since Haas's seems to be a bit more topological, although to be honest, I don't understand his proof yet.

I don't yet know how to do step 3, but it's certainly a reasonable result. This is in light of the fact that the Teichmüller space of 1-cusped torus may be modelled on a semi-algebraic set given by the defining equation for Markoff triples (with a minor renormalisation) [Hua14, Prop 3.8].

4. Generalisations

Generalisation 1: conjugation.

The monodromy representation ρ_0 given for S_0 is by no means canonical, and we can obtain an immediate generalisation by conjugating this monodromy representation by a matrix

$$g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in SL_2(\mathbb{R}).$$

Let the *skewed lattice* $g \cdot (\mathbb{Z} \times \mathbb{Z})$ denote the rank 2 Abelian group

$$g \cdot (\mathbb{Z} \times \mathbb{Z}) = \{m(\alpha, \gamma) + n(\beta, \delta) \mid m, n \in \mathbb{Z}\}.$$

Proposition 4. *Given an arbitrary real quadratic form*

$$f(x, y) = ax^2 + bxy + cy^2 \textit{ with discriminant } \Delta(f) = b^2 - 4ac,$$

there are pairs $(x, y) \in g \cdot (\mathbb{Z} \times \mathbb{Z})$ *such that*

$$0 < \frac{|f(x, y)|}{\sqrt{\Delta(f)}} \leqslant \frac{1}{3} \textit{ except when } f \textit{ is a } \text{skewed Markoff form.}$$

There are countably many skewed Markoff forms $f_i(x, y)$ *and they each take the form:*

$$\begin{aligned} f_i(x, y) = &\left(\delta^2 p + \gamma\delta(2a - 3p) + \gamma^2(b - 3a)\right) x^2 \\ &+ (2\alpha\gamma(3a - b) + (\alpha\delta + \beta\gamma)(3p - 2a) - 2\beta\delta p)\, xy \\ &+ \left(\beta^2 p + \alpha\beta(2a - 3p) + \alpha^2(b - 3a)\right) y^2, \end{aligned}$$

where $0 < a < \frac{p}{2}, a \equiv \pm r \mod p, bp - a^2 = 1$ *and* (p, q, r) *is a* Markoff triple. *That is, a triple of positive integers satisfying:*

$$p^2 + q^2 + r^2 = 3pqr.$$

Proof. We know from Theorem 1 that for every matrix $M \in \rho_0(\pi_1(S_0)) \lneq SL_2(\mathbb{Z})$ that corresponds to a simple closed geodesic,

$$Q_M(m, n) > \tfrac{1}{3}\sqrt{\Delta(Q_M)} \text{ for every } (m, n) \in \mathbb{Z} \times \mathbb{Z}.$$

Since (1) holds for any $g \in SL_2(\mathbb{R})$,

$$\begin{aligned} \min_{(m,n)\in g\cdot(\mathbb{Z}\times\mathbb{Z})} \frac{Q_{gMg^{-1}}(m, n)}{\sqrt{\Delta(Q_{gMg^{-1}})}} &= \min_{(m,n)\in g\cdot(\mathbb{Z}\times\mathbb{Z})} \frac{Q_M(g^{-1} \cdot (m, n))}{\sqrt{(\mathrm{Tr}(gMg^{-1}))^2 - 4}} \\ &= \min_{(m',n')\in\mathbb{Z}\times\mathbb{Z}} \frac{Q_M(m', n')}{\sqrt{\Delta(Q_M)}} > \frac{1}{3}. \end{aligned}$$

By the same computations, quadratic forms which are not skewed Markoff forms have minima $\leqslant \frac{1}{3}$.

Denoting M component-wise by $\left[\begin{smallmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{smallmatrix}\right]$, we know that

$$Q_M(x, y) = m_{21}x^2 + (m_{22} - m_{11})xy - m_{12}y^2$$

and hence

$$m_{21} = p, \ m_{22} - m_{11} = 3p - 2a \text{ and } m_{12} = 3a - b \tag{2}$$

for some Markoff triple (p, q, r). Substituting (2) into $Q_{gMg^{-1}}$ yields the desired presentation for skewed Markoff forms. □

There are two other ideas that I'd like to try:

Generalisation 2: use 1-cusped tori which aren't isometric to S_0. This shouldn't be too hard, although there are two potential issues here: first, I'm not sure yet how to characterise the ideal points; the second lies in the fact that whereas $SL_2(\mathbb{Z})$ was a isometry supergroup of $\rho_0(\pi_1(S_0))$ that let us take any coprime pair (m, n) to $(1, 0)$ — I'm not sure yet if this will be an issue.

An update: I did manage to (more-or-less) do this. But it turns out that it's already a reasonably well-known [Sch76, Sch77].

Generalisation 3: use 3-cusped projective planes? I make this suggestion because of the Markoff quads that Norbury and I discovered/defined [HN13]. I'm not sure yet how to geometrically make use of these gadgets to obtain some type of Markoff theorem. Perhaps there's some scope for looking at pairs of ideal points or something?

References

[Coh55] Harvey Cohn, *Approach to Markoff's minimal forms through modular functions*, Ann. of Math. (2) **61** (1955), 1–12. MR 0067935 (16,801e) 1

[Coh71] ______, *Representation of Markoff's binary quadratic forms by geodesics on a perforated torus*, Acta Arith. **18** (1971), 125–136. MR 0288079 (44 #5277) 1

[Haa87] Andrew Haas, *The geometry of Markoff forms*, Number theory (New York, 1984–1985), Lecture Notes in Math., vol. 1240, Springer, Berlin, 1987, pp. 135–144. MR 894509 (88j:11037) 5

[Haa88] ______, *Diophantine approximation on hyperbolic orbifolds*, Duke Math. J. **56** (1988), no. 3, 531–547. MR 948532 (90c:11031) 5

[HN13] Yi Huang and Paul Norbury, *Simple geodesics and markoff quads*, preprint (2013), arXiv:1312.7089. 7

[Hua14] Yi Huang, *Moduli spaces of surfaces*, Ph.D. thesis, June 2014. 6

[Sch76] Asmus L. Schmidt, *Minimum of quadratic forms with respect to Fuchsian groups. I*, J. Reine Angew. Math. **286/287** (1976), 341–368. MR 0457358 (56 #15566) 7

[Sch77] ———, *Minimum of quadratic forms with respect to Fuchsian groups. II*, J. Reine Angew. Math. **292** (1977), 109–114. MR 0457359 (56 #15567) 7

[Ser85] Caroline Series, *The geometry of Markoff numbers*, Math. Intelligencer **7** (1985), no. 3, 20–29. MR 795536 (86j:11069) 1