*Based on pages 2-4 of "Introduction to the Weil Conjectures", by Runar Ile.*

Let $k$ be a finite field of order $|k| = q = p^\epsilon$, where $p$ is prime, and let $\bar{k}$ be the algebraic closure of $k$.[1] For $m \in \mathbb{Z}_{>0}$, let $k_m$ be the unique field extension of $k$ such that

$$[k_m : k] = m.$$

Then $|k_m| = q^m$ and

$$k = k_1 \subseteq k_m \subsetneq \bar{k}.$$

Let

$$X \subseteq \bar{k}\mathbb{P}^D$$

be a projective variety defined by equations with coefficients in $k$. For $m \in \mathbb{Z}_{>0}$, let $X(k_m)$ be the set of points in $X$ with coordinates in $k_m$, and let

$$N_m := |X(k_m)| \leq |k_m|^D.$$

The *zeta function* of $X$ over $k$ is given by the formal power series

$$Z(X,t) := \exp \sum_{m=1}^{\infty} N_m \frac{t^m}{m} \in 1 + t \cdot \mathbb{Q}[[t]]. \tag{1}$$

Note that this is a transformation of the usual zeta function, with $t = q^{-s}$:

$$\zeta(X,s) := \exp \sum_{m=1}^{\infty} N_m \frac{(q^{-s})^m}{m}. \tag{2}$$

**Theorem 1** (Weil conjectures)**:** *Assume that $X$ is nonsingular and $d$-dimensional.*

1. *(Rationality) There exist $P, Q \in \mathbb{Q}[t]$ such that*
$$Z(X,t) = \frac{P(t)}{Q(t)}.$$
2. *(Poincaré duality) Let $\chi$ be the Euler characteristic of $X$. Then*
$$Z\Big(X, \frac{1}{q^d t}\Big) = \pm q^{d\chi/2} t^\chi Z(X,t). \tag{3}$$
3. *(Riemann hypothesis)*
$$Z(X,t) = \frac{P_1(t) \cdot P_3(t) \cdots P_{2d-1}(t)}{P_0(t) \cdot P_2(t) \cdots P_{2d}(t)}, \tag{4}$$
*where*
   - $$P_0(t) := 1 - t, \qquad P_{2d} := 1 - q^d t.$$
   - *For $r = 1, 2, \ldots, 2d-1$,*
$$P_r(t) = \prod_{j=1}^{\beta_r} (1 - \alpha_{r,j} t),$$
*for some $\beta_r \in \mathbb{Z}_{\geq 0}$, and for some $\alpha_{r,1}, \ldots, \alpha_{r,\beta_r}$, where*
$$|\alpha_{r,j}| = q^{r/2}, \qquad j = 1, 2, \ldots, \beta_r.$$

[1] Note that $\bar{k}$ must be infinite.

*In particular, the roots of $P_r$ (as a function of s) lie on the* critical line
$$\left\{s \in \mathbb{C} : \mathrm{Re}(s) = \frac{r}{2}\right\}. \tag{5}$$

4. *(Betti numbers) For $r = 1, 2, \ldots, 2d-1$, our $\beta_r$ is the rth Betti number of X, i.e.*
$$\beta_r = \operatorname{rank} H_r(X). \tag{6}$$
*We might discuss the remainder of this conjecture another day. Note that*
$$\operatorname{rank} H_r(X) := \operatorname{rank} H_r(X, \mathbb{Z}) = \operatorname{rank} H_r(X, \mathbb{Q}), \tag{7}$$
*which follows from the universal coefficient theorem.*

**Example.**
$$X = \bar{k}\mathbb{P}^d = \sqcup_{j=0}^{d} \mathbb{A}_{\bar{k}}^{j}. \tag{8}$$
Then
$$X(k_m) = \sqcup_{j=0}^{d} \mathbb{A}_{k_m}^{j} \subseteq X, \tag{9}$$
so
$$N_m = 1 + q^m + \ldots + q^{dm} = \frac{q^{(d+1)m} - 1}{q^m - 1}.$$
Note that
$$X(k_m) = \frac{\mathbb{A}_{k_m}^{d+1} \setminus \{\mathbf{0}\}}{k_m^\times}, \tag{10}$$
where $k_m^\times$ acts on $\mathbb{A}_{k_m}^{d+1} \setminus \{\mathbf{0}\}$ by multiplication. It can be shown[2] that the Betti numbers are
$$\operatorname{rank} H_r(X) = \begin{cases} 1, & r = 0, 2, \ldots, 2d \\ 0, & \text{otherwise,} \end{cases} \tag{11}$$
and so $\chi = d + 1$. Let's run through the conjectures for this example.

1. In this case,
$$\begin{aligned} Z(X, t) &= \exp \sum_{m=1}^{\infty} N_m \frac{t^m}{m} = \exp \sum_{m=1}^{\infty} (1 + q^m + \ldots + q^{dm}) \frac{t^m}{m} = \exp \sum_{j=0}^{d} \sum_{m=1}^{\infty} \frac{(q_j t)^m}{m} \\ &= \exp \sum_{j=0}^{d} - \log(1 - q^j t) = \prod_{j=0}^{d} \frac{1}{1 - q^j t} = \frac{P(t)}{Q(t)}, \end{aligned} \tag{12}$$
where $P(t) = 1 \in \mathbb{Q}[t]$ and $Q(t) = 1 - q^j t \in \mathbb{Q}[t]$.
2. From equation (12),
$$\begin{aligned} Z\Big(X, \frac{1}{q^d t}\Big) = \prod_{j=0}^{d} \Big(1 - \frac{q^j}{q^d t}\Big)^{-1} &= \frac{q^{d(d+1)/2} t^{d+1}}{\prod_{j=0}^{d} (q^j t - 1)} = (-1)^{d+1} q^{d(d+1)/2} t^{d+1} Z(X, t) \\ &= \pm q^{d\chi/2} t^{\chi} Z(X, t), \end{aligned}$$
since $\chi = d + 1$.
3. Now $Z(X, t)$ is in the form of the Riemann hypothesis:
$$Z(X, t) = \frac{P_1(t) \cdot P_3(t) \cdots P_{2d-1}(t)}{P_0(t) \cdot P_2(t) \cdots P_{2d}(t)}.$$

[2] Intuitively, this is because $\mathbb{CP}^d$ is the disjoint union of a 0-cell, a 2-cell, ..., and a 2d-cell, now regarding it as a real manifold.

We now compute $\beta_1, \ldots, \beta_{2d-1}$. Perhaps the most elegant way to see that no cancellation may occur here is to define

$$\beta_0 := \beta_{2d} := 1, \qquad \alpha_{0,1} := 1, \qquad \alpha_{2d,1} := q^d, \tag{13}$$

so that, for $r = 0, 1, \ldots, 2d$,

$$P_r(t) = \prod_{j=1}^{\beta_r} (1 - \alpha_{r,j} t),$$

where

$$|\alpha_{r,j}| = q^{r/2}, \qquad j = 1, 2, \ldots, \beta_r.$$

Since there can be no cancellation, we use the equation

$$\prod_{j=0}^{d} \frac{1}{1 - q^j t} = \frac{P_1(t) \cdot P_3(t) \cdots P_{2d-1}(t)}{P_0(t) \cdot P_2(t) \cdots P_{2d}(t)} \tag{14}$$

to deduce the following:

- For $r = 1, 3, \ldots, 2d-1$,
$$P_r(t) = 1 \qquad \therefore \beta_r = 0. \tag{15}$$
- For $r = 2, 4, \ldots, 2d-2$,
$$\beta_r = 1. \tag{16}$$

4. Thus, we see that
$$\beta_r = \operatorname{rank} H_r(X), \qquad r = 1, 2, \ldots, 2d-1.$$

We also want to know whether or not the $k_m$ embed into one another.

**Lemma 2:** *$k_m \subseteq k_n$ if and only if $m \mid n$.*

*Proof.* Define

$$F : \bar{k} \to \bar{k}$$

$$x \mapsto x^q,$$

a Frobenius endomorphism.[3] In this case $F$ is an automorphism: injective because $\bar{k}$ is an integral domain and surjective because $\bar{k}$ is algebraically closed.

For $m \in \mathbb{Z}_{>0}$,

$$k_m = \{z \in \bar{k} : z^{q^m} = z\}, \tag{17}$$

since $\subseteq$ follows from $|k_m^{\times}| = q^m - 1$ and $\supseteq$ then follows from there being at most $q^m$ solutions to equation (17). Thus, if $m|n$ then $k_m \subseteq k_n$.

Conversely, assume that $m$ does not divide $n$. Assume, for the sake of contradiction, that $k_m \subseteq k_n$. Then there exist $t \in \mathbb{Z}_{\geq 0}$ and $r \in \{1, 2, \ldots, m-1\}$ such that

$$n = tm + r.$$

Thus, if $x \in k_m$ then $x \in k_n$ also, in which case (using the fact that $F^{tm}$ is bijective)

$$F^{tm}(x) = x = F^n(x) = F^{tm}F^r(x) \implies x = F^r(x). \tag{18}$$

[3]It is easy to check that $F$ is a well defined ring homomorphism.

This yields $q^m$ distinct solutions to the degree $q^r$ polynomial $x^{q^r} - x$, which is impossible. Thus, $m|n$.

Hence, $k_m \subseteq k_n$ if and only if $m|n$. Now we see that $k_m$ embeds into $k_n$ if and only if $m|n$, since $\bar{k}$ contains precisely one isomorphic copy of $k_m$, namely the one defined in equation (17). $\square$

Thus

$$k_{q^{1!}} \subseteq k_{q^{2!}} \subseteq k_{q^{3!}} \subseteq \ldots \subseteq \cup_{m=1}^{\infty} k_{q^{m!}}. \tag{19}$$

We can show that $k_{q^\infty} = \cup_{m=1}^{\infty} k_{q^{m!}} = \cup_{m=1}^{\infty} k_{q^m}$ is algebraically closed, thereby providing an explicit construction for the algebraic closure.

*Proof.* It's easy to check that it's a field. Let $g(X) \in k_{q^\infty}[X]$ be non-constant. To show: $g(X)$ has a root. There exists $n \in \mathbb{Z}_{>0}$ such that $g(X) \in k_{q^{n!}}[X]$. The spliting field of $g(X) \in k_{q^{n!}}[X]$ is a finite extension[4] of $k_{q^{n!}}[X]$, and is therefore $k_{q^e}[X]$ for some $e \in \mathbb{Z}_{>0}$. So $g(X)$ decomposes into linear factors, and therefore has a root. $\square$

Henceforth,

$$\bar{k} = k_{q^\infty} = \cup_{m=1}^{\infty} k_{q^m}. \tag{20}$$

In other words,

$$\overline{\mathbb{F}}_q = \cup_{m=1}^{\infty} \mathbb{F}_{q^m}. \tag{21}$$

**The Weil conjectures for Grassmannians** *Trithang Tran and Yi Huang*

**Definition 3:** *Fix a field $\mathbb{F}$. The Grassmannian $Gr(m,n)$ (or $Gr(m,n)(\mathbb{F})$ if we want to specify $\mathbb{F}$) as a set is the set of $m$-dimensional vector subspaces of $\mathbb{F}^n$.*

We can topologise $Gr(m,n)$ as follows. An *m-frame* in $\mathbb{F}^n$ is an $m$-tuple of linearly independent vectors in $\mathbb{F}^n$. Let $V_m(\mathbb{F}^n)$ be the collection of $m$-frames in $\mathbb{F}^n$. This is an open subset of $\mathbb{F}^n \times \ldots \times \mathbb{F}^n$ ($m$ times). Consider the canonical map

$$q : V_m(\mathbb{F}^n) \twoheadrightarrow Gr(m,n)$$

that sends a frame to the subspace it spans. We can use $q$ to give $Gr(m,n)$ the quotient topology.[5]

In order to check that $Gr(m,n)$ satisfies the Weil conjectures we need to first compute its zeta function. To do this we need to compute $|Gr(m,n)(\mathbb{F}_q)|$.

We first give a cell decomposition of $Gr(m,n)(\mathbb{F})$ for any field $\mathbb{F}$: the datum of an $m$-dimensional subspace of $\mathbb{F}^n$ may be represented by $m$ linearly independent vectors in $\mathbb{F}^n$ by taking an element of its $q$ preimage. This basis for the given subspace may

[4]The splitting field is always a finite extension. By the explicit construction, one can show that the degree is at most $d!$, where $d$ is the degree of the polynomial.

[5]Specifically, the open sets are of the form $q(U)$, where $U$ is open in $V_m(\mathbb{F}^n)$.

in turn be written as a $m \times n$ matrix with entries in $\mathbb{F}$, and any two such matrices obtained for the same subspace will have the same reduced row echelon form. Partition the points of $Gr(m,n)(\mathbb{F})$ according to the positions of the pivotal 1's for each point in the Grassmannian. Since the positions of the leading ones completely determine all the matrix entries above and to the left of themselves whilst leaving every other entry free to vary, each set in our partition of $Gr(m,n)$ has a natural bijection with $\mathbb{F}^j$, for some $j$ such that $0 \leq j \leq m(n-m)$. If we topologise the Grassmanian as above, this bijection is a homeomorphism, and our partition is a cell-decomposition of $Gr(m,n)$. This tells us that the dimension $d$ of $Gr(m,n)$ is $m(n-m)$.[6]

$$\begin{pmatrix} 0 & \dots & 0 & 1 & * & \dots & * & 0 & * & \dots & * & 0 & * & \dots & * \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & * & \dots & * & 0 & * & \dots & * \\ \vdots & & \vdots & & & & & \vdots & & & & \vdots & & & * \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 1 & * & \dots & * \end{pmatrix}$$

Figure 1: An example of a $m \times n$ matrix in reduced row echelon form. This corresponds to a particular cell of dimension equal to the number of $*$'s, which we can think of as free variables.

For each cell in this decomposition, the number of variable entries for the first to the $(n-m)$-th non-pivotal columns is a partition of the dimension $j$ (which is the number of $*$'s) of this cell into $n-m$ parts (the $r$th part is the number of $*$s in the $r$th non-pivotal column), each of size at most $m$ (the number of rows). In particular, the $j$-cells of $Gr(m,n)$ are in bijection with the set of length $\leq (n-m)$ partitions of $j$ with entries $\leq m$. Denoting $p(j)$ as the number of partitions of $j$ into at most $n-m$ parts, each of size at most $m$, we have that:

$$|Gr(m,n)(\mathbb{F}_q)| = \sum_{j=0}^{m(n-m)} p(j)q^j.$$

As an aside, these cells we have described are called *Schubert cells* and have a nice description in terms of *Schubert symbols* $\sigma = (\sigma_1, \dots, \sigma_m)$ where the $\sigma_i$ essentially keep track of where the leading entries of our matrix are. A more thorough description can be found in Chapter 6 of *Characteristic Classes* by Milnor and Stasheff.

We can now write down the zeta function:

$$\begin{aligned} Z(Gr(m,n),t) &= \exp \sum_{i \geq 1} |Gr(n,m)(\mathbb{F}_{q^i})| \frac{t^i}{i} \\ &= \exp \sum_{i \geq 1} \sum_{j=0}^{m(n-m)} p(j) q^{ij} \frac{t^i}{i} \\ &= \sum_{j=0}^{m(n-m)} p(j) \exp \log(1 - q^j t)^{-1} \\ &= \prod_{j=0}^{m(n-m)} \frac{1}{(1-q^j t)^{p(j)}}. \end{aligned}$$

[6] Importantly, this maximum $j = m(n-m)$ is attained (otherwise it wouldn't be the dimension), for instance take $\mathrm{diag}(1,m) \in M_{m,n}(\mathbb{F})$. Try to convince yourself that $m(n-m)$ is the maximum, just by playing around with it. Otherwise show that $j = mn - m(m-1)/2 - |\sigma|$, where $\sigma = (\sigma_1, \dots, \sigma_m)$ is the Schubert symbol; it then follows easily that $j \leq m(n-m)$, with equality only in the example given.

This also proves the rationality part of the Weil conjectures for $Gr(m,n)$.

To prove the second part of the Weil conjectures (functional form), we observe that
$$p(j) = p(d-j), \qquad j = 0, 1, \ldots, d, \tag{22}$$
since partitions $(a_1, \ldots, a_m)$ of $j$ correspond to partitions of $d-j$ via the bijection[7]
$$(a_1, \ldots, a_{n-m}) \longleftrightarrow (m - a_1, \ldots, m - a_{n-m}). \tag{23}$$
Let $X := Gr(m,n)$, and let $\chi$ be the Euler characteristic of $X$. We want to show that
$$Z\Big(X, \frac{1}{q^d t}\Big) = \pm q^{d\chi/2} t^\chi Z(X,t).$$
Using the fact that $p(j) = p(m(n-m) - j)$, and noting that
$$\chi = \sum_{j=1}^{d} p(j),$$
we bash:
$$\begin{aligned} Z\Big(X, \frac{1}{q^d t}\Big) &= \prod_{j=1}^{d} \frac{1}{(1 - \frac{q^{j-d}}{t})^{p(j)}} = \frac{t^{\sum_j p(j)} q^{\sum_j (d-j)p(j)}}{\prod_j (tq^{d-j} - 1)^{p(j)}} \\ &= \frac{t^\chi q^{\sum_j j p(d-j)}}{(-1)^\chi \prod_j (1 - tq^{d-j})^{p(d-j)}} = (-1)^\chi t^\chi q^{\sum_j j[p(d-j) + p(j)]/2} Z(X,t) \\ &= (-1)^\chi t^\chi q^{\sum_j [(d-j)p(j) + jp(j)]/2} Z(X,t) = (-1)^\chi t^\chi q^{d\chi/2} Z(X,t). \end{aligned}$$

For the next part of the Weil conjectures (Riemann hypothesis), we note that our first computation of the zeta function gives us the required form. Take $P_0(t) = 1 - t, P_{m(n-m)} = 1 - q^{m(n-m)}t$,
$$P_r(t) = \prod_{j=1}^{p(r)} (1 - q^{r/2} t)$$
for $1 < r < m(n-m)$ even, and $P_r = 1$ for $1 < r < m(n-m)$ odd. In particular, the Betti numbers are $\beta_r = p(r)$ for $r$ even, and $\beta_r = 0$ for $r$ odd.

Lastly, we note that $\chi(Gr(m,n)) = \chi = \sum_{r=0}^{2d} (-1)^r \beta_r$ which is the last part of the Weil conjectures. Note that this matches with our earlier description of the cellular structure of the Grassmannian.

**Dougal Davis** *Friday 13 January 2012*

Let $k$ be a field of size $q = p^\varepsilon$, where $p$ is prime and $\varepsilon \in \mathbb{Z}_{>0}$, and let $k_m$ be the field of size $q^m$ in $\bar{k}$. Let $X$ be an algebraic variety defined by polynomials with coefficients

[7] The partitions can have 0s at the end. Note the significance of limiting the size of a part to $m$.

in $k$ and (local) coordinates in $\bar{k}$.[8] Let

$$X(k_m) := \{\text{points in } X \text{ with coordinates in } k_m \subseteq k\}, \qquad N_m := |X(k_m)|.$$

The *Frobenius*

$$F : X \to X$$

is defined locally by

$$(a_1, \dots, a_n) \mapsto (a_1^q, a_2^q, \dots, a_n^q).$$

This is well defined locally because if $f \in k[a_1, \dots, a_n]$ then

$$f(a_1, \dots, a_n)^q = f(a_1^q, \dots, a_n^q). \tag{24}$$

Moreover, it can be verified that this map is well defined when overlapping charts are used. Note that if $a \in X$ then $a \in X(k_m)$ if and only if $F^m(a) = a$, which follows from previous Frebenius discussion. Now $X(k_m)$ is the set of fixed points of $F^m$, and $N_m$ is the number of fixed points of $F^m$.

Let us now assume that there is a cohomology theory $H^*(X)$ with properties similar to those of singular cohomology, and see which parts of the Weil conjectures we can deduce. For simplicity, let us assume that the coefficients are in $\mathbb{Q}$. From the Lefschetz fixed point formula,

$$N_m = \sum_{r=0}^{2d} (-1)^r \mathrm{Tr}(H^r(F^m)), \tag{25}$$

where $H^r(F^m) : H^r(X) \to H^r(X)$ is the map induced by $F^m : X \to X$ (via a contravariant functor).[9] The trace makes sense because it inputs a linear transformation from a finite-dimensional vector space (over $\mathbb{Q}$) to itself. Now

$$\begin{aligned} Z(X,t) &= \exp \sum_{m=1}^{\infty} N_m \frac{t^m}{m} \\ &= \exp \sum_{m=1}^{\infty} \frac{t^m}{m} \sum_{r=0}^{2d} (-1)^r \mathrm{Tr}(H^r(F^m)) \\ &= \exp \sum_{r=0}^{2d} (-1)^r \sum_{m=1}^{\infty} \frac{t^m}{m} \mathrm{Tr}(H^r(F^m)) \end{aligned}$$

Let $\beta_r := \dim(H^r(X))$ be the $r$th Betti number, for $r = 0, 1, \dots, 2d$, and let

$$P_r(t) := \det(1 - tH^r(F)) = \prod_{j=1}^{\beta_r} (1 - \alpha_{r,j} t), \tag{26}$$

where the $\alpha_{r,j}$ are the (repeated) eigenvalues of $H^r(F)$. The second equality may be more familiar as

$$\det\Big(\frac{1}{t} - H^r(F)\Big) = \prod_{j=1}^{\beta_r} \Big(\frac{1}{t} - \alpha_{r,j}\Big), \tag{27}$$

[8] This is a comment on local coordinates. A *variety* is a type of *prevariety*. Analagously to manifolds, a *prevariety* is a ringed space (whose structure sheaf is a sheaf of $k$-valued functions) that is locally an affine variety. An *affine variety* is an irreducible algebraic set, where an *algebraic set* is the zero locus of a set of polynomials. Note that the *dimension* of a variety is the minimum number of complex coordinates needed to define it locally.

[9] According to Nepa, this notation is consistent with standard category theory notation.

since each side is the characteristic polynomial of $H^r(F)$ evaluated at $\frac{1}{t}$. Note that

$$\mathrm{Tr}(H^r(F^m)) = \mathrm{Tr}(H^r(F)^m) = \sum_{j=1}^{\beta_r} \alpha_{r,j}^m. \tag{28}$$

Now

$$Z(X,t) = \exp \sum_{r=0}^{2d} (-1)^r \sum_{m=1}^{\infty} \frac{t^m}{m} \sum_{j=1}^{\beta_r} \alpha_{r,j}^m = \exp \sum_{r=0}^{2d} (-1)^r \sum_{j=1}^{\beta_r} \sum_{m=1}^{\infty} \frac{(\alpha_{r,j} t)^m}{m} \tag{29}$$

$$= \exp \sum_{r=0}^{2d} (-1)^r \sum_{j=1}^{\beta_r} -\log(1-\alpha_{r,j} t) = \prod_{r=0}^{2d} P_r(t)^{(-1)^{r+1}} = \frac{P_1(t)\cdots P_{2d-1}(t)}{P_0(t)\cdots P_{2d}(t)}. \tag{30}$$

As $P_r(t) \in \mathbb{Q}[t]$ for $r = 0, 1, \ldots, 2d$, we have proven rationality. We have also shown the Riemann hypothesis, aside from the fact that $|a_{r,j}| = q^{r/2}$ for all $r, j$. We have also shown the Betti number statement. We now derive the functional equation, which indeed comes from Poincaré duality, which again follows from our rather strong assumption concerning the existence of a cohomology theory with particular nice properties.

Given our assumptions, there should be a cup product with the property that if $r \in \{0, 1, \ldots, 2d\}$, $f : X \to X$ is continuous, $x \in H^r(X)$, and $x' \in H^{2d-r}(X)$, then

$$H^r(f)(x) \cup H^{2d-r}(f)(x') = H^{2d}(f)(x \cup x'). \tag{31}$$

Poincaré duality states the following.

(a) There exists an isomorphism

$$\eta : H^{2d}(X) \to \mathbb{Q}. \tag{32}$$

(b) The map

$$\Phi : H^r(X) \to H^{2d-r}(X)^*$$

given by

$$\Phi(x) : x' \mapsto \eta(x \cup x'), \qquad x \in H^r(X), x' \in H^{2d-r}(X)$$

is an isomorphism, where $^*$ denotes the dual vector space.

The following lemma seems plausible, and Dougal seems to think it's easy enough to prove.

**Lemma 4:** *Let $V$ be a finite-dimensional vector space, and let*

$$f : V^* \to V^*, \qquad g : V \to V$$

*be linear transformations such that if $w \in V^*$ then $f(w) = w \circ g$. Then*

$$\det(f) = \det(g).$$

Given the above, we proceed to deduce the functional equation. Fix $r \in \{0, 1, \ldots, 2d\}$, and define

$$H_r(F) : H^r(X) \to H^r(X)$$

by

$$H_r(F)(x) \cup x' = x \cup H^{2d-r}(F)(x'), \qquad x \in H^r(X), \qquad x' \in H^{2d-r}(X). \tag{33}$$

At this stage it is not obvious that equation (33) defines $H_r(F)$, but we will later see that it does. Applying $\eta$ to both sides yields

$$\Phi(H_r(F)(x))(x') = \Phi(x)(H^{2d-r}(F)(x')), \qquad x \in H^r(X),\ x' \in H^{2d-r}(X). \tag{34}$$

Hence,

$$\Phi(H_r(F)(x)) = \Phi(x) \circ H^{2d-r}(F), \qquad x \in H^r(X). \tag{35}$$

At this stage, note that applying $\Phi^{-1}$ to both sides of equation (35) yields

$$H_r(F)(x) = \Phi^{-1}(\Phi(x) \circ H^{2d-r}(F)), \qquad x \in H^r(X), \tag{36}$$

which shows that $H_r(F)$ is defined by equation (33).

Substituting $w = \Phi(x) \in H^r(x)$ into equation (35) yields

$$(\Phi \circ H_r(F) \circ \Phi^{-1})(w) = w \circ H^{2d-r}(F).$$

Now

$$(\Phi \circ (1 - tH_r(F)) \circ \Phi^{-1})(w) = (1 - t(\Phi \circ H_r \circ \Phi^{-1})(F))(w) \tag{37}$$

$$= w - t(w \circ H^{2d-r})(F) = w \circ (1 - tH^{2d-r}(F)). \tag{38}$$

Now lemma 4 yields[10]

$$\det(\Phi \circ (1 - tH_r(F)) \circ \Phi^{-1}) = \det(1 - tH^{2d-r}(F)) \tag{39}$$

$$\Leftrightarrow \det(1 - tH_r(F)) = P_{2d-r}(t). \tag{40}$$

If $x \in H^r(X)$ and $x' \in H^{2d-r}(X)$ then, by Poincaré duality,

$$(H_r(F) \circ H^r(F))(x) \cup x' = H^r(F)(x) \cup H^{2d-r}(F)(x') = H^{2d}(F)(x \cup x'). \tag{41}$$

By Poincaré duality, $H^{2d}(X) \cong \mathbb{Q}$, so $H^{2d}(F)$ must act by scalar multiplication; let $\deg(F)$ denote this scalar. We don't know why, but

$$\deg(F) = q^d, \qquad d := \dim(X). \tag{42}$$

Now

$$H_r(F) \circ H^r(F) = (\deg F)I = q^d I \qquad \therefore H_r(F) = q^d H^r(F)^{-1}. \tag{43}$$

Thus,

$$P_{2d-r}(t) = \det(1 - tH_r(F)) = \det(1 - q^d t H^r(F)^{-1}) = \prod_{j=1}^{\beta_r} \Big(1 - \frac{q^d t}{\alpha_{r,j}}\Big) \tag{44}$$

$$= (-1)^{\beta_r} \prod_{j=1}^{\beta_r} \frac{q^d t}{\alpha_{r,j}} \Big(1 - \frac{\alpha_{r,j}}{q^d t}\Big) = \frac{(-q^d t)^{\beta_r}}{\prod_{j=1}^{\beta_r} \alpha_{r,j}} P_r\Big(\frac{1}{q^d t}\Big). \tag{45}$$

$$\therefore P_r\Big(\frac{1}{q^d t}\Big) = \frac{(-1)^{\beta_r} \prod_{j=1}^{\beta_r} \alpha_{r,j}}{(q^d t)^{\beta_r}} P_{2d-r}(t). \tag{46}$$

Using equation (30),

$$Z\Big(X, \frac{1}{q^d t}\Big) = \prod_{r=0}^{2d} P_r\Big(\frac{1}{q^d t}\Big)^{(-1)^{r+1}} \tag{47}$$

$$= (-q^d t)^{\sum_{r=0}^{2d} (-1)^{r+1} \beta_r} \prod_{r=0}^{2d} \Big(\prod_{j=1}^{\beta_r} \alpha_{r,j}\Big)^{(-1)^{r+1}} \prod_{r=0}^{2d} P_{2d-r}(t)^{(-1)^{r+1}}. \tag{48}$$

[10]Perhaps we need another little lemma here, as well as lemma 4.

$$= \pm(q^d t)^\chi \sqrt{\prod_{r=0}^{2d} \Big( \prod_{j=1}^{\beta_r} \alpha_{r,j} \prod_{j=1}^{\beta_r} \alpha_{2d-r,j} \Big)^{(-1)^{r+1}}} \cdot Z(X,t), \tag{49}$$

where $\chi$ is the Euler characteristic of $X$. From the definition of $H_r(F)$, we see that $H_r(F)$ and $H^{2d-r}(F)$ have the same eigenvalues, for $r = 0, 1, \ldots, 2d$. Thus, we now have

$$Z\Big(X, \frac{1}{q^d t}\Big) = \pm(q^d t)^\chi \sqrt{\prod_{r=0}^{2d} \det(H^r(F))^{(-1)^{r+1}} \det(H_r(F))^{(-1)^{r+1}}} \cdot Z(X,t) \tag{50}$$

$$= \pm(q^d t)^\chi \sqrt{\prod_{r=0}^{2d} (q^{d\beta_r})^{(-1)^{r+1}}} \cdot Z(X,t) = \pm(q^d t)^\chi \sqrt{q^{-d\chi}} \cdot Z(X,t) \tag{51}$$

$$Z\Big(X, \frac{1}{q^d t}\Big) = \pm q^{d\chi/2} t^\chi Z(X,t), \tag{52}$$

which is the functional equation.

**Narthana Epa** *Friday 20 January 2012*

**Motivation.** Let $X$ be a topological space and let $U$ be an open subset of $X$. Let $\mathcal{F}$ be a functor sending open sets $U$ into some category $\mathscr{C}$. If $V \hookrightarrow U$ is an inclusion of open sets, then the morphism $\phi : \mathcal{F}(U) \to \mathcal{F}(V)$ is defined by restricting $f \mapsto f|_V$.

**Definition 5:** *A presheaf is a contravariant functor $\mathcal{F} : TOP(X) \to \mathscr{C}$, where*

$$Ob(TOP(X)) = \{U \subseteq X \mid U \text{ is open }\}, \text{and}$$
$$Mor(V,U) = \begin{cases} i : V \hookrightarrow U & \text{if } V \subseteq U \\ \varnothing & \text{if } V \nsubseteq U. \end{cases}$$

**Definition 6** (Presheaf)**:** *A sheaf is a presheaf $\mathcal{F}$ such that if $U \subseteq X$, and $\{U_i\}_{i \in I}$ is an open cover by subset of $U$, then*

$$\mathcal{F}(U) \xrightarrow{\alpha} \prod_{k \in I} \mathcal{F}(U_k) \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} \prod_{(i,j) \in I^2} \mathcal{F}(U_i \cap U_j),$$

*where*

1. *$\alpha$ sends $s \mapsto \prod_{k \in I} s|_{U_k}$ by the usual restrictions,*
2. *$\beta_1|_{U_k} : \mathcal{F}(U_k) \to \prod_{(i,k) \in I^2} \mathcal{F}(U_i \cap U_k)$, and*
3. *$\beta_2|_{U_k} : \mathcal{F}(U_k) \to \prod_{(k,j) \in I^2} \mathcal{F}(U_k \cap U_j)$*

*is an equaliser. That is,*

1. *$\alpha$ factors uniquely through every morphism that equates $\beta_1$ and $\beta_2$ by right composition*

*(as above). (Note that if $\mathscr{C}$ = **Set** or $\mathcal{A}b$ the this is equivalent to $\alpha$ being injective.); and*

2. $\beta_1 \circ \alpha = \beta_2 \circ \alpha$.

Alternatively,

**Definition 7** (Sheaf)**:** *A sheaf is a presheaf $\mathcal{F}$ such that if $U \subseteq X$, and $\{U_i\}_{i \in I}$ is an open cover by subset of $U$, then the followings are satisfied.*

1. *If $s, t \in \mathcal{F}(U)$ and $s|_{U_i} = t|_{U_i}$ for all $i \in I$ then $s = t$.*
2. *If $\{f_i\}$ is a collection with each $f_i \in \mathcal{F}(U_i)$ and $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $(i,j) \in I^2$, then there exists $f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$ for all $i \in I$.*

**Motivation.**

1. means that this patched function is unique.
2. means that functions on an open cover on an open set that agree on all intersections may be patched together to form a function on the whole open set.

**Towards Sheaf Cohomology.** Let $\mathcal{A}b(X)$ be the category of sheaves over $X$ with values in abelian groups. That is, $\mathcal{F} \in \mathcal{A}b(X) : Top(X) \to Ab$. The morphisms are natural transformations.

**Definition 8:** *Let $\mathscr{D}, \mathscr{C}$ be two categories and $\mathcal{F}, \mathcal{G} : \mathscr{D} \to \mathscr{C}$ be two contravariant functors. A natural transformation $\eta : \mathcal{F} \to \mathcal{G}$ is a collection $\{\eta_X \in Mor(\mathscr{C})\}_{X \in Ob(\mathscr{D})}$ of morphisms in $\mathscr{C}$ such that if $f \in Mor(\mathscr{D}) : X \to Y$, then the diagram*

$$\begin{array}{ccc} \mathcal{F}(Y) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(X) \\ \downarrow{\scriptstyle \eta_Y} & & \downarrow{\scriptstyle \eta_X} \\ \mathcal{G}(Y) & \xrightarrow[\mathcal{G}(f)]{} & \mathcal{G}(X) \end{array}$$

*commutes.*

Recall the definition of exact functor.

**Definition 9:** *Let $\mathcal{A}$ be an abelian category and $\mathcal{A}b$ be the category of abelian groups. Suppose that*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

*is an arbitrary short exact sequence in $\mathcal{A}$. A functor $\mathcal{F} : \mathcal{A} \to \mathcal{A}b$ is left exact if*

$$0 \longrightarrow \mathcal{F}(A) \longrightarrow \mathcal{F}(B) \longrightarrow \mathcal{F}(C)$$

*is exact in $\mathcal{A}b$. It is right exact if*

$$\mathcal{F}(A) \longrightarrow \mathcal{F}(B) \longrightarrow \mathcal{F}(C) \longrightarrow 0$$

*is exact in $\mathcal{A}b$. It is exact if it is both left and right exact.*

**Definition 10** (Injective object)**:** *An object $A$ of an abelian category $\mathcal{A}$ is injective if and only if the functor* $\mathrm{Hom}(-, A) : \mathcal{A} \to \mathcal{A}b$ *is exact.*

**Fact.** If $\mathcal{F} \in \mathcal{A}b(X)$, then there exists an exact sequence $0 \to \mathcal{F} \to I$, where $I$ is injective. That is, each object in $\mathcal{A}b(X)$ can be "embedded" in an injective object.

Note that the definition of exact functor extends to long exact sequences in $\mathcal{A}$ by notice the following. Suppose that

$$0 \longrightarrow A_0 \xrightarrow{\delta_0} A_1 \xrightarrow{\delta_1} A_2 \xrightarrow{\delta_2} A_3 \xrightarrow{\delta_3} \dots$$

is a long exact sequence. This is a concatenation of short exact sequences.

$$0 \longrightarrow \ker\delta_i \longrightarrow A_i \xrightarrow{\delta_i} \ker\delta_{i+1} \longrightarrow 0$$

for $i \geq 1$.

**Definition 11** (Right derived functor)**:** *Let $\mathcal{F} : \mathcal{A} \to \mathcal{A}b$ be a left exact functor. The right derived functors of $\mathcal{F}$, indexed by $i \in \mathbb{Z}$, are functors $R^i\mathcal{F} : \mathcal{A} \to \mathcal{A}b$ that send $A \mapsto H^i(\mathcal{F}(I_A^*))$, where*

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \dots$$

*is an exact sequence in $\mathcal{A}$ and $I_A^* = (I^n)_{n \in \mathbb{Z}_{\geq 0}}$ is a sequence of injective objects in $\mathcal{A}$.*

The right derived functor measures how close a left exact functor is to being exact (e.g. $H^i \equiv 0$ if $F$ is exact). Moreover, it is well defined because injective resolutions are chain-homotopic.

**Lemma 12:** *Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories. Then $R^0F = F$ as functors, in that they spit out isomorphic values.*

*Proof.* Let $A \in \mathcal{A}$. To show: $R^0F(A) = F(A)$. Let

$$0 \to A \xrightarrow{\varepsilon} I^0 \xrightarrow{d^0} \dots$$

be an injective resolution of $A$. Then

$$0 \to F(A) \xrightarrow{F(\varepsilon)} F(I^0) \xrightarrow{F(d^0)} \dots$$

yields

$$R^0F(A) = \ker(F(d^0)) = \operatorname{im}(F(\varepsilon)) \cong F(A).$$

□

We first observe that if $I$ is injective, then $R^i\mathcal{F} = 0$ for all $i > 0$, since it has the exact sequence $0 \to I \to I \to 0 \to 0 \to \dots$, and by uniqueness up to homotopy.

Let $\Gamma(X, -) : \mathcal{A}b(X) \to \mathcal{A}b$ be the *global sections*, defined by sending $\mathcal{F} \mapsto \mathcal{F}(X)$.

**Definition 13** (Sheaf cohomology)**:** *For each functor $\mathcal{F} \in \mathcal{A}b(X)$,*

$$H^i(X, \mathcal{F}) = R^i\Gamma(X, -)[\mathcal{F}]$$

**Narthana Epa** *Friday 27 January 2012*

To motivate using étale cohomology, we need to see why the Zariski topology is a bad choice. We shall see that the $i$th homology group is trivial for $i > 0$, but first we need some further background on sheaves.

**Theorem 14:** *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, let*
$$0 \to A' \to A \to A'' \to 0$$
*be an exact sequence in $\mathcal{A}$, and let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Note that $R^i\mathcal{F} : A \to Ab$ for $i \in \mathbb{Z}_{>0}$. Then there exist maps*
$$\partial^i : R^i\mathcal{F}(A'') \to R^{i+1}\mathcal{F}(A'), \qquad i \geq 0$$
*such that*
$$0 \to R^0\mathcal{F}(A') \to R^0\mathcal{F}(A) \to R^0\mathcal{F}(A'') \xrightarrow{\partial^0} R^1\mathcal{F}(A') \to \dots$$
*is exact.*

For instance, if $R^i\mathcal{F}$ is cohomology, then we get the LES for cohomology.

Let $X$ be a topological space, and let $A$ be an abelian group with the discrete topology. The *locally constant sheaf on $X$ with values in $A$* is
$$A_X : \mathrm{Top}(X) \to \mathcal{A}b$$
$$U \mapsto \{f : U \xrightarrow{\text{cts}} A\} \cong A^n,$$
where $n$ is the number of components of $X$ (since each component maps to a point).[11]

**Example 15:** *$X = \{*\}$. Then the locally constant sheaf on $X$ with values in $\mathbb{Z}$ is given by*
$$\mathbb{Z}_X(U) \cong \begin{cases} \mathbb{Z}, & \text{if } U = \{*\} \\ 0, & \text{if } U = \emptyset. \end{cases}$$
*We want an injective resolution, but this is easy because $\mathbb{Z}_X$ is injective:*

*Proof.* To show: $\mathbb{Z}_X$ is injective.
To show: $\mathrm{Hom}(-, \mathbb{Z}_X) : \mathcal{A}b(X) \to \mathcal{A}b$ is exact.
An arbitrary object $\mathcal{F} \in \mathcal{A}b(X)$ is of the form
$$\mathcal{F}(U) = \begin{cases} F, & \text{if } U = \{*\} \\ 0, & \text{if } U = \emptyset, \end{cases}$$
for some abelian group $F$. Let
$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \tag{53}$$
be an exact sequence in $\mathcal{A}b(X)$, and let $F, G, H$ be the corresponding abelian groups. Then
$$0 \to F \to G \to H \to 0 \tag{54}$$
is exact, so
$$0 \to \mathrm{Hom}(H, \mathbb{Z}) \to \mathrm{Hom}(G, \mathbb{Z}) \to \mathrm{Hom}(F, \mathbb{Z}) \to 0 \tag{55}$$
is exact, so
$$0 \to \mathrm{Hom}(\mathcal{H}, \mathbb{Z}_X) \to \mathrm{Hom}(\mathcal{G}, \mathbb{Z}_X) \to \mathrm{Hom}(\mathcal{F}, \mathbb{Z}_X) \to 0 \tag{56}$$
is exact. Thus, $\mathrm{Hom}(-, \mathbb{Z}_X)$ is exact, so $\mathbb{Z}_X$ is injective. $\square$

*Now*
$$0 \to \mathbb{Z}_X \to \mathbb{Z}_X \to 0 \to \dots$$

[11] This is as constant as a sheaf can get. It is the sheafification of the constant presheaf.

*is an injective resolution. Ignoring the first two objects and applying global sections yields*

$$\mathbb{Z} \to 0 \to \dots,$$

*as* $\Gamma(X, \mathbb{Z}_X) = \mathbb{Z}_X(X) = \mathbb{Z}$, *so*

$$H^i(X, \mathbb{Z}_X) = R^i\Gamma(X, -)[\mathbb{Z}_X] = H^i(\mathbb{Z}_X(I^*_{\mathbb{Z}})) = \partial_{i,0}. \tag{57}$$

Sheaf cohomology usually agrees with singular cohomology:

**Theorem 16:** *Let $X$ be a locally contractible topological space, and let $\mathbb{Z}_X$ be the locally constant sheaf on $X$ with values in $\mathbb{Z}$. Then*

$$H^i(X, \mathbb{Z}_X) \cong H^i_{\text{sing}}(X; Z), \qquad i \geq 0. \tag{58}$$

*Proof.* (sketch) Somehow it suffices to use an acyclic resolution, such as

$$0 \to \mathbb{Z}_X \to C_0^{\text{sing}}(X)_X \to \dots, \tag{59}$$

where $C_0^{\text{sing}} = \mathbb{Z}\{\Delta^0 \to X\}$. □

**Lemma 17:** *Let $X$ be an irreducible topological space, and let $A$ be a discrete topological space. Let $V$ be a non-empty open subset of $X$, and let $f : V \to A$ be a continuous function. Then $f$ is constant.*

*Proof.* It suffices to prove that $V$ is connected, since that would imply that $f(V)$ is connected and therefore comprises one point (as $A$ is discrete). Proof by contradiction: assume that $V$ is disconnected. Then there exist open sets $U_1, U_2 \subseteq X$ such that

$$V = (U_1 \cap V) \sqcup (U_2 \cap V).$$

Then

$$X = (X \setminus (U_1 \cap V)) \cup (X \setminus (U_2 \cap V))$$

expresses $X$ as a union of two proper closed sets, contradicting the assumption that $X$ is irreducible. Hence $V$ is connected, and it follows that $f$ is constant. □

A sheaf $\mathcal{F}$ on a topological space $X$ is *flabby* (or *flasque*) if

$$\mathcal{F}(i) : \mathcal{F}(U) \to \mathcal{F}(V) \tag{60}$$

is surjective for all inclusions $i : V \hookrightarrow U$ of open subsets of $X$.

Now we get to the point.

**Theorem 18:** *Let $X$ be an irreducible topological space, and let $\mathcal{F}_X$ be a locally constant sheaf on $X$. Then*

$$H^i(X, \mathcal{F}_X) \cong \{0\}, \qquad i > 0.$$

This follows immediately from the following two results.

**Lemma 19:** *Let $X$ be an irreducible topological space, and let $\mathcal{F}_X$ be a locally constant sheaf on $X$, with values in some discrete abelian group $A$. Then $\mathcal{F}_X$ is flasque.*

**Proposition 20:** *Let $X$ be a topological space, and let $\mathcal{F} \in \mathcal{A}b(X)$ be flasque. Then*

$$H^i(X, \mathcal{F}) = 0, \qquad i > 0.$$

*Proof of 19.* Let $i : V \hookrightarrow U$ be an inclusion of open subsets of $X$. By 17, any continuous function $V \to A$ must be constant, and therefore extends to a constant function $U \to A$. Thus,

$$\mathcal{F}(i) : \mathcal{F}(U) \to \mathcal{F}(V)$$

is surjective, so $\mathcal{F}$ is flasque. □

To prove 18, it remains to prove . We shall need the following three lemmata:

**Lemma 21:** *Let $\mathcal{I}$ be an injective sheaf on a topological space $X$. Then $I$ is flasque.*

*Proof of 21.* Let $U$ be an open subset of $X$, and let $\mathbb{Z}_X$ be the sheaf of locally constant $\mathbb{Z}$-valued maps on $X$.[12] Moreover, let $j_!(\mathbb{Z}_U)$ denote the smallest abelian subsheaf of $\mathbb{Z}_X$ that contains all $s \in \mathbb{Z}_X(V)$ for all open subsets $V$ of $U$. The map

$$\mathrm{Hom}(\mathbb{Z}_X, \mathcal{I}) \to \mathrm{Hom}(j_!(\mathbb{Z}_U), \mathcal{I})$$

induced by the embedding $j_!(\mathbb{Z}_U) \hookrightarrow \mathbb{Z}_X$ is surjective (as $\mathcal{I}$ is injective) and can be identified (check) with the restriction map $\mathcal{I}(X) \to \mathcal{I}(U)$.[13] □

**Lemma 22:** *Let*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

*be a SES in $\mathcal{A}b(X)$, and let $U \subseteq X$ be open. Then*

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0$$

*is a SES of abelian groups.*

**Lemma 23:** *Let*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

*be a SES in $\mathcal{A}b(X)$. If $\mathcal{F}$ and $\mathcal{G}$ are flasque, then $\mathcal{H}$ is flasque.*

Conditional on these standard technical results, we can complete the proof of 18.

*Proof of 18.* By the **fact**, and also using the first isomorphism theorem, there exists an exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \frac{\mathcal{I}}{\mathcal{F}} \to 0, \tag{61}$$

where $\mathcal{I} \in \mathcal{A}b(X)$ is injective. Applying 22 with $U = X$ yields an exact sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{I}) \to \Gamma\Big(X, \frac{\mathcal{I}}{\mathcal{F}}\Big) \to 0. \tag{62}$$

By 12, we now have a SES

$$0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{I}) \to H^0\Big(X, \frac{\mathcal{I}}{\mathcal{F}}\Big) \to 0. \tag{63}$$

Consider the LES

$$\ldots \to H^i(X, \mathcal{I}) \to H^i\Big(X, \frac{\mathcal{I}}{\mathcal{F}}\Big) \to H^{i+1}(X, \mathcal{F}) \to H^{i+1}(X, I) \to \ldots.$$

If $i > 0$ then $H^i(X, \mathcal{I})$ and $H^{i+1}(X, \mathcal{I})$ are trivial. Thus,

$$H^i\Big(X, \frac{\mathcal{I}}{\mathcal{F}}\Big) \cong H^{i+1}(X, \mathcal{F}), \qquad i > 0. \tag{64}$$

[12] This proof is taken directly from `http://maths-magic.ac.uk/admin/send/course_file.php?id=2901`. Incidentally, this website has a whole lot of free postgraduate lectures from British universities.

[13] I'm not sure how this identification works yet.

We now use induction to prove that $H^i(X, \mathcal{F})$ is trivial for $i > 0$. From 63, the map

$$\varphi : H^0(X, \mathcal{I}) \to H^0\Big(X, \frac{\mathcal{I}}{\mathcal{F}}\Big)$$

is surjective.[14] The LES then implies that $H^1(X, \mathcal{F})$ is trivial.

As $\frac{\mathcal{I}}{\mathcal{F}}$ is flasque (by 23), it must also be true that $H^1\Big(X, \frac{\mathcal{I}}{\mathcal{F}}\Big)$ is trivial. From 64, it then follows that $H^2(X, \mathcal{F})$ is trivial, and we continue. □

So what went wrong?

**Example 24:** *Consider $x \mapsto x^2 : \mathbb{C} \to \mathbb{C}$. By the IMT, we expect a local inverse (square root function) away from 0. Consequently, we want*

$$\mathbb{C} \setminus \textit{ branch cut}$$

*to be open in $\mathbb{C}$. Under the Zariski topology, however, it isn't open!*

This example suggests that the Zariski topology has too few open sets to produce a useful cohomology theory.

**Narthana Epa** *Friday 3 February 2012*

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We want to define étale cohomology. We will still use sheaf cohomology, but we will use affine open sets instead of Zariski open sets. In order to do this, we need to define étale morphisms between varieties. We achieve this by using the correspondence between morphisms between affine open sets and the induced ring homomorphisms. And so we begin in the proof-machined world of commutative algebra (all rings are assumed commutative and unital), but quickly find ourselves doing algebraic geometry using the language of schemes à la Grothendieck.

A ring homomorphism $\phi : R \to S$ induces an $R$-algebra structure on $S$. Such a map is *étale* if there exist $n \in \mathbb{Z}_{>0}$ and $f_1, \ldots, f_n \in R[x_1, \ldots, x_n]$ such that

$$S \cong \frac{R[x_1, \ldots, x_n]}{(f_1, \ldots, f_n)} \tag{65}$$

as $R$-algebras and

$$\Big[\det\Big(\frac{\partial f_i}{\partial x_j}\Big)\Big] \in S$$

is invertible.

**Example 25:** *The inclusion $R \hookrightarrow \mathbb{C}$ is étale because*

$$\mathbb{C} \cong \frac{\mathbb{R}[x]}{(x^2+1)}$$

[14] We're doing this rather naively. Flabby resolutions are clearly enough to define cohomology directly, since acyclic resolutions are (and injective $\implies$ flabby $\implies$ acyclic). Knowing this, all of this is immediate. Of course it's not easy to show that acyclic resolutions are chain-homotopic, but we didn't prove it for injective ones anyway.

*and*

$$[\det(\dots)] \cdot [-\frac{1}{2}x] = [2x] \cdot [-\frac{1}{2}x] = [-x^2] = [1]. \tag{66}$$

Let $\phi : R \to S$ and $\psi : R \to R'$ be ring homomorphisms. The *base change of $\phi$ by $\psi$* is the pushout

$$\begin{aligned} i_2 : R' &\to S \otimes_R R' \\ r &\mapsto 1 \otimes r. \end{aligned}$$

The commutative diagram is

$$\begin{array}{ccc} R & \xrightarrow{\phi} & S \\ \psi\big\downarrow & & \big\downarrow i_1 \\ R' & \xrightarrow[i_2]{} & S \otimes_R R', \end{array} \tag{67}$$

and the universal property is there but has not been drawn in.

**Theorem 26:**

*(a) The set of étale homomorphisms is closed under composition.*

*(b) The base change of an étale homomorphism by any homomorphism is étale.*

*Proof.*

(a) Consider étale homomorphisms

$$R \xrightarrow{\phi} S \xrightarrow{\psi} T$$

such that

$$S \cong \frac{R[x_1, \dots, x_n]}{(f_1, \dots, f_n)}, \qquad T \cong \frac{R[y_1, \dots, y_m]}{(g_1, \dots, g_m)}$$

For $i = 1, 2, \dots, m$, we define $g_i'$ by lifting the coefficients of $g_i \in S[y_1, \dots, y_m]$ by the quotient map

$$q : R[x_1, \dots, x_n] \to S.$$

Any lift will do, and now

$$T \cong \frac{R[x_1, \dots, x_n, y_1, \dots, y_m]}{(f_1, \dots, f_n, g_1', \dots, g_m')}. \tag{68}$$

Moreover,

$$[\det(\dots)] = [\det(\partial f_i / \partial x_j)][\det(\partial g_i' / \partial y_j)] \in T \tag{69}$$

is invertible.

(b) Let $\phi : R \to S$ be an étale homormorphism, and let $\psi : R \to R'$ be a ring homomorphism. Let

$$S \cong \frac{R[x_1, \dots, x_n]}{(f_1, \dots, f_n)}.$$

We need to check that

$$S \otimes_R R' \cong \frac{R'[x_1, \dots, x_n]}{(f_1', \dots, f_n')} \tag{70}$$

where $f_i'$ is $f_i$ with coefficients replaced by their images under $\psi$ (for $i = 1, 2, \ldots, n$). The isomorphism

$$\frac{R[x_1, \ldots, x_n]}{(f_1, \ldots, f_n)} \otimes R' \to \frac{R'[x_1, \ldots, x_n]}{(f_1', \ldots, f_n')} \tag{71}$$

adjusts the coefficient $r \mapsto \psi(r)r'$.

□

We now want to understand étale morphisms. Milne says something like, "an étale morphism is the algebraic geometry analogue of a local isomorphism of manifolds (differential geometry), an unbranched covering of Riemann surfaces (complex analysis), an an unramified extension (algebraic number theory). For varieties, it is possible to characterize étale morphisms geometrically; for arbitrary schemes, there is only commutative algebra." We can certainly see the local isomorphism relationship, because the invertibility condition for étale ring homormophisms was precisely the hypothesis for the inverse function theorem. Also, we could have gone the geometric path, but we instead went more broadly to general schemes via commutative algebra: the upside is that we know some commutative algebra, while the downside is that most of us don't know general schemes as well as we know varieties geometrically. See [7].

An *affine scheme* is a locally ringed space[15] isomorphic to $\mathrm{Spec}(A)$ for a ring $A$. Often the isomorphism to *Spec* is abbreviated to an equals sign.

A *scheme* is a locally ringed space $X$ such that every point has an open neighbourhood which is an affine scheme. The morphisms in the category of schemes are the morphisms of locally ringed spaces. A variety is a special case of a scheme (working with integral domains rather than general commutative unital rings). See [8].

Let $X$ be a scheme.

A subset $U \subseteq X$ is *affine open* if $U$ is an affine scheme. A morphism $f : X \to Y$ of schemes is *étale at* $x \in X$ if there exist affine open $\mathrm{Spec}(A) = U \subseteq X$ and $\mathrm{Spec}(B) = V \subseteq Y$ such that

- $x \in U$
- $f(U) \subseteq V$
- The induced ring homomorphism[16] $B \to A$ is étale.

We now define the category Ét$/X$. The objects are the étale morphisms of the form

$$f : V \to X,$$

where $V$ is a scheme. The morphisms between $f_1 : V_1 \to X$ and $f_2 : V_2 \to X$ are the étale morphisms $g : V_1 \to V_2$ such that $f_1 = f_2 \circ g$.

[15] As an example, if $X$ is an algebraic variety with the Zariski topology, and $U \subset X$ is open, then the ring of rational functions $O_X(U)$ is a locally ringed space. Generally, a *ringed space* $(X, O_X)$ is a topological space $X$ with a sheaf of rings on $X$, and a *locally ringed space* further insists that the stalks are local rings.

[16] Indeed the ring homomorphisms $B \to A$ are in bijective correspondence with the scheme morphisms[17] $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. Certainly $\phi : B \to A$ corresponds to the preimage map $\phi^{-1} : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$.

The *étale site* of $X$, denoted $X_{\text{ét}}$, is the category $\text{Ét}/X$ along together with all 'coverings' (additional data). Specifically, to each object $\phi : V \xrightarrow{\text{ét}} X$, associate the collection of all families of morphisms

$$\{\phi_i : U_i \xrightarrow{\text{ét}} V\}_{i\in I} \tag{72}$$

such that

$$\cup_{i\in I}\, \phi_i(U_i) = V. \tag{73}$$

Let $\mathcal{C}$ be a category. An *étale sheaf* of objects in $\mathcal{C}$ is a contravariant functor

$$\mathcal{F} : \text{Ét}(X) \to \mathcal{C}$$

such that if $\{g_i : U_i \to V\}_{i\in I}$ is a covering then

$$\mathcal{F}(U) \xrightarrow{\ \alpha\ } \prod_{k\in I}\mathcal{F}(U_k) \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} \prod_{(i,j)\in I^2}\mathcal{F}(U_i \times_V U_j) \tag{74}$$

is an equaliser, where $U_i \times_V U_j$ is the pullback (the universal property is there but not drawn):

$$\begin{array}{ccc} U_i \times_V U_j & \longrightarrow & U_j \\ \big\downarrow & & \big\downarrow \phi_j \\ U_i & \xrightarrow[\phi_i]{} & V. \end{array} \tag{75}$$

Note that a Zariski covering is a special type of étale covering. Define

$$H^r(X_{\text{ét}}, \mathcal{F}) := H^r(\Gamma(X, \mathcal{I})), \qquad r \in \mathbb{Z}_{>0}, \tag{76}$$

where $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \dots$ is an injective resolution of $\mathcal{F}$. The étale sheaf we will be the locally constant sheaf on $X_t$ with values in an abelian group. A problem with choosing $\mathbb{Z}$ as our abelian group is that we always get

$$H^1(X_{\text{ét}}, \mathbb{Z}) = 0. \tag{77}$$

We get more information by defining *l-adic cohomology*. Let $k$ be a finite field, $|k| = q = p^{\varepsilon}$, $p$ prime, and let $l \neq p$ be a prime. Let $X$ be a variety over $\bar{k}$. Define the *l-adic integers* and the *l-adic rationals* as usual:

$$\mathbb{Z}_l := \varprojlim_n \frac{\mathbb{Z}}{l^n\mathbb{Z}}, \tag{78}$$

and $\mathbb{Q}_l$ is defined as the field of fractions of $\mathbb{Z}_l$. Then

$$H^r(X, \mathbb{Z}_l) := \varprojlim_n H^r(X_{\text{ét}}, \mathbb{Z}/l^n\mathbb{Z}), \tag{79}$$

and the *l-adic cohomology* of $X$ is given by

$$H^r(X) := H^r(X, \mathbb{Q}_l) := H^r(X, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l. \tag{80}$$

**Richard Hughes** *Friday 10 February 2012*

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*Drawn from Chapter II of "Algebraic Geometry" (Hartshorne).*

## Our aim:

In order to understand étale cohomology and Deligne's proof of the Riemann hypothesis of Weil, we need a better understanding of schemes.

A specific goal for this talk is to set up the background required to determine how a morphism $f : \text{Spec}(A) \to \text{Spec}(B)$ induces a morphism $f^* : B \to A$.

## Revision of sheaves:

**Definition 27** (Presheaf)**:** *Let $X$ be a topological space. A* **presheaf** *$\mathcal{F}$ of abelian groups on $X$ consists of the data*

*(a) for every open subset $U \subseteq X$, an abelian group $\mathcal{F}(U)$, and*

*(b) for every inclusion $V \subseteq U$ of open subsets of $X$, a morphism of abelian groups $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$,*

*subject to the conditions*

*(0) $\mathcal{F}(\emptyset) = 0$, where $\emptyset$ is the empty set,*

*(1) $\rho_{UU}$ is the identity map $\mathcal{F}(U) \to \mathcal{F}(U)$, and*

*(2) if $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.*

NB: This is equivalent to saying that a presheaf is a contravariant functor $\mathcal{F} : \text{Top}(X) \to \mathcal{A}b$.

**Definition 28** (Sheaf)**:** *A presheaf $\mathcal{F}$ on $X$ is a* **sheaf** *if it satisfies:*

*(3) if $U$ is an open set, $\{U_i\}$ is an open covering of $U$, and $s \in \mathcal{F}(U)$ is an element such that $s|_{U_i} = 0$[18] for all $i$, then $s = 0$;*

*(4) if $U$ is an open set, $\{U_i\}$ an open cover of $U$, and we have $s_i \in \mathcal{F}(U_i)$ for each $i$ such that for every $i$ and $j$, $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there is an $s \in \mathcal{F}(U)$ such that $s_{U_i} = s_i$ for each $i$.*

NB: This is equivalent to saying that

$$\mathcal{F}(U) \longrightarrow \prod_{k \in I} \mathcal{F}(U_k) \rightrightarrows \prod_{(i,j) \in I^2} \mathcal{F}(U_i \cap U_j)$$

is an equaliser.

## Our favourite examples:

[18] Presumably we define $s|_{U_i} = \rho_{UV}(s)$, but I'm not sure if we ever explicitly stated this.

**Example 29** (The sheaf of regular functions on $X$, $\mathscr{O}$)**:**

*Let $X$ be a variety over $k$. For each open $U \subseteq X$, let $\mathscr{O}(U)$ be the ring of regular functions from $U$ to $k$. For each $V \subseteq U$ let $\rho_{UV} : \mathscr{O}(U) \to \mathscr{O}(V)$ be the usual restriction map. Then $\mathscr{O}$ is a sheaf of rings on $X$.*

**Example 30** (Constant sheaves on $X$)**:**

*Let $X$ be a topological space, $A$ an abelian group. Give $A$ the discrete topology, and for any open $U \subseteq X$ let $\mathscr{A}(U)$ be the group of all continuous maps $U \to A$. If $c$ is the number of connected components of $U$, then*

$$\mathscr{A}(U) = A^{\oplus c}.$$

*We call $\mathscr{A}$ the constant sheaf on $X$ determined by $A$.*

## New material for sheaves:

We need to introduce *stalks*, and discuss some properties of stalks and sheaves. This is required to understand how a morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$ induces a morphism $B \to A$ (probably modulo some conditions — see Nepa's talk last week for motivation).

**Definition 31** (Stalk)**:** *Let $\mathcal{F}$ be a presheaf on $X$, and let $P$ be a point of $X$. The* **stalk** *of $\mathcal{F}$ at $P$, $\mathcal{F}_P$, is the direct limit of the groups $\mathcal{F}(U)$ for all open $U \ni P$, via the restriction maps $\rho$.*

**Notes:**

1) There are notes on direct limits on Arun's webpage.

2) In our case, an element of $\mathcal{F}_P$ is a pair $\langle U, s\rangle$ where

   - $U$ is an open neighbourhood of $P$, and
   - $s \in \mathcal{F}(U)$.

   $\langle U, s\rangle$ and $\langle V, t\rangle$ define the same element of $\mathcal{F}_P$ if and only if there is an open neighbourhood or $P$, $W$, with $W \subseteq U \cap V$ such that $s|_W = t|_W$.

**Definition 32** (Morphism of sheaves)**:** *If $\mathcal{F}$ and $\mathcal{G}$ are presheaves on $X$, a* **morphism** *$\varphi : \mathcal{F} \to \mathcal{G}$ consists of a morphism of abelian groups $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set $U$, such that whenever $V \subseteq U$ is an inclusion,*

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \Big\downarrow{\scriptstyle \rho_{UV}} & & \Big\downarrow{\scriptstyle \rho_{UV}} \\ \mathcal{F}(V) & \xrightarrow[\varphi(V)]{} & \mathcal{G}(V) \end{array}$$

*commutes. An* **isomorphism** *is a morphism with a two-sided inverse.*

The next proposition demonstrates the local nature of a sheaf (it is false for presheaves).

**Proposition 33:** *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then $\varphi$ is an isomorphism if and only if the induced map on the stalk $\varphi_P : \mathcal{F}_P \to \mathcal{G}_P$ is an isomorphism for every $P \in X$.*

*Proof.* If $\varphi$ is an isomorphism each $\varphi(U)$ is an isomorphism, and thus so is the direct limit $\varphi_P$ (as Nepa mentioned, this part of the implication is true because 'direct limit' is a functor).

Conversely, assume $\varphi_P$ is an isomorphism for every $P \in X$. To show that $\varphi$ is an isomorphism it is sufficient to show that $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism for all $U$; then the inverse morphism $\psi$ can be defined as

$$\psi(U) = \varphi(U)^{-1} \quad \text{for each } U \subseteq X \text{ open.}$$

**Injectivity:**

Let $s \in \mathcal{F}(U)$ and suppose $\varphi(U)(s) \in \mathcal{G}(U)$ is 0. Then for every $P \in U$, the image $\varphi_P(s)$ of $\varphi(U)(s)$ in the stalk $\mathcal{G}_P$ is 0. Since $\varphi_P$ is injective for each $P$, $s_P = 0$ in $\mathcal{F}_P$ for each $P \in U$. $s_P = 0$ means that $s$ and 0 have the same image in $\mathcal{F}_P$, so there is an open neighbourhood of $P$, $W_P$, with $W_P \subseteq U$, such that $s|_{W_P} = 0$. Thus, by

$$U = \bigcup_{P \in U} W_P$$

and sheaf property (3), $s = 0$ on $U$.

**Surjectivity:**

Let $t \in \mathcal{G}(U)$. For each $P \in U$ let $t_P \in \mathcal{G}_P$ be its germ at $P$. Since $\varphi_P$ is surjective, there exists $s_P \in \mathcal{F}_P$ such that $\varphi_P(s_P) = t_P$. Let $s_P$ be represented by a section $s(P)$ on a neighbourhood $V_P$ of $P$. Then $\varphi(s(P))$ and $t|_{V_P}$ are elements of $\mathcal{G}(V_P)$ with the same germ at $P$.

Thus, replacing $V_P$ with a smaller neighbourhood if required, $\varphi(s(P)) = t|_{V_P}$ in $\mathcal{G}(V_P)$. $U$ is covered by the $V_P$, and on each $V_P$ we have a section $s(P) \in \mathcal{F}(V_P)$. For two points $P$ and $Q$, $s(P)|_{V_P \cap V_Q}$, $s(Q)|_{V_P \cap V_Q} \in \mathcal{F}(V_P \cap V_Q)$ are both sent to $t|_{V_P \cap V_Q}$ by $\varphi$.

By the injectivity of $\varphi$, $s(P)|_{V_P \cap V_Q} = s(Q)|_{V_P \cap V_Q}$; then by the glueing property of sheaves (4), there exists $s \in \mathcal{F}(U)$ such that $s|_{V_P} = s(P)$ for each $P$.

Finally, $\varphi(U)(s)$ and $t$ are in $\mathcal{G}(U)$, and for each $P$, $\varphi(U)(s)|_{V_P} = t|_{V_P}$, so by sheaf property (3) applied to $\varphi(U)(s) - t$, we conclude that $\varphi(U)(s) = t$.

□

## Spectrum of a ring:

For our purposes, all rings are **commutative** and **unital**.

**Definition 34:** *Let $A$ be a ring. Then define*

$$\mathrm{Spec}(A) = \{\textit{prime ideals of } A\}.$$

*If $\mathfrak{a}$ is an ideal of $A$, define $V(\mathfrak{a}) \subseteq \mathrm{Spec}(A)$ to be*

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(A) | \mathfrak{p} \supseteq \mathfrak{a}\}.$$

The following lemma determines some properties of $V$.

**Lemma 35:**

*(a) If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of $A$, then $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.*

*(b) If $\{\mathfrak{a}_i\}$ is any set of ideals of $A$, then $V(\sum \mathfrak{a}_i) = \bigcap V(\mathfrak{a}_i)$.*

*(c) If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals, $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$ if and only if $\sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}}$.*

*Proof.* Definition chasing exercise. For (c), recall that

$$\sqrt{\mathfrak{a}} = \bigcap_{\substack{\mathfrak{p} \in \mathrm{Spec}(A) \\ \mathfrak{p} \supseteq \mathfrak{a}}} \mathfrak{p}$$

□

**Example 36:**

*1)* $\mathrm{Spec}(\mathbb{C}[t]) = \mathbb{A}^1_{\mathbb{C}}$

$$V((t-i)^2(t+1)\mathbb{C}[t]) = \{(t-i)\mathbb{C}[t],\ (t+1)\mathbb{C}[t]\}$$

*2)* $\mathrm{Spec}(\mathbb{Z}) = \{p\mathbb{Z} | p \textit{ prime or zero}\}$

$$\begin{aligned} V(6\mathbb{Z}) &= V((2\mathbb{Z})(3\mathbb{Z})) \\ &= V(2\mathbb{Z}) \cup V(3\mathbb{Z}) \\ &= \{2\mathbb{Z},\ 3\mathbb{Z}\} \end{aligned}$$

$$V((0)) = \bigcup_{p \textit{ prime}} p\mathbb{Z} = \mathrm{Spec}(\mathbb{Z})$$

Define a topology on $\mathrm{Spec}(A)$ by taking subsets of the form $V(\mathfrak{a})$ to be the closed sets. Note:

- $V(A) = \emptyset$
- $V((0)) = \mathrm{Spec}(A)$
- The lemma gives us the union and intersection conditions.

**Definition 37** (Sheaf of rings on $\mathrm{Spec}(A)$)**:** *We wish to define a* **sheaf of rings $\mathscr{O}$ on** $\mathrm{Spec}(A)$.

*For each prime ideal $\mathfrak{p} \subseteq A$, let $A_\mathfrak{p}$ be the localisation of $A$ at $\mathfrak{p}$.*

*For open $U \subseteq \mathrm{Spec}(A)$, define $\mathscr{O}(U)$ to be the set of functions*

$$s : U \to \sqcup_{\mathfrak{p} \in U} A_\mathfrak{p}$$

*such that $s(\mathfrak{p}) \in A_\mathfrak{p}$ for each $\mathfrak{p}$, and such that $s$ is locally a quotient of elements of $A$.*

*This means that for each $\mathfrak{p} \subseteq U$, there is an open neighbourhood $V$ of $\mathfrak{p}$ with $V \subseteq U$, and elements $a, f \in A$ such that for each $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = \frac{a}{f} \in A_\mathfrak{q}$.*

**Notes:**

- These functions are closed under sums and products, and have identity the 1 section. So $\mathscr{O}(U)$ is a commutative unital ring.
- If $V \subseteq U$, the natural restriction $\mathscr{O}(U) \to \mathscr{O}(V)$ is a ring homomorphism, making $\mathscr{O}$ into a presheaf.
- The local nature of the definition makes $\mathscr{O}$ a sheaf.

**Definition 38** (Spectrum of a ring)**:** *Let $A$ be a ring. The* **spectrum** *of $A$ is the pair consisting of the topological space* $\mathrm{Spec}(A)$ *together with the sheaf of rings $\mathscr{O}$ defined above.*

**Definition 39:** *For $f \in A$ denote by $D(f)$ the open complement of $V((f))$.*

**Remark:** Open sets of the form $D(f)$ form a base for the topology of $\mathrm{Spec}(A)$. (Exercise)

**Proposition 40:** *Let $A$ be a ring, and $(\mathrm{Spec}(A), \mathscr{O})$ its spectrum.*

*(a) For any $\mathfrak{p} \in \mathrm{Spec}(A)$, the stalk $\mathscr{O}_\mathfrak{p}$ of the sheaf $\mathscr{O}$ is isomorphic to the local ring $A_\mathfrak{p}$.*

*(b) For any element $f \in A$, the ring $\mathscr{O}(D(f))$ is isomorphic to the localised ring $A_f$.*

*(c) In particular, $\Gamma(\mathrm{Spec}(A), \mathscr{O}) \cong A$.*

*Proof.* **(a)** Define a homomorphism

$$\begin{array}{rccc} \varphi : & \mathscr{O}_\mathfrak{p} & \to & A_\mathfrak{p} \\ & s & \mapsto & s(\mathfrak{p}). \end{array}$$

**Well-defined:**

If $s \sim t$ in $\mathscr{O}_\mathfrak{p}$, there is a neighbourhood $W \ni \mathfrak{p}$ on which $s|_W = t|_W$, and so $s(\mathfrak{p}) = t(\mathfrak{p})$.

**Surjectivity:**

An element of $A_\mathfrak{p}$ can be represented as a quotient $\frac{a}{f}$ with $a, f \in A$, $f \notin \mathfrak{p}$. Then $D(f)$ is an open neighbourhood of $\mathfrak{p}$, and $\frac{a}{f}$ defines a section of $\mathscr{O}$ over $D(f)$ whose value at $\mathfrak{p}$ is the given element.

**Injectivity:**

Let $U$ be a neighbourhood of $\mathfrak{p}$, and let $s, t \in \mathscr{O}(U)$ such that $s(\mathfrak{p}) = t(\mathfrak{p})$ at $\mathfrak{p}$. By shrinking $U$ if necessary we may assume that on $U$,

$$s = \frac{a}{f}, \quad t = \frac{b}{g}, \quad a, b, f, g \in A, \quad f, g \notin \mathfrak{p}.$$

Since $\frac{a}{f}, \frac{b}{g}$ have the same image in $A_{\mathfrak{p}}$, there is an element $h \notin \mathfrak{p}$ such that $h(ga - fb) = 0$ in $A$. Thus $\frac{a}{f} = \frac{b}{g}$ in every local ring $A_{\mathfrak{q}}$ such that $f, g, h \notin \mathfrak{q}$. This set is the open set $D(f) \cap D(g) \cap D(h)$, which contains $\mathfrak{p}$. So $s = t$ on this neighbourhood of $\mathfrak{p}$, and so they have the same stalk.

**(b)** and **(c)**

First, if $f = 1$, then $A_f = A$, and

$$D(1) = V((1))^c = V(A)^c = \emptyset^c = \text{Spec}(A),$$

so (b) says $A \cong \mathscr{O}(\text{Spec}(A)) =: \Gamma(\text{Spec}(A), \mathscr{O})$, i.e. (c) is a special case of (b).

Define a homomorphism

$$\begin{array}{rccc} \psi: & A_f & \to & \mathscr{O}(D(f)) \\ & \frac{a}{f^n} & \mapsto & s \end{array}$$

where $s$ assigns to each $\mathfrak{p}$ the image of $\frac{a}{f^n}$ in $A_{\mathfrak{p}}$.

**Injectivity:**

If $\psi\left(\frac{a}{f^n}\right) = \psi\left(\frac{b}{f^m}\right)$ then for every $\mathfrak{p} \in D(f)$, $\frac{a}{f^n}$ and $\frac{b}{f^m}$ have the same image in $A_{\mathfrak{p}}$. So there is an element $h \notin \mathfrak{p}$ such that $h(f^m a - f^n b) = 0$ in $A$.

Let $\mathfrak{a}$ be the annihilator of $f^m a - f^n b$. Then $h \in \mathfrak{a}$ and $h \notin \mathfrak{p}$, so $\mathfrak{a} \nsubseteq \mathfrak{p}$. This holds for any $\mathfrak{p} \in D(f)$, thus $V(\mathfrak{a}) \cap D(f) = \emptyset$.

Therefore $f \in \sqrt{\mathfrak{a}}$, so for some $l$ we have $f^l \in \mathfrak{a}$. Thus $f^l(f^m a - f^n b) = 0$, so $\frac{a}{f^n} = \frac{b}{f^m}$ in $A_f$.

**Surjectivity:**

Let $s \in \mathscr{O}(D(f))$. By definition of $\mathscr{O}$ we can cover $D(f)$ with open sets $V_i$ on which $s$ is represented by a quotient $\frac{a_i}{g_i}$ with $g_i \notin \mathfrak{p}$ for all $\mathfrak{p} \in V_i$; i.e. $V_i \subseteq D(g_i)$.

Since open sets of the form $D(h)$ form a base for the topology, $V_i = D(h_i)$ for some $h_i$. Since $D(h_i) \subseteq D(g_i)$, we have $V((h_i)) \supseteq V((g_i))$ and so by part (c) of our Lemma

$$\sqrt{(h_i)} \subseteq \sqrt{(g_i)}$$

and so $h_i^n \in (g_i)$ for some $n$. Thus $h_i^n = cg_i$, so $\frac{a_i}{g_i} = \frac{ca_i}{h_i^n}$. Since $D(h_i) = D(h_i^n)$ we may replace $h_i$ by $h_i^n$ and $a_i$ by $ca_i$, and assume that:

- $D(f)$ is covered by the open subsets $D(h_i)$, and
- $s$ is represented by $\frac{a_i}{h_i}$ on $D(h_i)$.

Now, $D(f) \subseteq \bigcup D(h_i)$ if and only if
$$V((f)) \supseteq \bigcap V((h_i)) = V\left(\sum (h_i)\right).$$
By our lemma this is equivalent to $f \in \sqrt{\sum(h_i)}$, so $f^n \in \sum(h^i)$ for some $n$. So $f^n$ can be expressed as a finite sum
$$f^n = \sum b_i h_i, \quad b_i \in A. \qquad (*)$$
So $D(f)$ can be covered by a finite number of $D(h_i)$. *Fix* a finite set $h_1, \ldots, h_r$ such that
$$D(f) \subseteq D(h_1) \cup \cdots \cup D(h_r).$$
Now, on $D(h_i) \cap D(h_j) = D(h_i h_j)$ we have two elements of $A_{h_i h_j}$ which represent $s$, $\frac{a_i}{h_i}$ and $\frac{a_j}{h_j}$. So, by injectivity of $\psi$, we must have
$$\frac{a_i}{h_i} = \frac{a_j}{h_j} \quad \text{in } A_{h_i h_j}.$$
So for some $n$, $(h_i h_j)^n (h_j a_i - h_i a_j) = 0$.

Since there are only finitely many indices involved, we may choose $n$ large enough to simultaneously work for all $i, j$. So
$$h_j^{n+1}(h_i^n a_i) - h_i^{n+1}(h_j^n a_j) = 0.$$
Replace each $h_i$ be $h_i^{n+1}$ and each $a_i$ by $h_i^n a_i$. Then $s$ is still represented on $D(h_i)$ by $\frac{a_i}{h_i}$, and now we have $h_j a_i = h_i a_j$ for all $i, j$.

Now, write $f^n = \sum b_i h_i$ as in (*)[19]. Let $a = \sum b_i a_i$. Then for each $j$, we have
$$h_j a = \sum_i b_i a_i h_j = \sum_i b_i h_i a_j = f^n a_j.$$
This says that $\frac{a}{f^n} = \frac{a_j}{h_j}$ on $D(h_j)$. So $\psi\left(\frac{a}{f^n}\right) = s$ everywhere, and $\psi$ is surjective. $\square$

## What have we done today?

- Sheaf revision.
- Definition of a **stalk**.
  - Local nature of a sheaf vs. a presheaf.
- Spec$(A)$ revision and topology.
- Definition of the **sheaf of rings** $\mathscr{O}$ **on** Spec$(A)$ and the **spectrum of** $A$.
- Correspondences between a ring and its spectrum.

## What should we do next time?

- Define the **direct image sheaf**.

[19]Note that this is the $n$ from when we determined $D(f)$ was finitely covered by some $D(h_i)$, **not** the $n$ just used in the replacement $h_i \leftrightarrow h_i^{n+1}$.

- Define the category of **locally ringed spaces** (this will make this correspondence $A \leftrightarrow (\mathrm{Spec}(A), \mathscr{O})$ functorial).
- State and prove Prop. 2.3 in Hartshorne [5], which describes the correspondence between morphisms $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$ and morphisms $B \to A$.
- Construct some specific examples for Prop. 2.3 to better understand the induced maps.
- Define **schemes**, and some important scheme related properties, e.g. **finite type**, **separated**, **proper**.

**Jeff Bailes** *Thursday 23 February 2012*

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Let $f : X \to Y$ be a continuous function. For a sheaf $\mathcal{F}$ on $X$, the *direct image sheaf on* $Y$, $f_*\mathcal{F}$, is given by

$$(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$$

for any open set $V \subseteq Y$ (exercise: this is a sheaf, by the gluing lemma). For a sheaf $\mathcal{G}$ on $Y$, the *inverse image sheaf on* $X$, $f^{-1}(\mathcal{G})$, is the sheaf associated to the presheaf

$$U \mapsto \varinjlim_{V \supseteq f(U)} \mathcal{G}(V).$$

Let $X$ be a topological space, and let $Z \subseteq X$ be a subspace with inclusion map $i : Z \hookrightarrow X$. Let $\mathcal{F}$ be a sheaf on $X$. The *restriction of* $\mathcal{F}$ *to* $Z$ is $\mathcal{F}|_Z := i^{-1}(\mathcal{F})$.

**Lemma 41:** *Any $P \in Z$ has the same stalk in $\mathcal{F}$ as in $\mathcal{F}|_Z$.*

A *ringed space* is a pair $(X, \mathcal{O}_X)$ such that $X$ is a topological space and $\mathcal{O}_X$ is a sheaf of rings on $X$. A *morphism of ringed spaces from* $(X, \mathcal{O}_X)$ *to* $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ such that $f : X \to Y$ is continuous and

$$f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$$

is a map of sheaves of rings on $Y$.

A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that every stalk of $\mathcal{O}_X$ is a local ring. The morphisms in the category of locally ringed spaces are the $(f, f^\#)$ as above such that if $P \in X$ then the induced map[20]

$$f^\#_P : (\mathcal{O}_Y)_{f(P)} \to (\mathcal{O}_X)_P$$

is a local homomorphism[21] of local rings.

We now describe how $f^\#_P$ is induced. Let $P \in X$. The sheaf morphism

$$f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$$

[20] This is explained in the next paragraph.

[21] A ring homomorphism $f : A \to B$ between local rings is a *local homomorphism* if $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$. In other words, it sends units to units and non-units to non-units (since the maximal ideal in a local ring comprises all non-units).

induces a ring homomorphism

$$f^{\#}(V) : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$$
$$x \mapsto (f^{\#}(V))(x) \in f_* \mathcal{O}_X(V) = \mathcal{O}_X(f^{-1}(V))$$

for each open subset $V \subseteq Y$.

Let $V$ range over open neighbourhoods of $f(P)$, so that $f^{-1}(V)$ ranges over a subset of the neighbourhoods of $P$. Taking direct limits yields

$$(\mathcal{O}_Y)_{f(P)} \to \varinjlim_{V} \mathcal{O}_X(f^{-1}(V)) \hookrightarrow (\mathcal{O}_X)_P, \tag{81}$$

and $f^{\#}_P$ is the composition.

**Proposition 42:** *A ringed space morphism $(f, f^{\#})$ is an isomorphism if and only if the following are true: $f$ is a homeomorphism and $f^{\#}$ is a sheaf isomorphism.*

The main result is part $(c)$ of the next proposition.

**Proposition 43:** *(a) If $A$ is a ring then $(\mathrm{Spec}(A), \mathcal{O})$ is a locally ringed space.*

*(b) If $\varphi : A \to B$ is a ring homomorphism, then $\varphi$ induces a natural*[22] *morphism of locally ringed spaces*

$$(f, f^{\#}) : (\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)}) \to (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}).$$

*(c) Let*

$$(f, f^{\#}) : (\mathrm{Spec}(B), \mathcal{O}_{\mathrm{Spec}(B)}) \to (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$$

*be a morphism of locally ringed spaces. Then it is induced by some homomorphism*

$$\varphi : A \to B.$$

*Proof.* (a) Let $A$ be a ring. The spectrum $(\mathrm{Spec}(A), \mathcal{O})$ of $A$ is a ringed space, since $\mathcal{O}$ is a sheaf of $\mathrm{Spec}(A)$. It is a locally ringed space by 40(a).

(b) Let $\varphi : A \to B$ be a ring homomorphism. Define

$$f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$
$$\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p}).$$

Note that for $\mathfrak{a} \lhd A$,

$$f^{-1}(V(\mathfrak{a})) = V(\varphi(\mathfrak{a})), \tag{82}$$

which shows that $f$ is continuous. Localizing at some $\mathfrak{p} \in \mathrm{Spec}(B)$ would yield

$$\varphi_{\mathfrak{p}} : A_{\varphi^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$$
$$\frac{x}{y} \mapsto \frac{\varphi(x)}{\varphi(y)}, \qquad \text{for } x \in A,\ y \in A \setminus \varphi^{-1}(\mathfrak{p}).$$

The sheaf morphism $f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is defined by

$$f^{\#}(V) : \mathcal{O}_{\mathrm{Spec}(A)}(V) \to \mathcal{O}_{\mathrm{Spec}(B)}(f^{-1}(V))$$
$$s \mapsto \sqcup_{\mathfrak{p} \in f^{-1}(V)} \varphi_{\mathfrak{p}} \circ s \circ f, \tag{83}$$

[22] What is meant here?

for open $V \subseteq \operatorname{Spec}(A)$. For any $\mathfrak{p} \in \operatorname{Spec}(A)$, the induced map $f^{\#}_{\mathfrak{p}}$ is locally $\varphi_{\mathfrak{p}}$,[23] which is a local homomorphism, therefore $(f, f^{\#})$ is a morphism of locally ringed spaces. It remains to show that $(f, f^{\#})$ is natural.

(c) Let

$$(f, f^{\#}) : (\operatorname{Spec}(B), \mathcal{O}_{\operatorname{Spec}(B)}) \to (\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)})$$

be a morphism of locally ringed spaces. Consider

$$f^{\#}(\operatorname{Spec}(A)) : \mathcal{O}_{\operatorname{Spec}(A)}(\operatorname{Spec}(A)) \to \mathcal{O}_{\operatorname{Spec}(B)}(\operatorname{Spec}(B)).$$

As

$$\mathcal{O}_{\operatorname{Spec}(A)}(\operatorname{Spec}(A)) = \Gamma(\operatorname{Spec}(A), \mathcal{O}_{\operatorname{Spec}(A)}) \cong A,$$

this yields $\varphi : A \to B$. For $\mathfrak{p} \in \operatorname{Spec}(B)$, this induces local homomorphisms

$$f^{\#}_{\mathfrak{p}} : (\mathcal{O}_{\operatorname{Spec}(A)})_{f(\mathfrak{p})} \to (\mathcal{O}_{\operatorname{Spec}(B)})_{\mathfrak{p}},$$

or equivalently[24]

$$\varphi_{\mathfrak{p}} : A_{f(\mathfrak{p})} \to B_{\mathfrak{p}},$$

such that

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow & & \downarrow \\ A_{f(\mathfrak{p})} & \xrightarrow[\varphi_{\mathfrak{p}}]{} & B_{\mathfrak{p}} \end{array} \tag{84}$$

commutes. To show: if $\mathfrak{p} \in \operatorname{Spec}(B)$ then $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$. Let $\mathfrak{p} \in \operatorname{Spec}(B)$. Locally (in a neighbourhood of $\mathfrak{p}$), $\varphi^{-1} = f^{\#} : A_{\varphi^{-1}(\mathfrak{p})} \to B_{\mathfrak{p}}$. As $f^{\#}$ is a local homomorphism, it follows that $\varphi^{-1}$ is a local homomorphism, so

$$\varphi^{-1}(\mathfrak{p}) = f(\mathfrak{p}). \tag{85}$$

From the construction in $B$, we see that $(f, f^{\#})$ is induced by $\varphi$.

□

An *affine scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to $\operatorname{Spec}(A)$ for some ring $A$. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$ in which every point has an open neighbourhood $U$ such that

$$(U, \mathcal{O}_X|_U)$$

is an affine scheme. With this notation, $X$ is the *underlying topological space* and $\mathcal{O}_X$ is the *structure sheaf* of the scheme.[25]

**Example 44:** *Let $k$ be a field. Then $(Spec(k), \mathcal{O})$ is an affine scheme with $Spec(k) = *$ and $\mathcal{O} = k$.*

[23] Consider $\langle s, U\rangle$ in the stalk. Pick $V \in \operatorname{Spec}(B)$ such that $\mathfrak{p} \in V \subseteq U$ and $s(\mathfrak{q}) = \frac{a}{b} \in A_{\mathfrak{q}}$ for $\mathfrak{q} \in V$. This is a local ring homomorphism because (1) $s \mapsto s \circ f$ is locally $c_{a/b} \mapsto c_{a/b}$ (which sends units to units and non-units to non-units, since units in $A_{\mathfrak{q}}$ are $\frac{a}{b}$ such that $a, b \in A \setminus \mathfrak{q}$) and (2) $\varphi_{\mathfrak{q}}$ is a local homomorphism for any $\mathfrak{q} \in f^{-1}(V)$.

[24] As described in (b), locally $f^{\#}_{\mathfrak{p}} = \varphi_{\mathfrak{p}}$.

[25] Often a scheme is denoted by $X$, and its underlying topological space by $\operatorname{sp}(X)$.

**Example 45:** *Let k be a field. We define the* affine line *over k to be* $\mathbb{A}^1_k := Spec(k[x])$. *There is a point* $\zeta \in \mathbb{A}^1_k$ *which corresponds to the zero ideal in* $k[x]$. *The closure of* $\zeta$ *is* $\mathbb{A}^1_k$ *and* $\zeta$ *is called a* generic point. *The other points correspond to the maximal ideals in* $k[x]$ *and are all closed points. These closed points are in one-to-one correspondence with the non-constant monic irreducible polynomials in* $k[x]$.

*In particular, if k is algebraically closed, then the closed points of* $\mathbb{A}^1_k$ *are in one-to-one correspondence with elements of k.*

$(\mathbb{A}^1_k, \mathcal{O}_{\mathbb{A}^1_k})$ *is a scheme.*

A scheme is called *connected* if its underlying topological space is connected. A scheme is called *irreducible* if its underlying topological space is irreducible.

A scheme $(X, \mathcal{O}_X)$ is called *reduced* if whenever $U \subset X$ is open then $\mathcal{O}_X(U)$ contains no nilpotent elements. It is called *integral* if every $\mathcal{O}_X(U)$ is an integral domain.

**Proposition 46:** *A scheme is integral if and only if it is both reduced and irreducible.*

A morphism of schemes $f : X \to Y$ is *locally of finite type* if there exists $\{V_i = Spec(B_i)\}_{i \in \mathcal{I}}$ a cover of $Y$ by open affine subsets, such that if $i \in \mathcal{I}$ then $f^{-1}(V_i)$ can be covered by open affine subsets $U_{i,j} = Spec(A_{i,j})$, where each $A_{i,j}$ is a finitely generated $B_i$-algebra.

The morphism $f : X \to Y$ is of *finite type* if whenever $i \in \mathcal{I}$ then $f^{-1}(V_i)$ can be covered by finitely many such $U_{i,j}$.

**Sam Chow** *Thursday 1 March 2012*

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**1.1**

Let $X$ be an algebraic variety over $\mathbb{Z}$. For $x \in |X|$, let $N(x)$ be the number of elements in the residue field $k(x)$ of $X$ in $x$.

Before we continue, let's try to understand what the last paragraph is saying.

$$X = \text{Spec}(A),$$

where

$$A = \frac{\mathbb{Z}[x_1, \dots, x_n]}{(f_1, \dots, f_s)}$$

is an integral domain and $f_1, \dots, f_s \in \mathbb{Z}[x_1, \dots, x_n]$. Define

$$|X| = \{\text{closed points in } X\} = \{\text{maximal ideals in } A\}.$$

Note that if $x \in |X|$ then $x = (a_1, \dots, a_n) \in \mathbb{Z}^n$ in local coordinates, where

$$f_i(a_1, \dots, a_n) = 0, \qquad \text{for } i = 1, 2, \dots, n.$$

For $x \in |X|$, the *residue field of* $X$ *in* $x$ is

$$k(x) = \frac{\mathcal{O}_x}{\mathfrak{m}_x},$$

where $\mathcal{O}$ is the ring of regular functions on $X$ and $\mathcal{O}_x$ is its germ at $x \in |X|$ (a local ring) and $\mathfrak{m}_x$ is its maximal ideal (indeed $k(x)$ is a field).

The *Hasse-Weil zeta function* of $X$ is

$$\zeta_X(s) = \prod_{x \in |X|} \frac{1}{1 - N(x)^{-s}}, \tag{86}$$

which converges for $\mathrm{Re}(s)$ sufficiently large.

**Example 47** ($X = \mathrm{Spec}(\mathbb{Z})$)**:** *The only maximal ideals are* $p\mathbb{Z}$*, where* $p$ *is a prime number. Then*

$$\mathcal{O}_{(p)} \cong \mathbb{Z}_{(p)},$$

*and* $\mathfrak{m}_{(p)} \cong p\mathbb{Z}_{(p)}$ *via the same isomorphism (restricted). Thus*

$$k((p)) \cong \frac{\mathbb{Z}_{(p)}}{p\mathbb{Z}_{(p)}} = \{[0/1], [1/1], \ldots, [p-1/1]\}, \tag{87}$$

*so* $N((p)) = p$*. Now*

$$\zeta_X(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} = \sum_{n=1}^{\infty} n^{-s} = \zeta(s), \tag{88}$$

*using the Euler product formula.*

We'll only really be studying algebraic varieties over $\mathbb{F}_q$, where $q$ is a power of the prime number $p$, noting that $p$ is the characteristic of $\mathbb{F}_q$. For $x \in |X|$, we write $q_x$ instead of $N(x)$. Put $\deg(x) = [k(x) : \mathbb{F}_q]$.[26] Then

$$q_x = N(x) = \#k(x) = q^{\deg(x)}.$$

Here we introduce the varaible $t = q^{-s}$. Put

$$Z(X, t) = \prod_{x \in |X|} \frac{1}{1 - t^{\deg(x)}}. \tag{89}$$

This converges if $|t|$ is sufficiently small, and we have

$$\zeta_X(s) = Z(X, q^{-s}). \tag{90}$$

### 1.2

Dwork and Grothendieck showed that $Z(X, t)$ is a rational function of $t$, i.e. there exist $P, Q \in \mathbb{Q}[t]$ such that

$$Z(X, t) = \frac{P(t)}{Q(t)}.$$

For Grothendieck, this is a corollary of general results in $l$-adic cohomology ($l \neq p$). This provides a cohomological interpretation of the zeroes and the poles of $Z(X, t)$, as well as a functional equation when $X$ is compact and smooth [some more history].

[26] We should check that $k(x)$ is a field extension of $\mathbb{F}_q$.

**1.3**

Let $X$ be an algebraic variety over an algebraically closed field of characteristic $p$ (we don't exclude the case $p = 0$). For prime $l \neq p$, Grothendieck defined $l$-adic cohomology groups $H^i(X, \mathbb{Q}_l)$. There are also cohomology groups with compact support, $H^i_c(X, \mathbb{Q}_p)$. If $X$ is compact, then $H^i(X, \mathbb{Q}_l) = H^i_c(X, \mathbb{Q}_p)$. The groups $H^i_c(X, \mathbb{Q}_l)$ are vector spaces of finite dimension over $\mathbb{Q}_l$, trivial for $i > 2\dim X$.

**1.4**

Let $X_0$ be a variety over $\mathbb{F}_q$, and let $X$ be the corresponding variety over $\overline{\mathbb{F}}_q$. Locally,

$$X_0 = \mathrm{Spec}(A_0), \qquad A_0 = \frac{\mathbb{F}_q[x_1, \ldots, x_n]}{(f_1, \ldots, f_s)}$$

and

$$X = \mathrm{Spec}(A), \qquad A = \frac{\overline{\mathbb{F}}_q[x_1, \ldots, x_n]}{(f_1, \ldots, f_s)},$$

where $f_1, \ldots, f_s \in \mathbb{F}_q[x_1, \ldots, x_n]$. Let $F : |X| \to |X|$ be the Frobenius. Locally,

$$(a_1, \ldots, a_n) \mapsto (a_1^q, \ldots, a_n^q).$$

Note that[27]

$$|X_0| = \frac{|X|}{\sim}, \tag{91}$$

where $x \sim y$ if and only if $x$ and $y$ are in the same orbit. For any subring $R$ of $\overline{\mathbb{F}}_q$, we define $X_0(R)$ to be the set of closed points of $X$ that have all local coordinates in $R$. We can identify $|X|$ with $X_0(\bar{\mathbb{F}}_q)$.

**Proposition 48:** *(a) By equation* (17),

$$|X|^F = X_0(\mathbb{F}_q).$$

*(b) Also by equation* (17),

$$|X|^{F^n} = X_0(\mathbb{F}_{q^n}), \qquad \text{for } n \in \mathbb{Z}_{>0}.$$

*(c) The set $|X_0|$ of closed points in $X_0$ identifies itself with the set $|X|_F$ of orbits of $F$ in $|X|$. The degree* $\deg(x)$ *of $x \in |X_0|$ is the number of elements in the corresponding orbit.*

*(d) From (b) and (c), we get the formula*

$$\#|X|^{F^n} = \#X_0(\mathbb{F}_{q^n}) = \sum_{x \in |X_0| : \deg(x) | n} \deg(x) \tag{92}$$

*(for $x \in |X_0|$ and* $\deg(x)|n$*, the point $x$ determines* $\deg(x)$ *points with coordinates in $\mathbb{F}_{q^n}$, all conjugated by $\mathbb{F}_q$).*

**1.5**

The morphism $F$ is finite, and in particular proper. It therefore induces maps

$$F^* : H^i_c(X, \mathbb{Q}_l) \to H^i_c(X, \mathbb{Q}_l).$$

[27] Dougal will prove this next week.

Grothendieck proved the Lefschetz trace formula,

$$\#|X|^F = \sum_{i=0}^{2d} (-1)^i \mathrm{Tr}(F^*, H_c^i(X, \mathbb{Q}_l)), \qquad d = \dim(X).$$

The right hand side, *a priori* an $l$-adic number, is an integer, and equal to theleft hand side. We note that this formula is reasonable because $dF = 0$, even at infinity ($X$ is not assumed to be compact); the equation $dF = 0$ implies that the fixed points of $F$ have multiplicity 1. An analagous formula holds for iterates of $F$:

$$\#|X|^{F^n} = \#X_0(\mathbb{F}_{q^n}) = \sum_i (-1)^i \mathrm{Tr}(F^{*n}, H_c^i(X, \mathbb{Q}_l)). \tag{93}$$

Apply the logarithmic derivative to equation (89):

$$\begin{aligned} \frac{t\frac{d}{dt}Z(X_0,t)}{Z(X_0,t)} &= t\frac{d}{dt}\log Z(X_0,t) = \sum_{x\in|X_0|} \frac{\deg(x)t^{\deg(x)}}{1-t^{\deg(x)}} \\ &= \sum_{x\in|X_0|}\sum_{n>0} \deg(x) t^{n\deg(x)} = \sum_{n>0} \#X_0(\mathbb{F}_{q^n})t^n. \end{aligned} \tag{94}$$

For $\phi$ a linear transformation on a vector spaces $V$, we have an identity on formal power series:

$$t\frac{d}{dt}\log(\det(1-\phi t, V)^{-1}) = \sum_{n>0} \mathrm{Tr}(\phi^n, V)t^n \tag{95}$$

(check it for $\dim(V) = 1$, and observe that both sides are additive in $V$ in a short exact sequence).[28] Substitute (93) into (94) and apply (95) to get

$$t\frac{d}{dt}\log Z(X_0,t) = \sum_i (-1)^i t\frac{d}{dt}\log\det(1-F^*t, H_c^i(X,\mathbb{Q}_l))^{-1},$$

so it follows that

$$Z(X_0,t) = \prod_i \det(1-F^*t, H_c^i(X,\mathbb{Q}_l))^{(-1)^{i+1}}. \tag{96}$$

The right hand side is in $\mathbb{Q}_l(t)$. The formula asserts that the Taylor series about $t = 0$, *a priori* a formal power series in $\mathbb{Q}_l[[t]]$ with constant term 1, is in $\mathbb{Z}[[t]]$, and is equal to the left hand hade, also considered as a power series in $t$. This formula is Grothendieck's cohomological interpretation of the function $Z(X,t)$. Our main result follows.[29]

### 1.6

**Theorem 49:** *Let $X_0$ be a projective non-singular (= smooth) variety over $\mathbb{F}_q$. For each i, the characteristic polynomial* $\det(t - F^*, H^i(X, \mathbb{Q}_l))$ *has coefficients independent of l (assuming that $l \neq p$). The complex roots $\alpha$ of this polynomial (the complex conjugates of the eigenvalues of $F^*$) have absolute values $|\alpha| = q^{i/2}$.*

**Dougal Davis** *Thursday 8 March 2012*

We address some of the issues arising out of Sam's talk. Rings are assumed to be

[28] Alternatively, change bases to make the matrix upper triangular.

[29] This is the Riemann hypothesis.

commutative and unital.

Let $\mathbb{F}$ be a field. A *scheme over* $\mathbb{F}$ is a scheme $Y_0$ with a morphism $Y_0 \to \mathrm{Spec}(\mathbb{F})$.[30] In the case where $Y_0 = \mathrm{Spec}(A_0)$, this corresponds to a ring homomorphism $\mathbb{F} \to A_0$, i.e. $A_0$ is a $\mathbb{F}$-algebra. If $\mathbb{K} \supseteq \mathbb{F}$ is a field extension, we can extend $Y_0$ to a $\mathbb{K}$-scheme $Y$: this is the pullback

$$\begin{array}{ccc} Y & \longrightarrow & Y_0 \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{K}) & \longrightarrow & \mathrm{Spec}(\mathbb{F}), \end{array}$$

where the universal property is there but not drawn. The map $\mathrm{Spec}(\mathbb{K}) \to \mathrm{Spec}(\mathbb{F})$ is induced by the inclusion $\mathbb{F} \hookrightarrow \mathbb{K}$. For affine schemes (here $Y = \mathrm{Spec}(A)$), this corresponds to the pushout

$$\begin{array}{ccc} A & \longleftarrow & A_0 \\ \uparrow & & \uparrow \\ \mathbb{K} & \longleftarrow & \mathbb{F}, \end{array}$$

i.e. $A = \mathbb{K} \otimes A_0$.

Let $\mathbb{F}_q$ be a finite field of size $q$ and with characteristic $p > 0$. Let $X$ be the algebraic variety obtained from $X_0$ by extension of the scalars to $\mathbb{F}_q$. Let $U_0 \subseteq X_0$ be an affine open set, and let $U = \iota^{-1}(U_0)$, where $\iota : X \to X_0$ is as in[31]

$$\begin{array}{ccc} X & \xrightarrow{\iota} & X_0 \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\overline{\mathbb{F}}_q) & \longrightarrow & \mathrm{Spec}(\mathbb{F}_q). \end{array}$$

Let

$$U_0 = \mathrm{Spec}(A_0), \qquad A_0 = \frac{\mathbb{F}_q[t_1, \ldots, t_n]}{(f_1, \ldots, f_s)},$$
$$U = \mathrm{Spec}(A), \qquad A = \frac{\overline{\mathbb{F}}_q[t_1, \ldots, t_n]}{(f_1, \ldots, f_s)}, \tag{97}$$

where $f_1, \ldots, f_s \in \mathbb{F}_q[t_1, \ldots, t_n]$.[32]

Define

$$\tilde{F} : A \to A$$

[30] The same construction holds for schemes over Noetherian rings. Indeed, this definition extends to schemes over schemes.

[31] This is not an inclusion map, but it is induced by one, as we shall see.

[32] We write $A_0$ in that form, and then the commutative diagram for the pushout (since $U$ and $U_0$ are affine schemes) dictates that $A = \overline{\mathbb{F}}_q \otimes A_0$.

$$\sum_i \alpha_i t^i \mapsto \sum_i \alpha_i^q t^i, \qquad \alpha_i \in A, \tag{98}$$

for finite sums, where if $i = (i_1, \dots, i_n) \in \mathbb{Z}^n_{\geq 0}$ then $t^i = t_1^{i_1} \cdots t_n^{i_n}$. In words, $\tilde{F}$ applies the Frobenius to the coefficients. Let

$$F : U \to U$$
$$x \mapsto \tilde{F}(x).$$

In particular, for maximal ideals,

$$F((t_1 - a_1, \dots, t_n - a_n)) = (t_1 - a_1^q, \dots, t_n - a_n^q). \tag{99}$$

"Pasting together" the maps $F : U \to U$ gives a well-defined morphism $F : X \to X$.

**Lemma 50:** *The morphism $\iota : X \to X_0$ induces a bijection*

$$\iota : |X|_F \to |X_0|$$
$$O \mapsto \iota(x) \qquad \text{if } x \in O,$$

*where $|X|_F$ denotes the set of orbits of closed points of $X$ under $F$. Moreover, if $x_0 \in |X_0|$ then*

$$\deg(x_0) = \#\iota^{-1}(x_0). \tag{100}$$

We spend the rest of the talk proving this, but let us first give an analogy.

**Example 51:** *Let $X_0 = \mathrm{Spec}(\mathbb{R}[t])$ and $X = \mathrm{Spec}(\mathbb{C}[t])$.*[33] *Here*

$$x_0 = (t^2 + 1) \in |X_0|$$

*factorizes into two prime ideals $(t - i)$ and $(t + i)$, which are in the same Galois orbit. The degree of $x_0$ is 2, which is the size of the corresponding Galois orbit.*

*Proof of lemma 50.* It suffices to prove the affine case, so write $X = U = \mathrm{Spec}(A)$. First notice that $\iota : U \to U_0$ is induced by the inclusion map[34]

$$\mathrm{inc} : A_0 \to A,$$

i,.e. if $x \in U$ then

$$\iota(x) = \mathrm{inc}^{-1}(x) = \{g \in A_0 \mid \mathrm{inc}(g) \in x\} = x \cap A_0. \tag{101}$$

To show:

(i) If $x \in U$ then $\iota \circ F(x) = \iota(x)$.

(ii) If $x \in |U|$ then $\iota(X) \in |U_0|$.

(iii) $\iota : |U|_F \to |U_0|$ is surjective.

(iv) $\iota : |U|_F \to |U_0|$ is injective.

(v) If $x_0 \in |U_0|$ then $\deg(x_0) = \#\iota^{-1}(x_0)$.

We now prove each part.

[33] These aren't finite fields, and some details are omittend, but the point is that Galois theory is what's driving it.

[34] In the above example, $(t - i) \cap \mathbb{R}[t] = (t^2 + 1)$, so $\iota((t - i)) = (t^2 + 1)$.

(i) Let $x \in U$. Then

$$\begin{aligned}\iota \circ F(x) &= F(x) \cap A_0 = \tilde{F}(x) \cap A_0 = \tilde{F}(x) \cap \tilde{F}(A_0)\\ &= \tilde{F}(x \cap A_0), \qquad \text{as } \tilde{F} \text{ is injective}\\ &= x \cap A_0 = \iota(x).\end{aligned}$$

(ii) Let $x \in |U|$. To show: $\iota(x) = |U_0|$. To show: $\iota(x)$ is a maximal ideal in $A_0$. To show:

$$\frac{A_0}{\iota(x)}$$

is a field. As $x$ is a maximal ideal in $A$, there exist $x_1, \ldots, x_n \in \overline{\mathbb{F}}_q$ such that

$$x = (t_1 - x_1, \ldots, t_n - x_n).$$

Now

$$\frac{A}{x} = \frac{\overline{\mathbb{F}}_q[t_1, \ldots, t_n]}{(t_1 - x_1, \ldots, t_n - x_n)} \cong \overline{\mathbb{F}}_q. \tag{102}$$

The isomorphism follows from the first isomorphism theorem, considering the evaluation map

$$\begin{aligned}\overline{\mathbb{F}}_q[t_1, \ldots, t_n] &\to \overline{\mathbb{F}}_q\\ g &\mapsto g|_{t_1 = x_1, \ldots, t_n = x_n}.\end{aligned}$$

The map $\mathrm{inc} : A_0 \to A$ induces an injective homomorphism

$$\mathrm{inc} : \frac{A_0}{\iota(x)} \to \frac{A}{x}. \tag{103}$$

Moreover, from the commutative diagram

$$\begin{array}{ccc} A & \xleftarrow{\mathrm{inc}} & A_0 \\ \uparrow & & \uparrow \\ \overline{\mathbb{F}}_q & \longleftarrow & \mathbb{F}_q, \end{array}$$

it follows that

$$\begin{array}{ccc} \frac{A}{x} & \xleftarrow{\mathrm{inc}} & \frac{A_0}{\iota(x)} \\ \uparrow & & \uparrow \\ \overline{\mathbb{F}}_q & \longleftarrow & \mathbb{F}_q, \end{array}$$

also commutes. Hence, $\mathrm{inc} : \frac{A_0}{\iota(x)} \to \frac{A}{x}$ is an isomorphism of $\mathbb{F}_q$-algebras, and is therefore also a ring homomorphism. Now $\frac{A_0}{\iota(x)}$ is a field, from the ring homomorphisms

$$\frac{A_0}{\iota(x)} \cong \frac{A}{x} \cong \overline{\mathbb{F}}_q. \tag{104}$$

Hence $\iota(x) \in |U_0|$.

(iii) Let $x_0 \in |U_0|$. To show: there exists $x \in |U|$ such that $\iota(x) = x_0$. Let $\{g_i\}_{i \in I}$ be a basis for $x_0$ as an $\mathbb{F}_q$-vector space, and extend this to a basis $\{g_i\}_{i \in I'}$ for $A_0$ as an $\mathbb{F}_q$-vector space. Then $\{g_i\}_{i\in I'}$ is a basis for the $\overline{\mathbb{F}}_q$-vector space

$$A = \overline{\mathbb{F}}_q \otimes A_0.$$

Hence,

$$(\mathrm{inc}(x_0)) = \Big\{ \sum_{i\in I} \alpha_i g_i \mid \alpha_i \in \mathbb{F}_q \Big\} \neq A, \tag{105}$$

as $x_0 \neq A_0$ implies that $I \neq I'$. As $A$ is a Noetherian ring, there therefore exists a maximal ideal $x \supseteq (\mathrm{inc}(x_0))$.[35] To show: $\iota(x) = x_0$. As $x \supseteq (\mathrm{inc}(x_0))$, it follows that $\iota(x) \supseteq x_0$. As $x_0$ is maximal and $\iota(x) \neq A_0$, it follows that $\iota(x) = x_0$. Hence $\iota : |U|_F \to |U_0|$ is surjective.

(iv) For $x \in U$, let $x_F = \{x, F(x), F(F(x)), \ldots\}$ (the orbit). Assume that $x, x' \in |U|$ satisfy $\iota(x) = \iota(x')$. To show: $x_F = x'_F$. Let

$$x = (t_1 - x_1, \ldots, t_n - x_n), \qquad x' = (t_1 - x'_1, \ldots, t_n - x'_n).$$

For $a \in \overline{\mathbb{F}}_q$, let $\deg(a) = \#a_F$. Let

$$g_i = \prod_{j=0}^{\deg(x_i)-1} (t_i - x_i^{q^j}), \qquad i = 1, 2, \ldots, n.$$

For $i = 1, 2, \ldots, n$, we see that $\tilde{F}(g_i) = g_i$, so $g_i \in A_0$. Hence

$$g_i \in x \cap A_0 = \iota(x) = \iota(x'), \tag{106}$$

so

$$\mathrm{inc}(g_i) \in x' = (t_1 - x'_1, \ldots, t_n - x'_n), \tag{107}$$

so

$$0 = g_i(x'_1, \ldots, x'_n) = \prod_{j=0}^{\deg(x_i)-1} (x'_i - x_i^{q^j}), \tag{108}$$

so $x'_i \in \{x_i, x_i^q, x_i^{q^2}, \ldots\}$. Let $S \subseteq (\overline{\mathbb{F}}_q)^n$ be the finite subset

$$S = \{x_1, x_1^q, \ldots\} \times \{x_2, x_2^q, \ldots\} \times \ldots \times \{x_n, x_n^q, \ldots\}.$$

As $S$ is finite, there exists $h \in A$ such that $h|_S$ is injective and $h(x_1, \ldots, x_n) = 0$. Let

$$h_{\tilde{F}} = \{h, \tilde{F}(h), \tilde{F}(\tilde{F}(h)), \ldots\},$$

a finite set,[36] and let

$$\tilde{h} = \prod_{h' \in h_{\tilde{F}}} h'.$$

Now $\tilde{F}(\tilde{h}) = \tilde{h}$, so $\tilde{h} \in \iota(x) = \iota(x')$, so

$$\tilde{h}(x'_1, \ldots, x'_n) = 0.$$

Thus, there exists $j \in \mathbb{Z}_{\geq 0}$ such that

$$\tilde{F}^j(h)(x'_1, \ldots, x'_n) = 0.$$

Choose $k \in \mathbb{Z}_{\geq 0}$ such that

$$j + k \equiv 0 \mod (\#h_{\tilde{F}} - 1), \tag{109}$$

in which case

$$\tilde{F}^{j+k}(h) = h. \tag{110}$$

[35]Use Zorn's lemma with the ascending chain condition.

[36]The 'degree' is finite on each coordinate, since $\overline{\mathbb{F}}_q = \cup_{m=1}^{\infty} \mathbb{F}_{q^m}$, from equation (21).

Now

$$h\Big((x_1')^{q^k},\dots,(x_n')^{q^k}\Big) = \tilde{F}^{j+k}(h)\Big((x_1')^{q^k},\dots,(x_n')^{q^k}\Big) \tag{111}$$

$$= \Big(\tilde{F}^{j}(h)(x_1',\dots,x_n')\Big)^{q^k} = 0. \tag{112}$$

As $(x_1')^{q^k},\dots,(x_n')^{q^k} \in S$, and as $h|_S$ is injective, and as $h(x_1,\dots,x_n) = 0$, this implies that

$$x_i = (x_i')^{q^k},$$

so $x_F = x_F'$. Hence, $\iota : |U|_F \to |U_0|$ is injective.

(v) Let $x_0 \in |U_0|$. To show: $\deg(x_0) = \#\iota(x_0)$. By Richard (40),

$$k(x_0) = \frac{(A_0)_{x_0}}{\mathfrak{m}_{x_0}} \cong \frac{A_0}{x_0}. \tag{113}$$

Let $x \in \iota^{-1}(x_0)$. As $\iota : |U|_F \to |U_0|$ is surjective, there exist $x_1,\dots,x_n \in \overline{\mathbb{F}}_q$ such that $x = (t_1 - x_1,\dots,t_n - x_n)$. The $\mathbb{F}_q$-algebra homomorphism

$$\begin{aligned} \frac{A_0}{x_0} &\to \mathbb{F}_q[x_1,\dots,x_n] \\ t_i &\mapsto x_i \end{aligned}$$

is an isomorphism of fields, by the first isomorphism theorem.[37] Hence,

$$\deg(x_0) = \left[\frac{A_0}{x_0} : \mathbb{F}_q\right] = \Big[\mathbb{F}_q[x_1,\dots,x_n] : \mathbb{F}_q\Big]. \tag{114}$$

For $i = 1,2,\dots,n$, let $m_i = \#\{x_i, x_i^q, \dots\}$. Then

$$m = \operatorname{lcm}(m_1,\dots,m_n)$$

is the smallest element of $\mathbb{Z}_{>0}$ such that

$$x_i \in \mathbb{F}_{q^m} \qquad \text{for } i = 1,2,\dots,n.$$

Then $\mathbb{F}_{q^m}$ is the smallest subfield of $\overline{\mathbb{F}}_q$ containing $x_1,\dots,x_n$, so

$$\mathbb{F}_q[x_1,\dots,x_n] = \mathbb{F}_{q^m}. \tag{115}$$

Hence,

$$\deg(x_0) = [\mathbb{F}_{q^m} : \mathbb{F}_q] = m. \tag{116}$$

Note that $F^j(x) = x$ if and only if $m|j$. Hence,

$$\deg(x_0) = m = \#\{x, F(x), F(F(x)), \dots\} = \#\iota^{-1}(x_0), \tag{117}$$

as $\iota : |U|_F \to |U_0|$ is injective.

□

**Dougal Davis** *Thursday 15 March 2012*

This is a continuation from the end of the 1/03/2012 session.

[37]Consider the evaluation map $A_0 \to \mathbb{F}_q[x_1,\dots,x_n]$. It's surjective, and the following holds. If $g \in A_0$ and $g(x_1,\dots,x_n) = 0$ then $g \in A_0 \cap x = \iota(x) = x_0$. Moreover, if $g \in x_0$ then $g(x_1,\dots,x_n) = 0$.

**Theorem 52:** *Let $X_0$ be the nonsingular projective variety over $\mathbb{F}_q$. For each $i$, the characteristic polynomial*

$$\det(t - F^*, H^i(X, \mathbb{Q}_l))$$

*has integer coefficients independent of $l$ (where $l \neq q$).*

*Futhermore, the complex roots $\alpha$ of the polynomial (i.e. the conjugates in $\mathbb{C}$ of the eigenvalues of $F^*$) have absolute value $|\alpha| = q^{\frac{1}{2}}$.*

**Remark:** If $a \in \overline{\mathbb{Q}}_l$, algebraic over $\mathbb{Q}$ then, there exists $f \in \mathbb{Q}[t]$ such that $f(a) = 0$, $f$ is not identically zero, and $f$ is irreducible. The conjugates in $\mathbb{C}$ of $a$ are the complex roots of $f$.

**Remark:** We will prove theorem 52 using the following lemma.

**Lemma 53:** *For each $i$ and each $l \neq p$ the eigenvalues of the endomorphisms $F^*$ of $H^i(X, \mathbb{Q}_l)$ are algebraic (over $\mathbb{Q}$), all of whose conjugates in $\mathbb{C}$, $\alpha$, have absolute value $|\alpha| = q^{\frac{1}{2}}$.*

*Proof.* of theorem 52. Assume lemma 53.

Recall that

$$Z(X_0, t) = \prod_{x_0 \in |X_0|} (1 - t^{\deg(x_o)})^{-s} \in \mathbb{Z}[[t]].$$

Hence, consider $Z(X_0, t)$ as a a formal power series with constant term 1:

$$Z(X_0, t) = \sum_{n \geq 0} a_n t^n \in \mathbb{Z}[[t]].$$

By 1.5.4 in Deligne's paper we have:

$$Z(X_0, t) = \prod_{x_0 \in |X_0|} \det(1 - F^* t, H^i_c(X, \mathbb{Q}_l))^{(-1)^{i+1}} \tag{118}$$

So the image of $Z(X_0, t)$ in $\mathbb{Q}_l[[t]] \supseteq \mathbb{Z}[[t]]$ is the Taylor series of a rational function. This holds if and only if the Hankel determinants

$$H_k = \det\big((a_{i+j+k})_{0 \leq i,j \leq M}\big)$$

vanish for $k > N$ for some $M, N \in \mathbb{Z}_{>0}$. However, this holds in $\mathbb{Q}_l$ if and only if it holds in $\mathbb{Q}$.

Thus,

$$Z(X_0, t) = \frac{P}{Q}, \qquad P, Q \in \mathbb{Z}[t],$$

where $P$ and $Q$ are relatively prime and each has positive constant term.

Now, by a lemma of Fatou, since $Z(X_0, t) \in \mathbb{Z}[t]$ and has constant term 1, both $P$ and $Q$ have constant term 1.

Let

$$P_i(t) = \det(1 - F^*t, H^i(X, \mathbb{Q}_l)) \\ = t^i \det\Big(\frac{1}{t} - F^*, H^i(X, \mathbb{Q}_l)\Big),$$

then by lemma 53, $P_0, \ldots, P_{2d}$ are relatively prime in pairs.

The RHS of equation 118 is therefore in the simplest form, and

$$P(t) = \prod_{i \text{ odd}} P_i(t), \qquad Q(t) = \prod_{i \text{ even}} P_i(t)$$

Let $K$ be the subfield of an algebraic closure $\overline{\mathbb{Q}}_l$ of $\mathbb{Q}_l$ generated over $\mathbb{Q}$ by the roots of $R(t) = P(t)Q(t)$. The roots of $P_i(t)$ are those roots of $R(t)$ with the property that all their conjugates in $\mathbb{C}$ have absolute value $q^{-\frac{1}{2}}$.

This set is stable under $\mathrm{Gal}(k/\mathbb{Q})$.

The polynomial $P_i(t)$ therefore has rational coefficients. Furthermore, by Gauss' lemma it has integer coeffiecients. (Another reason is because the roots of $P_i(t)$ are roots of $R(t)$ so must be algebraic integers.)

The description above of the roots of $P_i(t)$ is independent of $l$. The polynomial $P_i(t)$ is therefore independent of of $l$. Hence we have proved the theorem. □

**Sam Chow** *Thursday 29 March 2012*

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Let $X$ be a variety over a field $k$, and let $l$ be a prime number. We want to define constructible étale sheaves of $\mathbb{Q}_l$-vector spaces on $X$. Note that $X$ is a scheme, so we can give it the étale topology.

**Recall.** The objects of the category $\text{Ét}(X)$ are the étale morphisms of the form

$$f : V \to X,$$

where $V$ is a scheme (these are called $X$-schemes, or schemes defined over $X$). The morphisms between $f_1 : V_1 \to X$ and $f_2 : V_2 \to X$ are the étale morphisms $g : V_1 \to V_2$ such that $f_1 = f_2 \circ g$. The *étale site* of $X$, denoted $X_{\text{ét}}$, is the category $\text{Ét}(X)$ along together with all 'coverings' (additional data). Specifically, to each object $\phi : V \xrightarrow{\text{ét}} X$, associate the collection of all families of morphisms

$$\{\phi_i : U_i \xrightarrow{\text{ét}} V\}_{i \in I}$$

such that

$$\cup_{i \in I} \phi_i(U_i) = V.$$

An *étale presheaf* is a contravariant functor$\mathcal{F} : \text{Ét}(X) \to \text{Set}$. An *étale sheaf* is an *étale presheaf* such that. . . is an equaliser.

**Recall.** Let $f : X \to Y$ be a continuous function. For a sheaf $\mathcal{G}$ on $Y$, the *inverse image sheaf on* $X$, $f^{-1}(\mathcal{G})$, is the sheaf associated to the presheaf

$$U \mapsto \varinjlim_{V \supseteq f(U)} \mathcal{G}(V).$$

For schemes, we often consider sheaves of $\mathcal{O}$-modules, where $\mathcal{O}$ is the structure sheaf. Let $f : X \to Y$ be a morphism of locally ringed spaces (such as schemes). For a sheaf $\mathcal{G}$ of $\mathcal{O}_Y$-modules on $Y$, the *inverse image* of $\mathcal{G}$ is[38]

$$f^*(\mathcal{G}) = f^{-1}(\mathcal{G}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

Let $Z \subseteq X$ be a subscheme with inclusion morphism $i : Z \hookrightarrow X$, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on $X$. The *restriction of* $\mathcal{F}$ *to* $Z$ is $\mathcal{F}|_Z = i^*(\mathcal{F})$.

An étale sheaf $\mathcal{F}$ is *locally constant* if there exists a covering $(U_i \to X)$ such that $\mathcal{F}|_{U_i}$ is constant for each $i$.[39]

We abuse wording, for conciseness:

- By *sheaf*, we mean étale sheaf.
- By $\mathbb{Z}_l$*-sheaf*, we mean sheaf of $\mathbb{Z}_l$-modules.
- By $\mathbb{Q}_l$*-sheaf*, we mean sheaf of $\mathbb{Q}_l$-modules.

A $\mathbb{Z}_l$*-sheaf on* $X$ (or an *l-adic sheaf*) is a projective system (inverse system) $F = (F_n)_{n \in \mathbb{Z}_{\geq 0}}$ of sheaves (a priori, of abelian groups) on $X$ such that if $n \in \mathbb{Z}_{\geq 0}$ then $F_{n+1} \to F_n$ induces an isomorphism

$$\frac{F_{n+1}}{l^n F_{n+1}} \xrightarrow{\simeq} F_n.$$

For $n \in \mathbb{Z}_{\geq 0}$, $F_n$ is annihilated by $l^n$, and is therefore a $\frac{\mathbb{Z}}{l^n\mathbb{Z}}$-module. A sheaf of $\frac{\mathbb{Z}}{l^n\mathbb{Z}}$-modules is *locally constant* if it is locally constant as a sheaf of abelian groups. A $\mathbb{Z}_l$-sheaf $(F_n)_{n \in \mathbb{Z}_{\geq 0}}$ is *twisted constant* if each $F_n$ is locally constant. A $\mathbb{Q}_l$-sheaf $\mathcal{F}$ on $X$ is *twisted constant* if there exists a twisted constant $\mathbb{Z}_l$-sheaf $F$ on $X$ such that

$$\mathcal{F} \simeq F \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

as $\mathbb{Q}_l$-sheaves.

**Recall.** A subset $Y$ of a topological space $X$ is *locally closed* if it satisfies the following equivalent conditions:

(i) There exists an open set $U$ and a closed set $F$ such that $Y = U \cap F$.

(ii) Each $y \in Y$ has an open neighbourhood $U \subseteq X$ such that $U \cap Y$ is closed in $U$.

(iii) $Y$ is open in $\bar{Y}$.

[38]The old definition does not produce a sheaf of $\mathcal{O}_X$-modules on $X$.

[39]More precisely, $(U_i \to X)$ is a covering for the identity object, so each $U_i$ is a subset of $X$.

A $\mathbb{Q}_l$-sheaf $\mathcal{F}$ on $X$ is *constructible* if there exists a finite partition of $X$ into locally closed subsets $X_i$ such that $\mathcal{F}|_{X_i}$ is twisted constant for each $i$.

A *geometric point* of $X$ is a morphism $\bar{x} : \mathrm{Spec}(\bar{k}) \to X$. In the case of algebraic varieties that are complex manifolds, these are points in the ordinary sense.

Let $\bar{x}$ be a geometric point of $X$. An *étale neighbourhood* of $\bar{x}$, written as

$$(U, \bar{u}) \to (X, \bar{x}),$$

is a morphism $\bar{u} : \mathrm{Spec}(\bar{k}) \to U$ such that $\phi : U \to X$ is an étale morphism and

$$\bar{x} = \phi \circ \bar{u}.$$

Let $\mathcal{F}$ be an étale presheaf on $X$. The *stalk* of $\mathcal{F}$ at $\bar{x}$ is the direct limit[40]

$$\mathcal{F}_{\bar{x}} = \varinjlim_{(U,\bar{u}) \to (X,\bar{x})} \mathcal{F}(U),$$

over all étale neighbourhoods of $\bar{x}$ in $X$.[41]

Let $X$ be a set, and let $(Y_i)$ be an indexed family of topological spaces, with functions $f_i : X \to Y_i$. The *initial topology* on $X$ is the topology generated by the sets $f_i^{-1}(U)$, where $U$ is allowed to by any open set in $Y_i$. The initial topology is the coarsest topology on $X$ for which every $f_i$ is continuous. The *limit topology*, on an inverse limit of topological spaces, is the initial topology with respect to the inclusions.

Let $X$ be a connected Noetherian scheme, and let $\bar{x}$ be a geometric point of $X$. Let $C$ be the category of pairs $(Y, \pi)$ such that $\pi : Y \to X$ is a finite étale morphism, as a subcategory of $\text{Ét}(X)$. If $Y'$ factors through $Y$ as $Y' \to Y \to X$, then we obtain (how?) a group homomorphism $\mathrm{Aut}_C(Y') \to \mathrm{Aut}_C(Y)$. The *étale fundamental group* is

$$\pi_1(X, \bar{x}) = \varprojlim_{Y \in C} \mathrm{Aut}_C(Y),$$

with the limit topology, where each $\mathrm{Aut}_C(Y)$ is a discrete group.

Let $X$ be a topological space, and let $G$ be a topological group acting on $X$. The action is *continuous* if $G \times X \to X$ is continuous with respect to the product topology.

**Lemma 54:** *Assume that $X$ is connected, and let $\bar{x}$ be a geometric point of $X$. If $\mathcal{F}$ is a twisted constant $\mathbb{Q}_l$-sheaf on $X$, then $\pi_1(X, \bar{x})$ acts on the stalk $\mathcal{F}_{\bar{x}}$. Moreover, the fibre functor at $\bar{x}$ (i.e. $\mathcal{F} \mapsto \mathcal{F}_x$) is an equivalence between the category of twisted-constant $\mathbb{Q}_l$-sheaves on $X$ and the category of finite-dimensional $\mathbb{Q}_l$-vector spaces on which $\pi_1(X, \bar{x})$ acts continuously.*

People know a lot about the case $k = \mathbb{C}$. Connections between algebraic and analytic geometry were explored during the 1950s, and largely consolidated in GAGA.[42] Firstly, there is a functor

$$X \rightsquigarrow X^{\mathrm{an}}$$

from finite complex schemes to complex analytic spaces. If $X$ is a variety, we can define $X^{\mathrm{an}}$ 'locally' by using holomorphic functions instead of polynomials to define the structure sheaf. In this case, $X^{\mathrm{an}}$ is the set of closed points of $X$, endowed with the

[40] A priori, this is a general colimit. There are some étale cohomology notes by de Jong where he proves that it's directed.
[41] $U$ is shorthand for the $X$-scheme $\phi : U \to X$.
[42] Serre, Géométrie Algébrique et Géométrie Analytique.

*complex topology*, which makes the inclusion into $X$ continuous. We can get a ringed space $(X^{\text{an}}, \mathcal{O}_X^{\text{an}})$ which is an analytic space.[43] The theory runs much deeper.

Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The sheaf $\mathcal{F}$ is *of finite type* if the following condition holds: if $x \in X$ then there exists an open neighbourhood $U$ of $x$ such that $\mathcal{F}(U)$ is generated by finitely many sections (elements of $\mathcal{F}(U)$).

**Lemma 55:** *If $k = \mathbb{C}$, the constructible $\mathbb{Q}_l$-sheaves on $X$ correspond to the sheaves of $\mathbb{Q}_l$-vector spaces $\mathcal{F}$ on $X^{\text{an}}$ such that there exist a finite partition of $X$ into Zariski-locally closed subsets $X_i$ and, for each $i$, a local system of free $\mathbb{Z}_l$-modules of finite type $\mathcal{F}_i$ on $X_i$, with*

$$\mathcal{F}|_{X_i} = \mathcal{F}_i \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

Henceforth, we only consider constructible $\mathbb{Q}_l$-sheaves, and simply call them $\mathbb{Q}_l$-sheaves.

(1.10) Assume that $k$ is algebraically closed, and let $\mathcal{F}$ be a $\mathbb{Q}_l$-sheaf on $X$. Grothendieck defined $l$-adic cohomology groups $H^i(X < \mathcal{F})$ and $H^i_c(X, \mathcal{F})$. The $H^i_c(X, \mathcal{F})$ are finite-dimensional vector spaces over $\mathbb{Q}_l$, trivial for $i > 2\dim(X)$. For $k = \mathbb{C}$, the $H^i(X, \mathcal{F})$ and $H^i_c(X, \mathcal{F})$ are the usual cohomology groups of $X^{\text{an}}$, with coefficients in $\mathcal{F}$.[44]

**Richard Hughes** *Thursday 5 April 2012*

We give an equivalent definition of *étale*. Let $k$ be a field, and let $f : V \to \text{Spec}(k)$ is *étale* be a morphism of schemes. The morphism $f$ is *étale* if it is flat an unramified.

The morphism $f$ is *flat* if, for each $p \in V$, $f_p^\# : k \to \mathcal{O}_{V,p}$ is a flat map, i.e. $\mathcal{O}_{V,p}$ is a *flat k-module* with action $f_p^\#$, i.e. tensoring with $\mathcal{O}_{V,p}$ preserves exact sequences.[45] Recall that $\mathcal{O}_{\text{Spec}(k)} = k$. More precisely, it is the sheaf sending $\text{Spec}(k)$ to $k$.[46] The stalks are evidently $k$ (without abuse of terminology) at every point.

**Example 56:** *$\frac{\mathbb{Z}}{2\mathbb{Z}}$ is not a flat $\mathbb{Z}$-module, because*

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z}$$

*is injective yet*

$$\mathbb{Z} \otimes \frac{\mathbb{Z}}{2\mathbb{Z}} \xrightarrow{2 \otimes 1} \mathbb{Z} \otimes \frac{\mathbb{Z}}{2\mathbb{Z}}$$

*is not injective (as $1 \otimes 1 \mapsto 2 \otimes 1 = 1 \otimes 2 = 0$).*

Any free module is flat.[47] Consequently, any $k$-module is flat. Thus, $f : V \to \text{Spec}(k)$ is necessarily flat, so we concentrate on whether or not it is unramified.

[43] Like an analytic manifold, but allowing singularities. An *analytic manifold* is a manifold with analytic transition maps.
[44] Presumably he means singular cohomology with coefficients being values of $\mathcal{F}$ and $\mathcal{F}$ being a locally constant sheaf.
[45] It suffices to check that tensoring preserves injectivity – see Atiyah-Macdonald.
[46] Sheaves send the empty set to the trivial group.
[47] Let $A$ be a ring. A free $A$-module is isomorphic to $A^n$. If $M$ is an $A$-module, then $M \otimes A^n \simeq M$ by induction.

The morphism $f$ is *unramified* if, for each $p \in V$, the following conditions are met:

- $\mathcal{O}_{V,p}$ is finitely presented as a $k$-module (finite number of generators and relations).[48]
- $\mathcal{O}_{V,p}$ is a finite separable field extension of $k$.[49]

For interest, we also give the general definition of *unramified*. Let $g : V \to X$ be a morphism of locally ringed spaces. The morphism $g$ is *unramified* if, for each $p \in V$, the following conditions are met:

- $\mathcal{O}_{V,p}$ is finitely presented as a module over $\mathcal{O}_{X,g(p)}$.
- The set $\mathfrak{n} = g_p^{\#}(\mathfrak{m})$ is the maximal ideal in the local ring $\mathcal{O}_{V,p}$, where $\mathfrak{m}$ is the maximal ideal in the local ring $\mathcal{O}_{X,g(p)}$, and the induced map
$$\frac{\mathcal{O}_{X,g(p)}}{\mathfrak{m}} \hookrightarrow \frac{\mathcal{O}_{V,p}}{\mathfrak{n}} \tag{119}$$
is a finite separable field extension.

So at each point, $V$ is locally $\mathrm{Spec}(E)$, where $E/k$ is a finite separable extension. Thus, $f : V \to \mathrm{Spec}(k)$ is étale if and only if
$$V = \sqcup_\alpha \mathrm{Spec}(E_\alpha),$$
for finite separable field extensions $E_\alpha$ of $k$.

Under the Zariski topology, $\mathrm{Spec}(k)$ has only two open sets (too few, the usual problem). The idea is to use objects in $\acute{\mathrm{E}}\mathrm{t}(X)$ instead of Zariski open sets, and so étale sheaves are more interesting than ordinary ones.

### Inverse image sheaf

Let $f : X \to Y$ be a continuous function. For a sheaf $\mathcal{G}$ on $Y$, the *inverse image sheaf on* $X$, $f^{-1}(\mathcal{G})$, is the sheaf associated to the presheaf
$$U \mapsto \varinjlim_{V \supseteq f(U)} \mathcal{G}(V).$$

**Example 57** ($Y = \mathrm{Spec}(k)$)**:** *Let $A$ be an abelian group, and define $\mathcal{G}$ by $\mathcal{G}(\mathrm{Spec}(k)) = A$. Any $f : X \to \mathrm{Spec}(k)$ is continuous, so the presheaf is*
$$U \mapsto \mathcal{G}(\mathrm{Spec}(k)) = A. \tag{120}$$
*Thus, $f^{-1}(\mathcal{G})$ is the locally constant sheaf on X associated to $A$, i.e.*
$$U \mapsto \{f : U \xrightarrow{\text{cts}} A\} \cong A^n,$$

[48] Recall that $f^{\#} : k \to \mathcal{O}_{V,p}$. We say that $f$ is *locally finitely presented*.

[49] First and foremost, $\mathcal{O}_{V,p}$ must be a field for this condition to be satisfied. In that case $k \hookrightarrow \mathcal{O}_{V,p}$, since $f_p^{\#}$ is a local homomorphism of local rings. We won't discuss the definition of separable, but suffice it to say that pretty much any field extension that you can think of is separable.

*where n is the number of connected components of U.*

**Example 58:** *[$X = \mathrm{Spec}(k)$] Let $f : \mathrm{Spec}(k) \to Y$ be continuous, let $f(\mathrm{Spec}(k)) = \{P\}$, where $P \in Y$ (as $\mathrm{Spec}(k)$ is a single point), and let $\mathcal{G}$ be a sheaf on $Y$. Then*

$$\mathrm{Spec}(k) \mapsto \varinjlim_{P \in V} \mathcal{G}(V) = \mathcal{G}_p, \tag{121}$$

*i.e.* $f^{-1}(\mathcal{G}) = \mathcal{G}_p$.

### Inverse image for modules

Let $f : X \to Y$ be a morphism of locally ringed spaces. For a sheaf $\mathcal{G}$ of $\mathcal{O}_Y$-modules on $Y$, the *inverse image* of $\mathcal{G}$ is

$$f^*(\mathcal{G}) = f^{-1}(\mathcal{G}) \otimes_{f^{-1}(\mathcal{O}_Y)} \mathcal{O}_X.$$

**Example 59:** *Let $X = \mathrm{Spec}(k)$ and $Y = \mathrm{Spec}(k')$, where $k$ is a finite separable field extension of $k'$, and let $f : X \hookrightarrow Y$. Let $\mathcal{G}(\mathrm{Spec}(k])) = M$, which is a $k'$-vector space. Then*

$$f^*(\mathcal{G}) = M \otimes_{k'} k, \tag{122}$$

*i.e. $M$ as a $k$-vector space.*

**Example 60:** *Let $f : X \to \mathrm{Spec}(k)$ be a scheme morphism, where $k$ is a field, and let $\mathcal{G} = M$, where $M$ is a $k$-vector space. Let $M_X$ be the constant sheaf on $X$ with values in $M$. Then*

$$f^*(\mathcal{G}_1) = M_X \otimes_k \mathcal{O}_X. \tag{123}$$

### Inclusion morphism

Let $Z \subseteq X$ be a subscheme with inclusion $i : Z \hookrightarrow X$, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on $X$. The *restriction of $\mathcal{F}$ to $\mathbb{Z}$* is $\mathcal{F}|_Z = i^*(\mathcal{F})$.

**Example 61:** *Let $X = \mathrm{Spec}(\mathbb{C}[t]), Z = \mathrm{Spec}(\mathbb{C})$, and let*

$$\begin{aligned} i^\# : \mathbb{C}[t] &\hookrightarrow \mathbb{C} \\ t &\mapsto 0, \end{aligned}$$

*noting the following:*

- *By Richard40, $\mathcal{O}_X \cong \mathbb{C}[t]$ and $\mathcal{O}_Z \cong \mathbb{C}$.*
- *A map of locally ringed spaces is a pair $(i, i^\#)$, so here we're just specifying the second part of that data.*
- *By Jeff43 and Richard40, $i^\#_p$ and $i^\#$ are the same (see the commutative diagram (84)).*

*This gives $\mathbb{C}$ the structure of a $\mathbb{C}[t]$-module, which induces a $\mathbb{C}[t]_{t\mathbb{C}[t]}$-module structure: if $z \in \mathbb{C}, f, g \in \mathbb{C}[t]$ and $g \notin t\mathbb{C}[t]$ then*

$$\frac{f}{g} \cdot z = \frac{f(0)}{g(0)}. \tag{124}$$

*Moreover,*

$$i : Z \to X$$
$$\{0\} \mapsto (i^{\#})^{-1}(\{0\}) = t\mathbb{C}[t].$$

*Let $\mathcal{F}$ be the locally constant sheaf on* $\mathrm{Spec}(\mathbb{C}[t])$ *with values in* $\mathbb{C}[t]$.[50] *Then, using example 58 and Richard40,*

$$\mathcal{F}|_Z = i^*(\mathcal{F}) = i^{-1}(\mathcal{F}) \otimes_{i^{-1}(\mathcal{O}_X)} \mathbb{C} = \mathcal{F}_{t\mathbb{C}[t]} \otimes_{\mathbb{C}[t]_{t\mathbb{C}[t]}} \mathbb{C} \cong \mathbb{C}[t]^m \otimes_{\mathbb{C}[t]_{t\mathbb{C}[t]}} \mathbb{C} \cong \mathbb{C}, \quad (125)$$

*where $m \in \mathbb{Z}$ is irrelevant.*[51]

**Example 62:** *Let $F$ be the twisted constant $\mathbb{Z}_l$-sheaf on* $\mathrm{Spec}(k)$. *Then*

$$\mathcal{F} = F \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

*is a twisted constant $\mathbb{Q}_l$-sheaf, and is therefore constructible (as* $\mathrm{Spec}(k) = \{pt\}$*).*

## Étale fundamental group

We want to mimic the ordinary fundamental group, as the deck transformation group of the universal cover.

A scheme morphism $f : X \to Y$ is *finite* if $Y$ has an open coverby affine schemes

$$V_i = \mathrm{Spec}(B_i)$$

such that for each $i$,

$$f^{-1}(V_i) = U_i$$

is an affine open subscheme $\mathrm{Spec}(A_i)$, and the restriction of $f$ to $U_i$, which induces a ring homomorphism

$$B_i \to A_i,$$

makes $A_i$ a finitely generated module over $B_i$.

The subcategory $F\acute{\mathrm{E}}\mathrm{t}(X)$ of $\acute{\mathrm{E}}\mathrm{t}(X)$ has, as objects,

$$\{\pi : Y \xrightarrow{\text{finite, étale}} X\}.$$

Let $X$ be a variety over a field $k$, and let $\bar{x} : \mathrm{Spec}(\bar{k}) \to X$ be a geometric point. Define

$$F : F\acute{\mathrm{E}}\mathrm{t}(X) \to Set$$
$$Y \mapsto \mathrm{Hom}_X(\bar{x}, Y).$$

There is a projective system

$$\tilde{X} = (X_i)$$

of finite étale coverings of $X$, indexed by a directed set $I$, such that

$$F(Y) = \varinjlim_{i \in I} \mathrm{Hom}(X_i, Y), \quad (126)$$

functorially in $Y$ (for $Y \in F\acute{\mathrm{E}}\mathrm{t}(X)$). This defines $\tilde{X}$ up to isomorphism, and we call $\tilde{X}$ *"the" universal covering space of $X$.*

[50]By Richard40, $\mathcal{O}_{\mathrm{Spec}(\mathbb{C}[t])} \simeq \mathbb{C}[t]$.

[51]I read that $m = 1$, but I'm not so sure about the proof I saw.

We can choose $\tilde{X}$ such that each $X_i$ is *Galois over* $X$ (i.e. the degree over $X$ equals the order of $\mathrm{Aut}_X(X_i)$). A map $X_j \to X_i$ (with $i \leq j$) induces a homomorphism

$$\mathrm{Aut}_X(X_j) \to \mathrm{Aut}_X(X_i),$$

and we define

$$\pi_1(X, \bar{x}) = \mathrm{Aut}_X(\tilde{X}) = \varprojlim_i \mathrm{Aut}_X(X_i).$$

**Example 63:** *[$X = \mathrm{Spec}(k)$] The* separable closure *of $k$ in $\bar{k}$, denoted $k^{sep}$, is the unique separable extension of $k$ containing all separable extensions $K$ of $k$ such that $K \subseteq \bar{k}$. Somehow the choice of a geometric point $\bar{x}$ is just the choice of $k^{sep}$. Note that $k^{sep} = \bar{k}$ if and only if the field $k$ is* perfect *(every finite extension is separable). Most fields occurring in practice are perfect, so we will restrict to this case.*

*Let $\tilde{k} = (k_i)_{i \in I}$ be the projective system consisting of all finite extensions of $k$ contained in $k^{sep}$. We can equivalently work in the opposite category of $\acute{E}t(X)$, which comprises $k$-algebras, i.e.*

$$\mathrm{Aut}_X(\tilde{X}) = \mathrm{Aut}_k(\tilde{k}), \tag{127}$$

*where*

$$\mathrm{Aut}_k(\tilde{k}) = \varprojlim_i \mathrm{Aut}_{k-alg}(k_i). \tag{128}$$

*Then*

$$\pi_1(X, \bar{x}) = \mathrm{Aut}_k(\tilde{k}) = \varprojlim_i \mathrm{Gal}(k_i/k) = \mathrm{Gal}(k^{sep}/k), \tag{129}$$

*i.e. the "absolute Galois group". For instance,*

$$\pi_1(\mathrm{Spec}(\mathbb{R}), \mathrm{Spec}(\mathbb{C})) = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \frac{\mathbb{Z}}{2\mathbb{Z}}$$

*and*

$$\pi_1(\mathrm{Spec}(\mathbb{Q}), \mathrm{Spec}(\bar{\mathbb{Q}})) = \mathrm{Gal}(\bar{\mathbb{Q}}, \mathbb{Q}).$$

*We usually call $\bar{\mathbb{Q}}$ the* algebraic numbers.

**Joe Chan** *Thursday 12 April 2012*

**Comment**[SC] Every scheme $X$ has a unique morphism $X \to \mathrm{Spec}(\mathbb{Z})$, so every scheme is a $\mathbb{Z}$-scheme.[52] We've been working with what Macdonald calls preschemes, and calling them schemes; a *scheme* is a prescheme $X$ such that $X \to \mathrm{Spec}(\mathbb{Z})$ is separated.[53] Also, a scheme is a locally ringed space $(X, \mathcal{O}_X)$ with conditions, so it comes with a topology.

Let $X$ be a connected Noetherian[54] $k$-scheme, let $x : \mathrm{Spec}(\bar{k}) \to X$ be a geometric point, and let $\pi_1(X, x)$ be the étale fundamental group. If $x'$ is another geometric point,[55] then there is a *path*[56] between them,[57] which induces an isomorphism of

[52] Locally $X = \mathrm{Spec}(A)$, so the morphism is induced by $\mathbb{Z} \to A$.
[53] This probably isn't important for now, but I've been getting this creeping feeling that we're all missing something.
[54] covered by a finite number of $\mathrm{Spec}(A_i)$ and each $A_i$ is a Noetherian ring
[55] We may allow $x'$ to map from a different algebraic closure. Definitions vary.
[56] i.e. an isomorphism between the fibre functors.
[57] $X$ is Noetherian implies that it is locally path-connected. As $X$ is also connected, this implies that $X$ is path-connected.

profinite groups[58]

$$\pi_1(X, x) \to \pi_1(X, x').$$

Varying the path gives an inner automorphism, and indeed the isomorphism is canonical up to inner automorphism, so henceforth we omit the basepoint (assume that $X$ is Noetherian, say) and note that the étale fundamental group is well defined up to inner isomorphism.

If $f : X \to Y$ is a scheme morphism, then

$$f_* : \pi_1(X) \to \pi_1(Y)$$

is a group homomorphism, well defined up to inner automorphism.

Let $X$ be *normal*, i.e.

- *integral*, i.e. *reduced* (each $A_i$ has no nonzero nilpotents) and irreducible;
- each $A_i$ is integrally closed.

Recall that, for a field $k$,

$$\pi_1(\mathrm{Spec}(k)) = \mathrm{Gal}(k^{sep}/k).$$

One example to bear in mind is, for any prime $p$,

$$\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) = \hat{\mathbb{Z}},$$

where

$$\hat{\mathbb{Z}} = \varprojlim_n \frac{\mathbb{Z}}{n\mathbb{Z}} \cong \prod_p \mathbb{Z}_p = (\mathrm{Fr}_{\mathbb{F}_p,g}),$$

where $\mathrm{Fr}_{\mathbb{F}_p,g}$ is the geometric Frobenius (defined soon).

If $E \geq k$ is a finite field extension, then there is a map

$$\pi_1(\mathrm{Spec}(E)) \to \pi_1(\mathrm{Spec}(k))$$
$$\mathrm{Fr}_{E,g} \mapsto \mathrm{Fr}_{k,g}^{[E:k]}.$$

This is confusing. Actually I can't see why the $\mathrm{Gal}(E^{sep}/E)$ is cyclic unless $E$ has finite characteristic. Let's go to that case, but let's note the following first:

**Theorem 64:** *If $A$ and $B$ are $C$-algebras, then the diagram of affine schemes*

$$\begin{array}{ccc} \mathrm{Spec}(A \otimes_C B) & \longrightarrow & \mathrm{Spec}(B) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(C) \end{array}$$

*makes* $\mathrm{Spec}(A \otimes_C B)$ *into a fibre product:*

$$\mathrm{Spec}(A \otimes_C B) = \mathrm{Spec}(A) \times_{\mathrm{Spec}(C)} \mathrm{Spec}(B). \tag{130}$$

A scheme $X$ has *characteristic* $p$ (where $p$ is prime) if $p\mathcal{O}_X = \mathcal{O}_X$, i.e. there exists a map $X \to \mathrm{Spec}(\mathbb{F}_p)$ such that $X \to \mathrm{Spec}(\mathbb{Z})$ factors through it.

[58] Topological groups assembled from finite groups.

Let $k = \mathbb{F}_p$, and let $X$ be a $k$-scheme (then $X$ has characteristic $p$). The *absolute Frobenius* of $X$,

$$\mathrm{Fr}_X : X \to X,$$

is the morphism that is the identity on $|X|$ and the $p$th power map on $\mathcal{O}_X$, e.g. if $X = \mathrm{Spec}(A)$ then

$$\begin{aligned} \mathrm{Fr}_X : A &\to A \\ a &\mapsto a^p. \end{aligned}$$

Fix $\bar{k}/k$ and extend scalars:

$$\bar{X} = \mathrm{Spec}(\bar{k}) \times_{\mathrm{Spec}(k)} X. \tag{131}$$

Locally,

$$\bar{A}_i = \mathrm{Spec}(\bar{k}) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(A) = \mathrm{Spec}(\bar{k} \otimes_k A_i).$$

The *relative Frobenius* is

$$\mathrm{Fr}_r = \mathrm{id}_{\mathrm{Spec}(\bar{k})} \times_{\mathrm{Spec}(k)} \mathrm{Fr}_X.$$

The *geometric Frobenius* is

$$\mathrm{Fr}_g = \mathrm{Fr}^{-1}_{\mathrm{Spec}(\bar{k})} \times_{\mathrm{Spec}(k)} \mathrm{id}_X.$$

The *arithmetic Frobenius* is

$$\mathrm{Fr}_a = \mathrm{Fr}_{\mathrm{Spec}(\bar{k})} \times_{\mathrm{Spec}(k)} \mathrm{id}_X.$$

**Example 65:** $X = \mathrm{Spec}(A) = \mathrm{Spec}(k[t_1, \ldots, t_n])$. *Then*

$$\bar{X} = \mathrm{Spec}(\bar{k} \times_k A) = \mathrm{Spec}(\bar{k}[t_1, \ldots, t_n]).$$

*Now*

$$\begin{aligned} \mathrm{Fr}_r : t_i &\mapsto t_i^p \\ \mathrm{Fr}_a : a_i &\mapsto a_i^p \\ \mathrm{Fr}_g : a_i &\mapsto a_i^{1/p} \\ \mathrm{Fr}_X : a_i &\mapsto a_i^p \\ t_i &\mapsto t_i^p \end{aligned}$$

*Here* $\mathrm{Fr}_g$ *acts on* $\bar{k}$ *by the inverse of the Frobenius automorphism* $\mathrm{Fr}_a$.

For sheaf cohomology,

$$\mathrm{Fr}_r^* = \mathrm{Fr}_g^* : H^\bullet(\bar{X}_{\text{ét}}, \mathcal{F}) \to H^\bullet(\bar{X}_{\text{ét}}, \mathcal{F}). \tag{132}$$

In particular, for $l$-adic cohomology ($l$ prime),

$$\mathrm{Fr}_r^* = \mathrm{Fr}_g^* : H^\bullet(\bar{X}_{\text{ét}}, \mathbb{Q}_l) \to H^\bullet(\bar{X}_{\text{ét}}, \mathbb{Q}_l). \tag{133}$$

For connected $k$-schemes $X$ and $S$, define

$$(X/k)(S) = \mathrm{Hom}(S, X).$$

Let $\mathcal{F}$ be a smooth, rank $r$, $\bar{\mathbb{Q}}_l$-sheaf on $X$. By 54, this gives a continuous $\mathbb{Q}_l$-representation

$$\Lambda_{\mathcal{F}} : \pi_1(X) \to GL(r, \bar{\mathbb{Q}}_l).$$

Here $k = \mathbb{F}_p$ is finite; we assume also that $X$ is connected, normal, and finite (as a $k$-scheme).

**Lemma 66:** *In this context, let $n \in \mathbb{Z}_{>0}$, and let $k_n \subseteq \bar{k}$ be a degree $n$ extension of $k$. Then the set*

$$(X/k)(k_n)$$

*is finite.*

For $n \in \mathbb{Z}_{>0}$, define

$$S_n(X/k, \mathcal{F}) = \sum_{x \in (X/k)(k_n)} \mathrm{Tr}(\Lambda_F(\mathrm{Fr}_{k_n \xrightarrow{x} X, g})) = \sum_{x \in (X/k)(k_n)} \mathrm{Tr}(\mathrm{Fr}_x | \mathcal{F}).$$

The *L-function attached to* $\mathcal{F}$ is the formal power series $L(X/k, \mathcal{F})(T) \in 1 + T\bar{\mathbb{Q}}_l[[T]]$ given by

$$L(X/k, \mathcal{F})(T) = \exp \sum_{n=1}^{\infty} S_n(X/k, \mathcal{F}) \frac{T^n}{n} = \prod_{\mathfrak{p} \in |X|} \frac{1}{\det(1 - T^{\deg(\mathfrak{p})} \Lambda_{\mathcal{F}})(\mathrm{Fr}_{\mathfrak{p}}))}, \tag{134}$$

where $\deg(\mathfrak{p})$ is the degree of the corresponding Frobenius orbit.

This is analagous to (for a variety $X_0$ over a finite field)

$$Z(X_0, t) = \exp \sum_{m=1}^{\infty} N_m \frac{t^m}{m} \in 1 + t \cdot \mathbb{Q}[[t]],$$

where we have

$$Z(X_0, t) = \prod_{x \in |X_0|} \frac{1}{1 - t^{\deg(x)}}. \tag{135}$$

To see this,

$$[t^m] \sum_{x \in |X_0|} -\log(1 - t^{\deg(x)}) = [t^m] \sum_{x \in |X_0|} \sum_{r=1}^{\infty} \frac{t^{r \deg x}}{r} = \sum_{x \in |X_0| : \deg(x) | m} \frac{\deg(x)}{m} = \frac{N_m}{m}, \tag{136}$$

from equation (92).

**Theorem 67** (Deligne's target theorem)**:** *Let $U$ be a smooth geometrically-connected curve over a finite field $k$. Let $l$ be a prime, invertible in $k$, and let $\mathcal{F}$ be a smooth $\bar{\mathbb{Q}}_l$-sheaf on $U$ that is $l$-prime of weight $w$.*

**Trithang Tran** *Thursday 19 April 2012*

Before we start, let's briefly touch upon some earlier parts of Deligne's 'Weil I' that we've skipped for now.

**2.2**

Let $l$ be a prime number, and let $k$ be an algebraically closed field of characteristic $p \neq l$ (here $p$ may be 0). We first define $\mathbb{Z}_l(1)$ and $\mathbb{Q}_l(1)$, which are isomorphic (as groups) to $\mathbb{Z}_l$ and $\mathbb{Q}_l$ respectively. For $n \in \mathbb{Z}_{>0}$, let $\frac{\mathbb{Z}}{l^n}(1)$ be the group of $l^n$th roots of unity in $k$. Each $\frac{\mathbb{Z}}{l^n}(1)$ is cyclic, and of order $n$, and is therefore isomorphic to the group $\frac{Z}{l^n\mathbb{Z}}$.[59] We get a projective system (inverse system) by

$$\ldots \to \frac{Z}{l^2}(1) \xrightarrow{x \mapsto x^l} \frac{Z}{l}(1), \tag{137}$$

so let

$$\mathbb{Z}_l(1) = \varprojlim_n \frac{Z}{l^n}(1) \qquad \text{and} \qquad \mathbb{Q}_l(1) = \mathbb{Z}_l(1) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

To see that $\mathbb{Z}_l(1)$ is a $\mathbb{Z}_l$-module, note that $\mathbb{Z}_l(1) \simeq \mathbb{Z}_l$ as groups, and in particular there is a canonical choice of embedding

$$\mathbb{Z}_l \hookrightarrow \mathbb{Z}_l(1).$$

Note that

$$\mathbb{Q}_l(1) \simeq \mathbb{Q}_l$$

as vector spaces over $\mathbb{Q}_l$. We now work in the category of vector spaces over $\mathbb{Q}_l$, and implicitly regard duals and tensors as living in this category. For $r \in \mathbb{Z}_{>0}$, let[60]

$$\mathbb{Q}_l(r) = \mathbb{Q}_l(1)^{\otimes r}.$$

Let $\mathbb{Q}_l(0) = \{0\}$, and for $r \in \mathbb{Z}_{<0}$ let

$$\mathbb{Q}_l(r) = \mathbb{Q}_l(-r)^\vee$$

be the dual vector space over $\mathbb{Q}_l$. The spaces $\mathbb{Q}_l(r)$ are all isomorphic, but they will be acted upon differently.

At this stage we summarise Deligne (1.11) to (1.13), and hope that somebody will cover this material in a future talk.

### 1.11

Let $X$ be a variety on $\mathbb{F}_q$, where $q = p^\varepsilon$ for some prime $p$ and $\varepsilon \in \mathbb{Z}_{>0}$. Let $X$ be the variety on $\bar{\mathbb{F}}_q$ obtained by extension of scalars. Let $\mathcal{F}_0$ be an étale sheaf of sets on $X_0$, and let $\mathcal{F}$ be its inverse image sheaf on $X$. Recall from Dougal's 8 March talk that $\mathbb{F}_q \hookrightarrow \bar{\mathbb{F}}_q$ induces $\mathrm{Spec}(\bar{\mathbb{F}}_q) \to \mathrm{Spec}(\mathbb{F}_q)$, which gives $U \to U_0$ by

$$\begin{array}{ccc} U & \longrightarrow & U_0 \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\bar{\mathbb{F}}_q) & \longrightarrow & \mathrm{Spec}(\mathbb{F}_q). \end{array}$$

The $\mathcal{F}$ is the inverse image sheaf using $U \to U_0$.

Let $F : X \to X$ be the Frobenius (locally raises each coordinate to the $q$th power). We show that the sheaf $F^*\mathcal{F}$ is isomorphic to $\mathcal{F}$ via a canonical isomorphism $F^* : F^*\mathcal{F} \to \mathcal{F}$, and there follows a description of this isomorphism [omitted here]. This construction generalises to $\mathbb{Q}_l$-sheaves.

### 1.12

[59] To see this, note that if $t < n$ and $y \in \frac{Z}{l^n}(1)$ has order $l^t$ then any $l$th root of $y$ has order $l^{t+1}$. Repeating this process if necessary gives a generator, which generates $l^n$ distinct elements of $\frac{Z}{l^n}(1)$, and we know there cannot be more because $z^{l^n} - 1$ is a degree $l^n$ polynomial.

[60] This tensor product is probably over $\mathbb{Z}$ because Deligne hasn't yet mentioned regarding $\mathbb{Q}_l(1)$ as anything but a group, and also because he seems to specify the base ring when it's needed. Still, we need to work with this in explicitly if we want to understand it properly.

Let $X_0$ be a variety on $\mathbb{F}_q$, and let $\mathcal{F}_0$ be a $\mathbb{Q}_l$-sheaf on $X_0$. Let $(X, \mathcal{F})$ be obtained from $(X_0, \mathcal{F}_0)$ be extension of scalars from $\mathcal{F}_q$ to $\bar{\mathcal{F}}_q$. Let $F : X \to X$ and $F^* : F^*\mathcal{F} \to \mathcal{F}$ be as in (1.11). The morphism $F$ is finite, so $F^*$ defines an endomorphism[61]

$$F^* : H^i_c(X, \mathcal{F}) \to H^i_c(X, F^*\mathcal{F}) \to H^i_c(X, \mathcal{F}), \qquad i \in \mathbb{Z}_{\geq 0}. \tag{138}$$

For $x \in |X|$, there is a linear transformation

$$F^*_x : \mathcal{F}_{F(x)} \to \mathcal{F}_x$$

induced by $F^* : F^*\mathcal{F} \to \mathcal{F}$. For $x \in |X|^F$, this is an endomorphism of $\mathcal{F}_x$. Grothendieck proved the Lefschetz formula

$$\sum_{x \in |X|^F} \mathrm{Tr}(F^*_x, \mathcal{F}_x) = \sum_i (-1)^i \mathrm{Tr}(F^*, H^i_c(X, \mathcal{F})), \qquad i \in \mathbb{Z}_{\geq 0}.$$

An analagous formula holds for iterates of $F$, as we now describe. The $n$th iterate of $F^*$ (for $n \in \mathbb{Z}_{>0}$) defines a morphism $F^{*n}_x : \mathcal{F}_{F^n(x)} \to \mathcal{F}_x$. For $x \in |X|^{F^n}$, this is an endomorphism, and

$$\sum_{x \in |X|^{F^n}} \mathrm{Tr}(F^{*n}_x, \mathcal{F}_x) = \sum_i (-1)^i \mathrm{Tr}(F^{*n}, H^i_c(X, \mathcal{F})), \qquad i \in \mathbb{Z}_{\geq 0}. \tag{139}$$

**1.13**

Let $x_0 \in |X_0|$ with the previous setup. From (1.4), $|X_0|$ is the space of Frobenius orbits (see equation (91)). Let $Z \subseteq |X|$ be the Frobenius orbit corresponding to $x_0$, and let $x \in Z$. Recall that $|Z| = \deg(x_0)$. As $x_0$ is fixed by $F^{*\deg(x_0)}$, we let

$$F^*_{x_0} = F^{*\deg(x_0)}_x : \mathcal{F}_x \to \mathcal{F}_x,$$

and put

$$\det(1 - F^*_{x_0} t, \mathcal{F}_0) = \det(1 - F^*_{x_0} t, \mathcal{F}_x).$$

Up to local isomorphism[62], the pair $(\mathcal{F}_x, F^*_{x_0})$ is independent of $x$, which justifies the notation. Analagous notation will henceforth be used for other functions of $(\mathcal{F}_x, F^*_{x_0})$.

Now we arrive at the present.

**3.1**

Let $U_0$ be a curve over $\mathbb{F}_q$ that is the complement of a finite number of closed points in $\mathbb{P}^1$, and let be the curve over $\mathbb{F}_q$ induced by $U$. Let $\mathcal{F}_0$ be a twisted constant $\mathbb{Q}_l$-sheaf on $\mathcal{F}_0$, and let $\mathcal{F}$ be the inverse image sheaf on $U$. Recall that we can regard $\mathcal{F}_0$ as a finite-dimensional vector space over $\mathbb{Q}_l$ on which $\pi_1(U, u)$ acts continuously for every $u \in |U|$. As I understand it, the geometric points are the closed points, with our setup.

Let $\beta \in \mathbb{Q}$. The sheaf $\mathcal{F}_0$ is of *weight* $\beta$ if the following condition is met: if $x \in |U_0|$ then the eigenvalues of $F_x$ acting on $\mathcal{F}_0$[63] are algebraic numbers, all of whose conjugates in $\mathbb{C}$ have absolute value $q_x^{\beta/2}$, where $q_x = q^{\deg(x)}$. For example, $Q_l(r)$ is

[61]Not sure how this works, but we encountered this in on of Sam's previous talks, because $F$ is (in particular) proper.
[62]Not quite sure what is meant here. Incidentally, étale morphisms are supposed to be analagous to local isomorphisms!
[63]This is an abuse of language, but it is explained in (1.13).

of weight $-2r$ for $r \in \mathbb{Z}$.[64]

**Theorem 68:** *Let $\beta \in \mathbb{Z}$. Assume the following:*

*(i) $\mathcal{F}_0$ is endowed with a non-degenerate alternating bilinear form*
$$\psi : \mathcal{F}_0 \otimes \mathcal{F}_0 \to \mathbb{Q}_l(-\beta).$$

*(ii) The image of $\pi_1(U, u)$ in $GL(\mathcal{F}_u)$ is an open subgroup of the symplectic group $\mathrm{Sp}(\mathcal{F}_u, \psi_u)$.*

*(iii) For $x \in |U_0|$, the polynomial $\det(1 - F_x t, \mathcal{F}_0)$ has rational coefficients.*

*Then $\mathcal{F}$ has weight $\beta$.*

Let's unpack these assumptions a little. The stalk $\mathcal{F}_u$ is a direct limit of finite-dimensional vector spaces. We know that $\pi_1(U, u)$ is a continuous representation of the finite-dimensional vector space $\mathcal{F}_u$, so $\pi_1(U, u) \subseteq GL(\mathcal{F}_u)$. The bilinear form $\psi$ is really a collection of bilinear forms
$$\psi(U) : \mathcal{F}_0(V) \otimes \mathcal{F}_0(V) \to \mathbb{Q}_l(-\beta), \qquad \text{for } V = V \xrightarrow{\text{ét}} X \in \text{Ét}(X). \tag{140}$$
Note that $\mathcal{F}_0(V) \otimes \mathcal{F}_0(V) = (\mathcal{F}_0 \otimes \mathcal{F}_0)(V)$. We get, for $x \in U_0$, a map $\psi_x$ on the stalk
$$(\mathcal{F}_0 \otimes \mathcal{F}_0)_x = \mathcal{F}_x \otimes \mathcal{F}_x.$$
Using $U \to U_0$, the form $\psi$ analagously induces
$$\psi_x : \mathcal{F}_x \otimes \mathcal{F}_x \to \mathbb{Q}_l(-\beta). \tag{141}$$
The *symplectic group* is
$$\mathrm{Sp}(\mathcal{F}_u, \psi_u) = \{M \in \mathrm{GL}(\mathcal{F}_u) : \text{if } v, w \in \mathcal{F}_u \text{ then } \psi_u(Mv, Mw) = \psi_u(v, w)\}. \tag{142}$$
This has a topology (possibly the subspace topology from $\mathrm{GL}(\mathcal{F}_u)$), and forms a topological group. Finally, one way to see the canonical embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_l$ is via $\mathbb{Q}_l = \mathbb{Z}_l \otimes \mathbb{Q}$. It is important to remember that the characteristic of $\mathbb{Q}_l$ is 0 and not $l$.

To prove the theorem, we can and will assume that $U$ is affine and $\mathcal{F} \neq 0$. We set up the proof with some lemmata:

**Lemma 69:** *Let $k \in \mathbb{Z}_{>0}$. For $x \in |U_0|$, the logarithmic derivative*
$$t\frac{d}{dt}\log(\det(1 - F_x t^{\deg(x)}, \mathcal{F}_0^{\otimes 2k})^{-1})$$
*is a formal power series with coefficients in $\mathbb{Q}_{\geq 0}$.*

Let $x \in |U_0|$. From 95,
$$t\frac{d}{dt}\log(\det(1 - F_x t^{\deg(x)}, \mathcal{F}_0^{\otimes 2k})^{-1}) = \sum_{n=1}^{\infty} \mathrm{Tr}(F_x^n, \mathcal{F}_0^{\otimes 2k}) t^n.$$
For $n \in \mathbb{Z}_{>0}$, the coefficient of $t^n$ is
$$\mathrm{Tr}(F_x^n, \mathcal{F}_0^{\otimes 2k}) = \mathrm{Tr}(F_x^n, \mathcal{F}_0)^{2k} \in \mathbb{Q}_{\geq 0},$$
since (iii) implies that $\mathrm{Tr}(F_x^n, \mathcal{F}_0) \in \mathbb{Q}$ (recall that $\det(1 - F_x t, \mathcal{F}_0) = \prod_j (1 - \alpha_j t)$, where the $\alpha_j$ are the eigenvalues of $F_x$).

[64] Where does this come from?

**Lemma 70:** *The local factors* $\det(1 - F_x t^{\deg(x)}, \mathcal{F}_0^{\otimes 2k})^{-1}$ *are formal power series with coefficients in* $\mathbb{Q}_{\geq 0}$.

The logarithmic derivative merely removes the constant term. However,

$$f(t) = \log(\det(1 - F_x t^{\deg(x)}, \mathcal{F}_0^{\otimes 2k})^{-1})$$

has no constant term, so it is equal to its logarithmic derivative. Then

$$\det(1 - F_x t^{\deg(x)}, \mathcal{F}_0^{\otimes 2k})^{-1} = e^{f(t)},$$

and we can use the chain rule and product rule to show that the iterated derivatives are $\geq 0$. To see that $f(t)$ has no constant term:

$$\det(1 - F_x t^{\deg(x)}, \mathcal{F}_0^{\otimes 2k})^{-1} = \prod_j (1 - \alpha_j t^{\deg(x)})^{-1},$$

so
$$f(t) = \log(\det(1 - F_x t^{\deg(x)}, \mathcal{F}_0^{\otimes 2k})^{-1}) = -\sum_j \log(1 - \alpha_j t^{\deg(x)}).$$

Here the $\alpha_j$ are the eigenvalues of $F_x : \mathcal{F}_0^{\otimes 2k} \to \mathcal{F}_0^{\otimes 2k}$, and now $f(0) = 0$.

**Lemma 71:** *For* $i \in \mathbb{Z}_{>0}$, *let* $f_i = \sum_{n=0}^{\infty} a_{i,n} t^n$ *be a formal power series with* $a_{i,0} = 1$ *and* $a_{i,n} \in \mathbb{R}_{\geq 0}$ *for* $n \in \mathbb{Z}_{>0}$. *Assume that the order of* $f_i - 1$ *tends of infinity as* $i \to \infty$, *and let* $f = \prod_{i=1}^{\infty} f_i$. *Then the radius of converges of* $f_i$ *is at least that of* $f$ *(for* $i \in \mathbb{Z}_{>0}$*).*

In other words, if $f_i(z)$ diverges, for some $i \in \mathbb{Z}_{>0}$ and $z \in \mathbb{C}$, then $f$ diverges. To see this, note that

$$|f_i(z)| \leq f_i(|z|),$$

so it suffices to prove it for $z \in \mathbb{R}_{\geq 0}$. This follows from the fact that

$$a_n \geq a_{i,n},$$

where $f(t) = \sum_{n=0}^{\infty} a_n t^n$.

**Lemma 72:** *Under the assumptions of lemma 71, if* $f$ *and* $f_i$ *are the Taylor series of meromorphic functions, then*

$$\inf\{|z| : f(z) = \infty\} \leq \inf\{|z| : f_i(z) = \infty\}.$$

This follows from lemma 71.

We also quote (2.10):

**2.10**

For $\mathcal{F}$ a $\mathbb{Q}_l$-sheaf over an algebraic variety $X$ over an algebraically closed field $k$, define $\mathcal{F}(r) = \mathcal{F} \otimes \mathbb{Q}_l(r)$.

**Lemma 73:** *Let* $X$ *be a smooth, connected curve over an algebraically closed field* $k$, *let* $x \in |X|$, *and let* $\mathcal{F}$ *be a twisted constant* $\mathbb{Q}_l$*-sheaf. Then*

*(i) If* $X$ *is affine then* $H_c^0(X, \mathcal{F}) = 0$.

*(ii)*
$$H_c^2(X, \mathcal{F}) = (\mathcal{F}_x)_{\pi_1(X,x)}(-1).$$

*Part (i) just says that $\mathcal{F}$ has no section with finite support.*

*Proof of Theorem 68.* Let $k \in \mathbb{Z}_{>0}$. For each partition $P$ of $\{1, 2, \ldots, 2k\}$ into two-element subsets $\{i_\alpha, j_\alpha\}$ (for $\alpha = 1, 2, \ldots, k$, and let $i_\alpha < j_\alpha$ for all $\alpha$), define

$$\psi_P : \mathcal{F}_0^{\otimes 2k} \to \mathbb{Q}_l(-k\beta)$$
$$x_1 \otimes \ldots \otimes x_{2k} \mapsto \psi(x_{i_1}, x_{j_1}) \otimes \ldots \otimes \psi(x_{i_k}, x_{j_k}).$$

Let $x \in |U|$. Assumption (ii) implies that $\pi_1(U, u)$ is Zariski-dense in $\mathrm{Sp}(\mathcal{F}_u, \psi_u)$, so the coinvariants of $\pi_1(U, u)$ in $\mathcal{F}_u^{\otimes 2k}$ are the coinvariants in $\mathcal{F}_u^{\otimes 2k}$ of the full symplectic group.

Let $\mathcal{P}$ be the set of partitions $P$ (as described above). For appropriate $\mathcal{P}' \subseteq \mathcal{P}$, depending on $\dim(\mathcal{F}_u)$, the $\psi_P$ for $P \in \mathcal{P}'$ define an isomorphism[65]

$$(\mathcal{F}_u^{\otimes 2k})_{\pi_1} = (\mathcal{F}_u^{\otimes 2k})_{\mathrm{Sp}} \xrightarrow{\sim} \mathbb{Q}_l(-k\beta)^{\mathcal{P}'}, \tag{143}$$

where

$$x_1 \otimes \ldots \otimes x_{2k} : \mathcal{P}' \to \mathbb{Q}_l(-k\beta)$$
$$P \mapsto \psi_P(x_1 \otimes \ldots \otimes x_{2k}).$$

This induces an isomorphism

$$(\mathcal{F}_u^{\otimes 2k})_{\pi_1}(-1) = (\mathcal{F}_u^{\otimes 2k})_{\pi_1} \otimes \mathbb{Q}_l(-1) \xrightarrow{\sim} \mathbb{Q}_l(-k\beta - 1)^{\mathcal{P}'}, \tag{144}$$

where

$$x_1 \otimes \ldots \otimes x_{2k} \otimes y : \mathcal{P}' \to \mathbb{Q}_l(-k\beta - 1)$$
$$P \mapsto \psi_P(x_1 \otimes \ldots \otimes x_{2k}) \otimes y.$$

Let $N$ be the number of elements in $\mathcal{P}'$. Using (2.10),[66]

$$H_c^2(U, \mathcal{F}^{\otimes 2k}) = (\mathcal{F}_u^{\otimes 2k})_{\pi_1}(-1) \simeq \mathbb{Q}_l(-k\beta - 1)^{\mathcal{P}'} \simeq \mathbb{Q}_l(-k\beta - 1)^N. \tag{145}$$

James will complete the proof next week. □

**James Withers** *Thursday 26 April 2012*

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Let's get a bit of background before completing the proof.

**1.14**

This extends the material covered in (1.11) to (1.13). Define $Z(X_0, \mathcal{F}_0, t) \in \mathbb{Q}_l[[t]]$ by the product

$$Z(X_0, \mathcal{F}_0, t) = \prod_{x \in |X_0|} \det(1 - F_x^* t^{\deg(x)}, \mathcal{F}_0)^{-1}. \tag{146}$$

If this is confusing, recall that $\det(1 - F_x^* t^{\deg(x)}, \mathcal{F}_0) = \det(1 - F_x^* t^{\deg(x)}, \mathcal{F}_x)$, that the

[65] Herman Weyl. The classical groups, Princeton University Press, chap. VI, s1 (analogous to chap. V).

[66] Are we totally cheating here? (2.10) requires $U$ to be smooth and connected.

stalk $\mathcal{F}_x$ is a finite-dimensional vector space, and that $F_x^* = F_x^{*\deg(x)} : \mathcal{F}_x \to \mathcal{F}_x$. Here $\deg(x)$ is the size of the Frobenius orbit of $x \in |X_0|$, so $F^{*\deg(x)}(x) = x$. Recall that we regard $|X_0|$ as the space of Frobenius orbits in $|X|$.

Following (1.5.3), the logarithmic derivative of $Z$ is

$$t\frac{d}{dt}\log Z(X_0, \mathcal{F}_0, t) = \frac{t\frac{d}{dt}Z(X_0, \mathcal{F}_0, t)}{Z(X_0, \mathcal{F}_0, t)} = \sum_{n=1}^{\infty} \sum_{x \in |X|^{F^n} = X_0(\mathbb{F}_{q^n})} \mathrm{Tr}(F_x^{*n}, \mathcal{F}_0)t^n. \tag{147}$$

If $\mathcal{F}$ is the locally constant sheaf with values in $\mathbb{Q}_l$, we recover (1.1.2):

$$Z = \prod_{x \in |X_0|} (1 - t^{\deg(x)})^{-1}.$$

To see this, use the handy formula

$$\det(1 - \phi t) = \prod (1 - \alpha t), \tag{148}$$

over eigenvalues $\alpha$ of the linear transformation $\phi$, counted with multiplicity. This can be surmised from

$$\det\Big(\frac{1}{t} - \phi\Big) = \prod \Big(\frac{1}{t} - \alpha\Big), \tag{149}$$

which holds because each side is the characteristic polynomial of $\phi$ evaluated at $\frac{1}{t}$.

Recall equation (96):

$$\sum_{x \in |X|^{F^n}} \mathrm{Tr}(F_x^{*n}, \mathcal{F}_x) = \sum_i (-1)^i \mathrm{Tr}(F^{*n}, H_c^i(X, \mathcal{F})), \qquad i \in \mathbb{Z}_{\geq 0}.$$

Substituting equation (159) into equation (161), we find by the same calculation as (1.5) the following generalization of equation (96):

$$Z(X_0, \mathcal{F}_0, t) = \prod_i \det(1 - F^* t, H_c^i(X, \mathcal{F}))^{(-1)^{i+1}}. \tag{150}$$

This is an identity in $\mathbb{Q}_l[[t]]$.

**1.15**

This part is a dictionary for translating geometric language into the language of Galois theory. Some details are omitted here, as we rush to the point. Let $\varphi \in \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ be the Frobenius $x \mapsto x^q$. We can check that

$$F^* = \varphi^{-1}$$

in $\mathrm{End}(H_c^*(X, \mathcal{F}))$. The *geometric Frobenius* is $F = \varphi^{-1} \in \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$. Then

$$F^* = F. \tag{151}$$

For $x \in |X_0|$, let $x \in |X|$ be a point in the orbit $x$. Then $\mathcal{F}_x = (\mathcal{F}_0)_x$, and

$$\mathcal{F}_x^* = \mathcal{F}_x \in \mathrm{End}(\mathcal{F}_x). \tag{152}$$

where $F_x^* = F_x^{*\deg(x)} \in \mathrm{End}(\mathcal{F}_x)$. Significantly, equation (164) becomes

$$Z = (X_0, \mathcal{F}_0, t) = \prod_{x \in |X_0|} \det(1 - F_x t^{\deg(x)}, \mathcal{F}_0)^{-1} = \prod_i \det(1 - Ft, H_c^i(X, \mathcal{F}))^{(-1)^{i+1}}. \tag{153}$$

**2.2** (continued)

Let $X$ be a smooth variety purely of dimension $n$ over $k$, where the characteristic of the field $k$ is not $l$. The *orientation sheaf* of $X$ in $l$-adic cohomology is the locally constant $\mathbb{Q}_l$-sheaf with values in $\mathbb{Q}_l(n)$. The *fundamental class* is a morphism

$$\mathrm{Tr} : H_c^{2n}(X, \mathbb{Q}_l(n)) \to \mathbb{Q}_l,$$

or, alternatively,

$$\mathrm{Tr} : H_c^{2n}(X, \mathbb{Q}_l) \to \mathbb{Q}_l(-2n).$$

**2.3**

**Theorem 74:** *[Poincaré duality] If $X$ is proper and smooth, and purely of dimension $n$, then the bilinear form*

$$\mathrm{Tr}(\cdot \cup \cdot) : H^i(X, \mathbb{Q}_l) \otimes H^{2n-i}(X, \mathbb{Q}_l) \to \mathbb{Q}_l(-n)$$

*is a perfect pairing. As $\mathbb{Q}_l(-n) \simeq \mathbb{Q}_l$, the pairing identifies $H^i(X, \mathbb{Q}_l)$ with the dual of $H^{2n-i}(X, \mathbb{Q}_l(n))$. In particular, $H^{2n}(X, \mathbb{Q}_l)$ is identified with $\mathbb{Q}_l(-n)$.*

**2.4**

Let $X_0$ be a smooth and proper variety over a finite field $\mathbb{F}_q$, purely of dimension $n$[67], and let $X$ be the variety over $\bar{\mathbb{F}}_q$ be gotten from $X_0$ by extension of scalars. From (2.3), which Sam will cover next week, we deduce the following. If $(\alpha_j)$ are the eigenvalues of the geometric Frobenius $F$ acting on $H^i(X, \mathbb{Q}_l)$ then the eigenvalues of $F$ acting on $H^{2n-i}(X, \mathbb{Q}_l)$ are $(q^n \alpha_j^{-1})$.

**2.5**

For simplicity, assume that $X$ is connected. We can phrase (2.4) in geometric (as opposed to Galois) language:

1. The cup product between $H^i(X, \mathbb{Q}_l)$ and $H^{2n-i}$ is a perfect pairing with values in $H^{2n}(X, \mathbb{Q}_l)$, which has dimension 1.
2. The cup product commutes with $F^*$.
3. The morphism $F$ is finite and of degree $q^n$: on $H^{2n}(X, \mathbb{Q}_l)$, the map $F^*$ is multiplication by $q^n$. From (2.3), this is also the action of $F^*$ on $\mathbb{Q}_l(-n)$.
4. The eigenvalues of $F^*$ have the property (2.4).

We now arrive at the present. We'll start by finishing the proof of Theorem 68.

**3.7** (continued)

*Proof.* Remember we're assuming that $U$ is affine and that $\mathcal{F} \neq 0$. We applied (2.10), so we need $U$ to be smooth and connected. Connectedness probably isn't an issue, because we can consider each component separately and see what happens. How do we get around smoothness?

[67] i.e. each component has dimension $n$, for instance a curve is purely of dimension 1.

Henceforth assume further that $U$ is smooth and connected. From 164,

$$Z(U_0,\mathcal{F}_0^{\otimes 2k},t) = \prod_i \det(1-F^*t, H_c^i(U,\mathcal{F}^{\otimes 2k}))^{(-1)^{i+1}}$$

$$= \frac{\det(1-F^*t, H_c^1(U,\mathcal{F}^{\otimes 2k}))}{\det(1-F^*t,0)\det(1-F^*t,\mathbb{Q}_l(-k\beta-1)^N)}$$

$$= \frac{\det(1-F^*t, H_c^1(U,\mathcal{F}^{\otimes 2k}))}{\det(1-F^*t,\mathbb{Q}_l(-k\beta-1)^N)}.$$

The eigenvalue of $F^*$ acting on $\mathbb{Q}_l(-k\beta-1)$ is $q^{k\beta+1}$ (from (2.5)), so

$$\det(1-F^*t,\mathbb{Q}_l(-k\beta-1)^N) = \det(1-F^*t,\mathbb{Q}_l(-k\beta-1))^N = (1-q^{k\beta+1}t)^N. \quad (154)$$

Thus,

$$Z(U_0,\mathcal{F}_0^{\otimes 2k},t) = \frac{\det(1-F^*t, H_c^1(U,\mathcal{F}^{\otimes 2k}))}{(1-q^{k\beta+1}t)^N}. \quad (155)$$

Thus, the only only pole of $Z$ is $t = \frac{1}{q^{k\beta+1}}$. Recall from the definition that

$$Z(U_0,\mathcal{F}_0^{\otimes 2k},t) = \prod_{x\in|U_0|} \det(1-F_x^* t^{\deg(x)},\mathcal{F}_0^{\otimes 2k})^{-1}.$$

Consider some $x \in |U_0|$. Let $\alpha$ be an eigenvalue of $F_x$ acting on $\mathcal{F}_0$, and let $\gamma$ be a conjugate of $\alpha$ in $\mathbb{C}$.[68] Then $\alpha^{2k}$ is an eigenvalue of $F_x$ acting on $\mathcal{F}_0^{\otimes 2k}$, so

$$\frac{1}{\alpha^{2k/\deg(x)}}$$

is a pole of $\det(1-F_x t^{\deg(x)},\mathcal{F}^{\otimes 2k})^{-1}$, so $\gamma$ is also a pole, since the determinant is a polynomial. Now Lemma 72 implies that

$$\frac{1}{\gamma^{2k/\deg(x)}} \geq \frac{1}{q^{k\beta+1}},$$

so

$$|\gamma| \leq q_x^{\frac{\beta}{2}+\frac{1}{2k}}.$$

Sending $k\to\infty$ yields

$$|\gamma| \leq q_x^{\beta/2}.$$

The existence of $\psi$ ensures that $q_x^\beta\alpha^{-1}$ is also an eigenvalue of $F_x$ acting on $\mathcal{F}_0$ (how?). Moreover, $q_x^\beta\gamma^{-1}$ is a conjugate of $q_x^\beta\alpha^{-1}$ (easy to explicitly work out the minimal polynomial in terms of that of $\alpha$). Consequently,

$$|q_x^\beta\gamma^{-1}| \leq q_x^{\beta/2},$$

so

$$|\gamma| \geq q_x^{\beta/2}.$$

Whence, $|\gamma| = q_x^{\beta/2}$. We conclude that $\mathcal{F}$ has weight $\beta$. □

### 3.8

**Corollary 75:** *Let $\alpha$ be an eigenvalue of $F^*$ acting on $H_c^1(U,\mathcal{F})$. Then $\alpha$ is an algebraic*

[68]Note that $\alpha$ is an algebraic integer because its reciprocal is a root of $\det(1-F_x t)$, which has rational coefficients by assumption.

*number, and all of its conjugates $\gamma \in \mathbb{C}$ satisfy*

$$|\gamma| \leq q^{\frac{\beta+1}{2}+\frac{1}{2}}.$$

*Proof.* The formula (164) for $\mathcal{F}_0$ reduces to[69]

$$Z(U_0, \mathcal{F}_0, t) = \det(1 - F^* t, H^1_c(U, \mathcal{F})). \tag{156}$$

Assumption (iii) shows that $Z(U_0, \mathcal{F}_0, t)$ has rational coefficients,[70] so

$$\det(1 - F^* t, H^1_c(U, \mathcal{F}))$$

has rational coefficients. As $1/\alpha$ is a root of this determinant, it follows that $1/\alpha$, and thus $\alpha$, is algebraic.

To complete the proof, it suffices to prove that $Z \neq 0$ for $|t| < q^{-\frac{\beta}{2}-1}$.[71] Let

$$|t| < q^{-\frac{\beta}{2}-1}.$$

It suffices to prove that the infinite product

$$\frac{1}{Z} = \prod_{x \in |X_0|} \det(1 - F_x^* t^{\deg(x)}, \mathcal{F}_0)$$

converges absolutely. Let $N$ be the rank of $\mathcal{F}$. For $x \in |U_0|$, put

$$\det(1 - F_x^* t^{\deg(x)}, \mathcal{F}_0) = \prod_{i=1}^{N} (1 - \alpha_{i,x} t^{\deg(x)}),$$

where the $\alpha_{i,x}$ are eigenvalues of $F_x^*$ acting on $\mathcal{F}_0$. The infinite product converges absolutely if and only if the infinite sum

$$\sum_{i=1}^{N} \sum_{x \in |U_0|} -\alpha_{i,x} t^{\deg(x)}$$

converges absolutely. Fixing $i$, it suffices to prove that

$$\sum_{x \in |U_0|} |\alpha_{i,x} t^{\deg(x)}|$$

converges. From Theorem 68,

$$|\alpha_{i,x}| = q_x^{\beta/2}.$$

Let $\varepsilon > 0$, and let $|t| = q^{-\frac{\beta}{2}-1-\varepsilon}$. Then

$$\sum_{x \in |U_0|} |\alpha_{i,x} t^{\deg(x)}| = \sum_x q_x^{-1-\varepsilon}.$$

On the affine line, there are $q^n$ points in $\mathbb{F}_{q^n}$, so there are at most $q^n$ closed points of degree $n$ (for each $n \in \mathbb{Z}_{>0}$). Thus,

$$\sum_x q_x^{-1-\varepsilon} \leq \sum_{n=1}^{\infty} q^n \cdot q^{n(-1-\varepsilon)} = \sum_{n=1}^{\infty} q^{-n\varepsilon} < \infty, \tag{157}$$

[69] How does the second cohomology term disappear?

[70] Let $f \in \mathbb{Q}[[t]]$. Then $1/f \in \mathbb{Q}[[t]]$, since we can exlicitly compute its iterated derivatives at 0 and they are rational.

[71] Suppose we show this. As $\frac{1}{\alpha}$ is a root of the polynomial $Z$, it follows that $\frac{1}{\gamma}$ is too, for any conjugate $\gamma \in \mathbb{C}$ of $\alpha$. Then $|\frac{1}{\gamma}| \geq q^{-\frac{\beta}{2}-1}$, so $|\gamma| \leq q^{\frac{\beta}{2}+1}$.

as $|q^{-\varepsilon}| < 1$. We conclude that all conjugates $\gamma \in \mathbb{C}$ of $\alpha$ satisfy

$$|\gamma| \leq q^{\frac{\beta+1}{2}+\frac{1}{2}}.$$

□

**Corollary 76:** *Let $j_0 : U_0 \hookrightarrow \mathbb{P}^1_{\mathbb{F}_q}$ and $j : U \hookrightarrow \mathbb{P}^1$ be the canonical inclusion maps, and let $\alpha$ be an eigenvalue of $F^*$ acting on $H^1(\mathbb{P}^1, j_*\mathcal{F})$. The $\alpha$ is an algebraic number, and all of its conjugates $\gamma \in \mathbb{C}$ satisfy*

$$q^{\frac{\beta+1}{2}-\frac{1}{2}} \leq |\gamma| \leq q^{\frac{\beta+1}{2}+\frac{1}{2}}.$$

**Sam Chow** *Thursday 3 May 2012*

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### 1.11

Let $X$ be a variety on $\mathbb{F}_q$, where $q = p^\varepsilon$ for some prime $p$ and $\varepsilon \in \mathbb{Z}_{>0}$. Let $X$ be the variety on $\bar{\mathbb{F}}_q$ obtained by extension of scalars. Let $\mathcal{F}_0$ be an étale sheaf of sets on $X_0$, and let $\mathcal{F}$ be its inverse image sheaf on $X$. Recall from Dougal's 8 March talk that $\mathbb{F}_q \hookrightarrow \bar{\mathbb{F}}_q$ induces $\mathrm{Spec}(\bar{\mathbb{F}}_q) \to \mathrm{Spec}(\mathbb{F}_q)$, which gives $U \to U_0$ by

$$\begin{array}{ccc} U & \longrightarrow & U_0 \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\bar{\mathbb{F}}_q) & \longrightarrow & \mathrm{Spec}(\mathbb{F}_q). \end{array}$$

The $\mathcal{F}$ is the inverse image sheaf using $U \to U_0$.

Let $F : X \to X$ be the Frobenius (locally raises each coordinate to the $q$th power). We show that the inverse image sheaf $F^*\mathcal{F}$ is isomorphic to $\mathcal{F}$ via a canonical isomorphism $F^* : F^*\mathcal{F} \to \mathcal{F}$, and there follows a description of this isomorphism [omitted here]. This construction generalises to $\mathbb{Q}_l$-sheaves.

### 1.12

Let $X_0$ be a variety on $\mathbb{F}_q$, and let $\mathcal{F}_0$ be a $\mathbb{Q}_l$-sheaf on $X_0$. Let $(X, \mathcal{F})$ be obtained from $(X_0, \mathcal{F}_0)$ be extension of scalars from $\mathcal{F}_q$ to $\bar{\mathcal{F}}_q$. Let $F : X \to X$ and $F^* : F^*\mathcal{F} \to \mathcal{F}$ be as in (1.11). The morphism $F$ is finite, so $F^*$ defines an endomorphism[72]

$$F^* : H^i_c(X, \mathcal{F}) \to H^i_c(X, F^*\mathcal{F}) \to H^i_c(X, \mathcal{F}), \qquad i \in \mathbb{Z}_{\geq 0}. \tag{158}$$

For $x \in |X|$, there is a linear transformation

$$F^*_x : \mathcal{F}_{F(x)} \to \mathcal{F}_x$$

induced by $F^* : F^*\mathcal{F} \to \mathcal{F}$. For $x \in |X|^F$, this is an endomorphism of $\mathcal{F}_x$. Grothendieck proved the Lefschetz formula

$$\sum_{x \in |X|^F} \mathrm{Tr}(F^*_x, \mathcal{F}_x) = \sum_i (-1)^i \mathrm{Tr}(F^*, H^i_c(X, \mathcal{F})), \qquad i \in \mathbb{Z}_{\geq 0}.$$

[72] Not sure how this works, but we encountered this in on of Sam's previous talks, because $F$ is (in particular) proper.

An analagous formula holds for iterates of $F$, as we now describe. The $n$th iterate of $F^*$ (for $n \in \mathbb{Z}_{>0}$) defines a morphism $F_x^{*n} : \mathcal{F}_{F^n(x)} \to \mathcal{F}_x$. For $x \in |X|^{F^n}$, this is an endomorphism, and

$$\sum_{x \in |X|^{F^n}} \mathrm{Tr}(F_x^{*n}, \mathcal{F}_x) = \sum_i (-1)^i \mathrm{Tr}(F^{*n}, H_c^i(X, \mathcal{F})), \qquad i \in \mathbb{Z}_{\geq 0}. \tag{159}$$

**1.13**

Let $x_0 \in |X_0|$ with the previous setup. From (1.4), $|X_0|$ is the space of Frobenius orbits (see equation (91)). Let $Z \subseteq |X|$ be the Frobenius orbit corresponding to $x_0$, and let $x \in Z$. Recall that $|Z| = \deg(x_0)$. As $x_0$ is fixed by $F^{*\deg(x_0)}$, we let

$$F_{x_0}^* = F_x^{*\deg(x_0)} : \mathcal{F}_x \to \mathcal{F}_x,$$

and put

$$\det(1 - F_{x_0}^* t, \mathcal{F}_0) = \det(1 - F_{x_0}^* t, \mathcal{F}_x).$$

Up to local isomorphism[73], the pair $(\mathcal{F}_x, F_{x_0}^*)$ is independent of $x$, which justifies the notation. Analagous notation will henceforth be used for other functions of $(\mathcal{F}_x, F_{x_0}^*)$.

**1.14**

This extends the material covered in (1.11) to (1.13). Define $Z(X_0, \mathcal{F}_0, t) \in \mathbb{Q}_l[[t]]$ by the product

$$Z(X_0, \mathcal{F}_0, t) = \prod_{x \in |X_0|} \det(1 - F_x^* t^{\deg(x)}, \mathcal{F}_0)^{-1}. \tag{160}$$

If this is confusing, recall that $\det(1 - F_x^* t^{\deg(x)}, \mathcal{F}_0) = \det(1 - F_x^* t^{\deg(x)}, \mathcal{F}_x)$, that the stalk $\mathcal{F}_x$ is a finite-dimensional vector space, and that $F_x^* = F_x^{*\deg(x)} : \mathcal{F}_x \to \mathcal{F}_x$. Here $\deg(x)$ is the size of the Frobenius orbit of $x \in |X_0|$, so $F^{*\deg(x)}(x) = x$. Recall that we regard $|X_0|$ as the space of Frobenius orbits in $|X|$.

Following (1.5.3), the logarithmic derivative of $Z$ is

$$t \frac{d}{dt} \log Z(X_0, \mathcal{F}_0, t) = \frac{t \frac{d}{dt} Z(X_0, \mathcal{F}_0, t)}{Z(X_0, \mathcal{F}_0, t)} = \sum_{n=1}^{\infty} \sum_{x \in |X|^{F^n} = X_0(\mathbb{F}_{q^n})} \mathrm{Tr}(F_x^{*n}, \mathcal{F}_0) t^n. \tag{161}$$

If $\mathcal{F}$ is the locally constant sheaf with values in $\mathbb{Q}_l$, we recover (1.1.2):

$$Z = \prod_{x \in |X_0|} (1 - t^{\deg(x)})^{-1}.$$

To see this, use the handy formula

$$\det(1 - \phi t) = \prod (1 - \alpha t), \tag{162}$$

over eigenvalues $\alpha$ of the linear transformation $\phi$, counted with multiplicity. This can be surmised from

$$\det\Big(\frac{1}{t} - \phi\Big) = \prod \Big(\frac{1}{t} - \alpha\Big), \tag{163}$$

which holds because each side is the characteristic polynomial of $\phi$ evaluated at $\frac{1}{t}$.

[73]Not quite sure what is meant here. Incidentally, étale morphisms are supposed to be analagous to local isomorphisms!

Recall equation (96):

$$\sum_{x\in|X|^{F^n}} \mathrm{Tr}(F_x^{*n}, \mathcal{F}_x) = \sum_i (-1)^i \mathrm{Tr}(F^{*n}, H_c^i(X, \mathcal{F})), \qquad i \in \mathbb{Z}_{\geq 0}.$$

Substituting equation (159) into equation (161), we find by the same calculation as (1.5) the following generalization of equation (96):

$$Z(X_0, \mathcal{F}_0, t) = \prod_i \det(1 - F^* t, H_c^i(X, \mathcal{F}))^{(-1)^{i+1}}. \tag{164}$$

This is an identity in $\mathbb{Q}_l[[t]]$.

**1.15**

This part is a dictionary for translating geometric language into the language of Galois theory. Some details are omitted here, as we rush to the point. Let $\varphi \in \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ be the Frobenius $x \mapsto x^q$. We can check that

$$F^* = \varphi^{-1}$$

in $\mathrm{End}(H_c^*(X, \mathcal{F}))$. The *geometric Frobenius* is $F = \varphi^{-1} \in \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$. Then

$$F^* = F. \tag{165}$$

For $x \in |X_0|$, let $x \in |X|$ be a point in the orbit $x$. Then $\mathcal{F}_x = (\mathcal{F}_0)_x$, and

$$\mathcal{F}_x^* = \mathcal{F}_x \in \mathrm{End}(\mathcal{F}_x). \tag{166}$$

where $F_x^* = F_x^{*\deg(x)} \in \mathrm{End}(\mathcal{F}_x)$. Significantly, equation (164) becomes

$$Z = (X_0, \mathcal{F}_0, t) = \prod_{x\in|X_0|} \det(1 - F_x t^{\deg(x)}, \mathcal{F}_0)^{-1} = \prod_i \det(1 - Ft, H_c^i(X, \mathcal{F}))^{(-1)^{i+1}}. \tag{167}$$

**2.2**

Let $l$ be a prime number, and let $k$ be an algebraically closed field of characteristic $p \neq l$ (here $p$ may be 0). We first define $\mathbb{Z}_l(1)$ and $\mathbb{Q}_l(1)$, which are isomorphic (as groups) to $\mathbb{Z}_l$ and $\mathbb{Q}_l$ respectively. For $n \in \mathbb{Z}_{>0}$, let $\frac{Z}{l^n}(1)$ be the group of $l^n$th roots of unity in $k$. Each $\frac{Z}{l^n}(1)$ is cyclic, and of order $n$, and is therefore isomorphic to the group $\frac{Z}{l^n\mathbb{Z}}$.[74] We get a projective system (inverse system) by

$$\ldots \to \frac{Z}{l^2}(1) \xrightarrow{x\mapsto x^l} \frac{Z}{l}(1), \tag{168}$$

so let

$$\mathbb{Z}_l(1) = \varprojlim_n \frac{Z}{l^n}(1) \qquad \text{and} \qquad \mathbb{Q}_l(1) = \mathbb{Z}_l(1) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l.$$

To see that $\mathbb{Z}_l(1)$ is a $\mathbb{Z}_l$-module, note that $\mathbb{Z}_l(1) \simeq \mathbb{Z}_l$ as groups, and in particular there is a canonical choice of embedding

$$\mathbb{Z}_l \hookrightarrow \mathbb{Z}_l(1).$$

Note that

$$\mathbb{Q}_l(1) \simeq \mathbb{Q}_l$$

as vector spaces over $\mathbb{Q}_l$. We now work in the category of vector spaces over $\mathbb{Q}_l$, and

[74] To see this, note that if $t < n$ and $y \in \frac{Z}{l^n}(1)$ has order $l^t$ then any $l$th root of $y$ has order $l^{t+1}$. Repeating this process if necessary gives a generator, which generates $l^n$ distinct elements of $\frac{Z}{l^n}(1)$, and we know there cannot be more because $z^{l^n} - 1$ is a degree $l^n$ polynomial.

implicitly regard duals and tensors as living in this category. For $r \in \mathbb{Z}_{>0}$, let[75]

$$\mathbb{Q}_l(r) = \mathbb{Q}_l(1)^{\otimes r}.$$

Let $\mathbb{Q}_l(0) = \{0\}$, and for $r \in \mathbb{Z}_{<0}$ let

$$\mathbb{Q}_l(r) = \mathbb{Q}_l(-r)^\vee$$

be the dual vector space over $\mathbb{Q}_l$. The spaces $\mathbb{Q}_l(r)$ are all isomorphic, but they will be acted upon differently.

The *orientation sheaf* of $X$ in $l$-adic cohomology is the locally constant $\mathbb{Q}_l$-sheaf with values in $\mathbb{Q}_l(n)$. The *fundamental class* is, for each $n \in \mathbb{Z}_{>0}$, a morphism

$$\mathrm{Tr} : H_c^{2n}(X, \mathbb{Q}_l(n)) \to \mathbb{Q}_l,$$

or, alternatively,

$$\mathrm{Tr} : H_c^{2n}(X, \mathbb{Q}_l) \to \mathbb{Q}_l(-2n).$$

**2.3**

**Theorem 77:** *[Poincaré duality] If $X$ is proper and smooth, and purely of dimension $n$, then the bilinear form*

$$\mathrm{Tr}(\cdot \cup \cdot) : H^i(X, \mathbb{Q}_l) \otimes H^{2n-i}(X, \mathbb{Q}_l) \to \mathbb{Q}_l(-n)$$

*is a perfect pairing. As $\mathbb{Q}_l(-n) \simeq \mathbb{Q}_l$, the pairing identifies $H^i(X, \mathbb{Q}_l)$ with the dual of $H^{2n-i}(X, \mathbb{Q}_l(n))$. In particular, $H^{2n}(X, \mathbb{Q}_l)$ is identified with $\mathbb{Q}_l(-n)$.*

**2.4**

Let $X_0$ be a smooth and proper variety over a finite field $\mathbb{F}_q$, purely of dimension $n$[76], and let $X$ be the variety over $\bar{\mathbb{F}}_q$ be gotten from $X_0$ by extension of scalars. From (2.3), we deduce the following. If $(\alpha_j)$ are the eigenvalues of the geometric Frobenius $F$ acting on $H^i(X, \mathbb{Q}_l)$ then the eigenvalues of $F$ acting on $H^{2n-i}(X, \mathbb{Q}_l)$ are $(q^n \alpha_j^{-1})$.

Let's think about how to prove this. Dougal's argument (13 January) gives a unique

$$F_* : H^i(X, \mathbb{Q}_l) \to H^i(X, \mathbb{Q}_l)$$

such that

$$F_* x \cup y = x \cup F^* y$$

for $x \in H^i(X, \mathbb{Q}_l)$ and $y \in H^{2n-i}(X, \mathbb{Q}_l)$. If $x$ is an eigenvector of $F^*$ acting on $H^i(X, \mathbb{Q}_l)$ with eigenvalue $\alpha$, then

$$\alpha F_* x \cup y = F_x F^* x \cup y = F^* x \cup F^* y = F^*(x \cup y) = q^n x \cup y,$$

for $y \in H^{2n-i}(X, \mathbb{Q}_l)$, so $\alpha \neq 0$, and $F_* x = q^n \alpha^{-1} x$. It therefore suffices to show that $F_* \curvearrowright H^i(X, \mathbb{Q}_l)$ and $F^* \curvearrowright H^{2n-i}(X, \mathbb{Q}_l)$ have the same eigenvalues.

For $x \in H^i(X, \mathbb{Q}_l)$, let $x^\vee \in H^{2n-i}(X, \mathbb{Q}_l)$ be the dual element (fix a basis and dual

[75]This tensor product is probably over $\mathbb{Z}$ because Deligne hasn't yet mentioned regarding $\mathbb{Q}_l(1)$ as anything but a group, and also because he seems to specify the base ring when it's needed. Still, we need to work with this in explicitly if we want to understand it properly.

[76]i.e. each component has dimension $n$, for instance a curve is purely of dimension 1.

basis). Define

$$F : H^i(X, \mathbb{Q}_l) \to H^i(X, \mathbb{Q}_l)$$
$$x \mapsto F^*(x^\vee)^\vee.$$

Define a bilinear form on $H^i(X, \mathbb{Q}_l)$ by

$$\langle x, y \rangle = \mathrm{Tr}(x \cup y^\vee). \tag{169}$$

This is symmetric:

$$\langle x, y \rangle = \mathrm{Tr}(x \cup y^\vee) = y^\vee(x) = y^T x = x^T y = \mathrm{Tr}(y \cup x^\vee) = \langle y, x \rangle. \tag{170}$$

For $x, y \in H^i(X, \mathbb{Q}_l)$,

$$\langle F_* x, y \rangle = \mathrm{Tr}(F_* x \cup y^\vee) = \mathrm{Tr}(x \cup F^*(y^\vee)) = \langle x, Fy \rangle, \tag{171}$$

so $F_*$ and $F$ are adjoint. Thus $F_*$ has the same eigenvalues as $F$, which has the same eigenvalues as $F^* \curvearrowright H^{2n-i}(X, \mathbb{Q}_l)$.

**2.5**

For simplicity, assume that $X$ is connected. We can phrase (2.4) in geometric (as opposed to Galois) language:

1. The cup product between $H^i(X, \mathbb{Q}_l)$ and $H^{2n-i}$ is a perfect pairing with values in $H^{2n}(X, \mathbb{Q}_l)$, and the latter has dimension 1.
2. The cup product commutes with $F^*$.
3. The morphism $F$ is finite and of degree $q^n$: on $H^{2n}(X, \mathbb{Q}_l)$, the map $F^*$ is multiplication by $q^n$. From (2.3), this is also the action of $F^*$ on $\mathbb{Q}_l(-n)$.
4. The eigenvalues of $F^*$ have the property (2.4).

**2.6**

Put

$$\chi(X) = \sum_i (-1)^i \dim H^i(X, \mathbb{Q}_l).$$

If $n$ is odd, the form $\mathrm{Tr}(\cdot \cup \cdot)$ on $H^n(X, \mathbb{Q}_l)$ is alternating, since

$$x \cup x = (-1)^{n^2}(x \cup x), \qquad x \in H^n(X, \mathbb{Q}_l). \tag{172}$$

**Fact.** If $V$ admits a nondegenerate alternating bilinear form then $\dim(V)$ is even.

It follows that $n\chi(X)$ is always even, since

$$\chi(X) = (-1)^n \dim H^n(X, \mathbb{Q}_l) + 2 \sum_{i=0}^{n-1} (-1)^i \dim H^i(X, \mathbb{Q}_l). \tag{173}$$

From (1.5.4), (2.3), and (2.4), we deduce that

$$Z\Big(X_0, \frac{1}{q^n t}\Big) = \varepsilon \cdot q^{n\chi/2} t^\chi Z(X_0, t),$$

where $\varepsilon = \pm 1$. Dougal showed us this on 13 January 2012. If $n$ is even, let $N$ be the multiplicity of the eigenvalue $q^{n/2}$ of $F^*$ acting on $H^n(X,\mathbb{Q}_l)$. Then

$$\varepsilon = \begin{cases} 1, & \text{if } n \text{ is odd} \\ (-1)^N, & \text{if } n \text{ is even.} \end{cases}$$

To derive this, let $\beta_i = \dim H^i(X,\mathbb{Q}_l)$. Then

$$\begin{aligned} Z\Big(X_0, \frac{1}{q^n t}\Big) &= \prod_{i,j}\Big(1-\frac{\alpha_{i,j}}{q^n t}\Big)^{(-1)^{i+1}} = \prod_{i,j}\Big(-\frac{\alpha_{i,j}}{q^n t}\Big)^{(-1)^{i+1}} \cdot \prod_{i,j}\Big(1-\frac{q^n t}{\alpha_{i,j}}\Big)^{(-1)^{i+1}} \\ &= \prod_{i,j}\Big(-\frac{q^n t}{\alpha_{i,j}}\Big)^{(-1)^{i}} \cdot \prod_{i,j}\Big(1-\frac{q^n}{\alpha_{2n-i,j}}t\Big)^{(-1)^{i+1}} \\ &= t^\chi Z(X_0,t)\cdot \prod_{i,j}\Big(-\frac{q^n}{\alpha_{i,j}}\Big)^{(-1)^{i}} \\ &= t^\chi Z(X_0,t)\cdot \prod_{j}\Big(-\frac{q^n}{\alpha_{n,j}}\Big)^{(-1)^{n}} \cdot \prod_{i=0}^{n-1}\prod_{j}\Big(-\frac{q^n}{\alpha_{i,j}}\Big)^{(-1)^{i}}\Big(-\frac{q^n}{\alpha_{2n-i,j}}\Big)^{(-1)^{i}} \\ &= t^\chi Z(X_0,t)\cdot \prod_{i=0}^{n-1} q^{n(-1)^i\beta_i}\cdot \prod_{j}\Big(-\frac{q^n}{\alpha_{n,j}}\Big)^{(-1)^{n}}. \end{aligned}$$

Consider $F^* \curvearrowright H^n(X,\mathbb{Q}_l)$. If $n$ is odd, then $\mathrm{Tr}(\cdot\cup\cdot)$ is alternating, so the eigenvalues pair up, and

$$Z\Big(X_0,\frac{1}{q^n t}\Big) = q^{n\chi/2}t^\chi Z(X_0,t).$$

To show that the eigenvalues pair up, suppose $x \in H^n(X,\mathbb{Q}_l)$ is an eigenvector with eigenvalue $\lambda$. For $y \in H^n(X,\mathbb{Q}_l)$,

$$y\cup\lambda x^\vee = \lambda x\cup y^\vee = F^*x\cup y^\vee = y\cup F^*(x^\vee), \tag{174}$$

so $F^*x^\vee = \lambda x^\vee$. As $\mathrm{Tr}(\cdot\cup\cdot)$ is alternating, $x$ and $x^\vee$ are linearly independent.

If $n$ is even, then there may be unpaired eigenvalues, which must be $q^{n/2}$ or $-q^{n/2}$. Thus,

$$Z\Big(X_0,\frac{1}{q^n t}\Big) = \varepsilon\cdot q^{n\chi/2}t^\chi Z(X_0,t),$$

where $\varepsilon = -1$ if $q^{n/2}$ is unpaired and $\varepsilon = 1$ otherwise.

**2.7**

We will need other forms of the duality theorem. The case of curves will suffice for our purposes. If $\mathcal{F}$ is a $\mathbb{Q}_l$-sheaf on a variety $X$ over an algebraically closed field $k$, we write $\mathcal{F}(r)$ for the sheaf $\mathcal{F}\otimes\mathbb{Q}_l(r)$. This is (non-canonically) isomorphic to $\mathcal{F}$.

**2.8**

**Theorem 78:** *Assume that $X$ is smooth, purely of dimension $n$, and $\mathcal{F}$ twisted constant. Let $\mathcal{F}^\vee$ be the dual of $\mathcal{F}$. The bilinear form*

$$\begin{aligned} \mathrm{Tr}(\cdot\cup\cdot): H^i(X,\mathcal{F})\otimes H_c^{2n-i}(X,\mathcal{F}^\vee(n)) &\to H_c^{2n}(X,\mathcal{F}\otimes\mathcal{F}^\vee(n)) \\ &\to H_c^{2n}(X,\mathbb{Q}_l(n)) \to \mathbb{Q}_l \end{aligned}$$

*is a perfect pairing.*

**2.9**

Assume that $X$ is connected, and let $x \in |X|$. The functor $\mathcal{F} \mapsto \mathcal{F}_x$ is an equivalence between the category of twisted constant $\mathbb{Q}_l$-sheaves and the category of finite-dimensional $\mathbb{Q}_l$-vector spaces on which $\pi_1(X,x)$ acts continuously. This identifies $H^0(X,\mathcal{F})$ with the invariants of $\pi_1(X,x)$ acting on $\mathcal{F}_x$:

$$H^0(X,\mathcal{F}) \xrightarrow{\simeq} \mathcal{F}_x^{\pi_1(X,x)}. \tag{175}$$

For $X$ smooth, connected, and dimension $n$, (2.8) now gives

$$H_c^{2n}(X,\mathcal{F}) = H^0(X,\mathcal{F}^\vee(n))^\vee = (\mathcal{F}_x^\vee(n)^{\pi_1(X,x)})^\vee.$$

The duality exchanges invariants (largest invariant subspace) and coinvariants (largest invariant quotient). The formula becomes

$$H_c^{2n}(X,\mathcal{F}) = (\mathcal{F}_x)_{\pi_1(X,x)}(-n).$$

We will only use this for $n = 1$.

**2.10**

Scholium. Let $X$ be a smooth, connected curve over an algebraically closed field $k$, let $x \in |X|$, and let $\mathcal{F}$ be a twisted constant $\mathbb{Q}_l$-sheaf. Then

(i) $H_c^0(X,\mathcal{F}) = 0$ if $X$ is affine.

(ii) $H_c^2(X,\mathcal{F}) = (\mathcal{F}_x)_{\pi_1(X,x)}(-1)$.

The assertion (i) simply indicates that $\mathcal{F}$ does not have a section with finite support.

**2.11**

Let $X$ be a projective curve, smooth and connected over an algebraically closed field $k$. Let $U$ be an open subset of $X$, the complement of a finite subset $S \subseteq |X|$. Let $j$ be the inclusion $U \hookrightarrow X$, and let $\mathcal{F}$ be a twisted constant $\mathbb{Q}_l$-sheaf on $U$. Let $j_*$ be the direct image constructible sheaf of $\mathcal{F}$. Its fibre at $x \in S$ is of rank less than or equal to the rank at a general point; this is the space of invariants under the local monodromy group.

**2.12**

**Theorem 79:** *The bilinear form*

$$\begin{aligned}\mathrm{Tr}(\cdot \cup \cdot) : & H^i(X, j_*\mathcal{F}) \otimes H^{2-i}(X, j_*\mathcal{F}^\vee(1)) \to H^2(X, j_*\mathcal{F} \otimes j_*\mathcal{F}^\vee(1)) \\ & \to H^2(X, j_*(\mathcal{F}\otimes\mathcal{F}^\vee)(1)) \to H^2(X, j_*\mathbb{Q}_l(1)) = H^2(X,\mathbb{Q}_l(1)) \to \mathbb{Q}_l\end{aligned}$$

*is a perfect pairing.*

**2.13**

It will be convenient to have at our disposal the $\mathbb{Q}_l$-sheaf $\mathbb{Q}_l(r)$ for general schemes $X$ where $l$ is invertible. The point is to define the $\frac{\mathbb{Z}}{l^n}(1)$. By definition, $\frac{\mathbb{Z}}{l^n}(1)$ is the étale sheaf of $l^n$th roots of unity.

A *generic point* in a topological space $X$ is a point that is dense in $X$. Under the Zariski topology, a set is irreducible if and only if it has a generic point. Weil's notion was obsoleted in 1957 by Zariski. Let $R$ be a discrete valuation ring. Then $\text{Spec}(R)$ comprises two points: a *generic point*, $\{0\}$, and the *special point* (or *closed point*), which is the unique maximal ideal (note that every prime ideal is maximal in a DVR). For maps to $\text{Spec}(R)$, this gives a notion of the *special fibre* and the *generic fibre*. Such language can describe local Lefschetz theory, and we present this first for $\mathbb{C}$ and then in greater generality.

**4.1**

On $\mathbb{C}$, the results of local Lefschetz theory are as follows. Let $D = \{z \in \mathbb{C} : |z| < 1\}$, let $D^* = D - \{0\}$, and let $f : X \to D$ be a morphism of analytic spaces.[77] We assume that

(a) $X$ is non-singular, and purely of dimension $n+1$;

(b) $f$ is proper;

(c) $f$ is smooth away from a point $x$ in the special fibre $f^{-1}(0)$;

(d) $x$ is a nondegenerate quadratic point of $x$.

Let us unpack these assumptions a little.

(a) $X$ is *non-singular*, meaning that it's a complex manifold,[78] and of *pure dimension* $n+1$, meaning that each component has dimension $n+1$.

(b) $f$ is *proper*, meaning that the preimage of a compact set is compact.

(c) This means that if $x \in X$ and $f(x) \neq 0$ then $f$ is holomorphic at $x$.

(d) This means that $f'(x) = 0$ and that the Hessian $H$ is nonsingular, i.e. $\det(H) \neq 0$.

Let $t \in D^*$ and $X = f^{-1}(t)$ be "the" general fibre. To the previous data, we associate

(a/) Specialization morphisms

$$\text{sp} : H^i(X_0, \mathbb{Z}) \to H^i(X_t, \mathbb{Z}),$$

defined as follows: $X_0$ is a deformation retract of $X$, and sp is the composition

$$H^i(X_0, \mathbb{Z}) \xleftarrow{\sim} H^i(X, \mathbb{Z}) \to H^i(X_t, \mathbb{Z}).$$

[77] An analytic space $X$ is a generalization of analytic manifolds that allows for singularities. The ring of analytic functions $X \to \mathbb{C}$ makes such a space into a locally ringed space. The category of analytic spaces is a subcategory of locally ringed spaces. Analytic spaces are analytic varieties patched together.

[78] We're effectively making the assumption that $X$ is a complex manifold (biholomorphic transition maps), so we don't really need to know what a singularity is. Roughly, a point $x$ is *nonsingular* if it has a neighbourhood that is isomorphic to $(U, \mathcal{O}_U)$ for some integral domain $U$.

(b/) Monodromy transformations

$$T : H^i(X_t, \mathbb{Z}) \to H^i(X_t, \mathbb{Z}),$$

which describe the effect on the singular cycles of $X_t$ as $t$ "winds around 0". This is also the action of the positive generator of $\pi_1(D^*, t)$ on $H^i(X_t, \mathbb{Z})$, the latter being the fibre of the local system $R^i f_* \underline{\mathbb{Z}}|_{D^*}$ over $t$ (here the underscore denotes locally constant sheaf).

Lefschetz theory describes (a/) and (b/) in terms of the *vanishing cycle* $\delta \in H^n(X_t, \mathbb{Z})$, which we do not define here.[79] For $i \neq n, n+1$, we have an isomorphism

$$H^i(X_0, \mathbb{Z}) \xrightarrow{\sim} H^i(X_t, \mathbb{Z}).$$

For $i = n, n+1$, we have an exact sequence

$$0 \to H^n(X_0, \mathbb{Z}) \to H^n(X_t, \mathbb{Z}) \xrightarrow{x \mapsto (x,\delta)} \mathbb{Z} \to H^{n+1}(X_0, \mathbb{Z}) \to H^{n+1}(X_t, \mathbb{Z}) \to 0. \quad (176)$$

Presumably $(x, \delta)$ is an inner product inherited from the complex structure, though this is just a guess. In this case there would need to be some canonical way to pick a basis, perhaps generalizing this footnote.[80] For $i \neq n$, the monodromy $T$ is the identity. For $i = n$, we have

$$Tx = x \pm (x, \delta)\delta. \quad (177)$$

The following table provides data depending on $n$ mod 4:

| $n$ mod 4 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| sign | - | - | + | + |
| $(\delta, \delta)$ | 2 | 0 | 2 | 0 |
| $T\delta$ | $-\delta$ | $\delta$ | $-\delta$ | $\delta$ |

The monodromy transformation $T$ preserves the intersection form $Tr(\cdot \cup \cdot)$ on $H^n(X_t, \mathbb{Z})$. I would interpret this as follows: if $x, y \in H^n(X_t, \mathbb{Z})$ then

$$\mathrm{Tr}(Tx \cup Ty) = \mathrm{Tr}(x \cup y). \quad (178)$$

For $n$ odd, $T$ is a symplectic transvection.[81] For $n$ even, $T$ is an orthogonal symmetry (equivalently: symmetric isometry, or self-inverse isometry).

That concludes our discussion of (4.1). We will generalise this in (4.2), but for now we just discuss some arbitrary aspects of this. The disk $D$ is replaced by the spectrum $S$ of a Henselian[82] DVR $A$ with algebraically closed residue field. The morphism $f : X \to S$ is *proper* if and only if it is: separated, of finite type, and universally closed.

Recall that $f : X \to S$ is *of finite type* if there exists a cover $\{V_i = \mathrm{Spec}(B_i)\}$ of $S$ such that each $f^{-1}(V_i)$ can be covered by a finite number of $U_{ij} = \mathrm{Spec}(A_{ij})$ such that each $A_{ij}$ is finitely generated as a $B_i$-algebra. The morphism $f : X \to S$ is *separated and universally closed* if the following condition is met.

[79] [SGA 7] XIII and XV is the general reference given for vanishing cycles. Alternatively, see Lefschetz[10].

[80] http://mathoverflow.net/questions/56082/vanishing-cycles-in-a-nutshell

[81] A *transvection* (or *shear matrix*) is a matrix with 1s along the diagonal and precisely one other nonzero entry. A matrix is *symplectic* if it preserves a given bilinear form. There is a natural Hermitian form upon choosing a basis. Note that the symplectic group is generated by the set of symplectic transvections.

[82] Hensel's lemma holds: a simple root of a polynomial modulo a prime ideal $\mathfrak{p}$ lifts to a unique root modulo any power of $\mathfrak{p}$.

Let $R$ be a valuation ring with field of fractions $K$ (so $R$ is an integral domain, and if $x \in K$ then $x \in R$ or $x^{-1} \in R$), and let $i : \mathrm{Spec}(K) \to \mathrm{Spec}(R)$ be induced by the canonical inclusion $R \hookrightarrow K$. If the following diagram commutes, then there exists a unique morphism $\mathrm{Spec}(R) \to X$ that commutes with the diagram:

$$\begin{array}{ccc} \mathrm{Spec}(K) & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(R) & \longrightarrow & S. \end{array} \tag{179}$$

We conclude with the statement of a crucial theorem in the context of the Weil conjectures.

**Theorem 80** (existence of Lefschetz pencils)**:** *Let $X$ be a smooth complete surface over an algebraically closed field $k$. Then there exists*

- *a surface $X^*$ that is obtained from $X$ by blowing up a finite number of points, and*
- *a map $\pi : X^* \to \mathbb{P}^1$,*

*such that condition $(L)$ is satisfied. Condition L is as follows.*

*(a) $\pi$ is proper and flat. Moreover, there exists a section*

$$s : \mathbb{P}^1 \to X^*$$

*of $\pi$.*

*(b) Any generic fibre of $\pi$ is a smooth curve.*

*(c) The closed fibres (fibres over special points) are connected with at most one node (ordinary double point).*

**Joe Chan** *Thursday 17 May 2012*

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We begin with some intuition on vanishing cycles.

**Picture 1, from mathoverflow**

Let $(X_t)_{t\in\mathbb{C}}$ be a family of complex nonsingular plane cubics degenerating to a nodal cubic $X_0$. We can construct a basis of real curves $\alpha_t, \beta_t \in H_1(X_t, \mathbb{Z})$ such that $\beta_t \to \beta_0 \in H_1(X_0, \mathbb{Z})$ and $\alpha_t \to 0$ as $t \to 0$. Transporting these cycles around a loop in the $t$-plane, we get a new basis $T(\alpha_t), T(\beta_t)$. This is related to the old basis by the Picard-Lefschetz formula,

$$T(\alpha_t) = \alpha_t$$
$$T(\beta_t) = \beta_t \pm (\alpha_t \cdot \beta_t)\alpha_t.$$

The effect is to cut along $\alpha_t$, twist around $\beta_t$, and re-glue.

How does this generalise to higher homology?

**Picture 2, from Freitag and Kiehl**

Let $\Lambda_X$ be a sheaf over $X$ with Lefschetz pencil $\tilde{X}$ and Lefschetz fibration $f : \tilde{X} \to \mathbb{P}'$. Let $\Lambda = \frac{Z}{r\mathbb{Z}}$, where $r$ is invertible on $\mathbb{P}'$ under $f : \tilde{X} \to \mathbb{P}'$. Let $s, \eta \in \tilde{X}$ be a closed point and a generic point respectively. There is a specialisation map

$$sp : (R^p f_* \Lambda_{\tilde{X}})_s \to (R^p f_* \Lambda_{\tilde{X}})_\eta$$

between the stalks. The "local nature" is $f : X \to \text{Spec}(R)$, where $R$ is a strictly Henselian DVR.

A *discrete valuation ring* (DVR) is a local PID that is not a field. A local ring $R$ with maximal ideal $\mathfrak{m}$ is *Henselian* if Hensel's lemma holds; in other words, if $P \in R[X]$ is a monic polynomial, then any factorization of its image in $\frac{R}{\mathfrak{m}}[X]$ into a product of coprime monic polynomials can be lifted to a factorization in $R[X]$. A Henselian ring is *strict* if its residue field is separably closed.

**Example 81** ($p$-adic integers)**:** *Assume that $X$ is pure of real dimension $n+1$ (where $n$ is odd). Let $f : X \to \text{Spec}(R)$, where $R$ is a strict Henselian DVR. Assume that $f$ is flat, proper, and smooth away from the node. Recall that*

$$sp : H^i(X_s, \Lambda) \to H^i(X_\eta, \Lambda)$$

*is an isomorphism for $i \neq n, n+1$. For $p = n$, the morphism $sp$ is injective, and*

$$coker(sp) = \langle \delta \rangle.$$

*Intuitively, $H^n(X_s, \Lambda)$ has a basis of $\beta$s, and $H^n(X_\eta, \Lambda)$ has a basis of $\alpha$s.*

Recall the following existence theorem.

**Theorem 82** (existence of Lefschetz pencils)**:** *Let $X$ be a smooth complete surface over an algebraically closed field $k$. Then there exists*

- *a surface $X^*$ that is obtained from $X$ by blowing up a finite number of points, and*
- *a map $\pi : X^* \to \mathbb{P}^1$,*

*such that condition $(L)$ is satisfied. Condition $L$ is as follows.*

*(a) $\pi$ is proper and flat. Moreover, there exists a section*

$$s : \mathbb{P}^1 \to X^*$$

*of $\pi$.*

*(b) Any generic fibre of $\pi$ is a smooth curve.*

*(c) The closed fibres (fibres over special points) are connected with at most one node (ordinary double point).*

We give a brief description of a *blowup* at a point. Given a node, we want to "straighten" the curve locally. In particular, we want to resolve a neighbourhood around the node so that we don't have two tangent directions at the point. To achieve this, we replace the point $P$ by the space of directions through $P$. There is a more algebraic picture for schemes, but for now consider blowing up a point $P$ in a plane. The space of directions though $P$ is called the *exceptional divisor*, which is the projectivised normal space at $P$, and is isomorphic to $\mathbb{P}^1$.

We give a brief description of projective duality, in preparation for defining Lefschetz pencils. Let $k$ be a field and $m \in \mathbb{Z}_{>0}$. There is a bijection between points in $k\mathbb{P}^m$ and hyperplanes in $k^{m+1}$ given by

$$[u_0 : \ldots : u_m] \mapsto \{(x_0, \ldots, x_n) : u_0 x_0 + \ldots + u_n x_n = 0\}. \tag{180}$$

In other words, the dot product is zero. We can check that this criterion is well defined, though the dot product need not be an inner product.

We now define Lefschetz pencils (see [7]). Let $k$ be a field, let $m \in \mathbb{Z}_{>0}$, and consider

$$(k\mathbb{P}^m)^\vee = Gr(m, k^{m+1}).$$

A *pencil of hyperplanes in* $\mathbb{P}^m$ is a line in $(k\mathbb{P}^m)^\vee$. Let $D$ be such a pencil. If $H_0, H_\infty \in D$ are distinct, then

$$D = \{\alpha H_0 + \beta H_\infty \mid [\alpha : \beta] \in k\mathbb{P}^1\}.$$

The *axis* of $D$ is

$$\cap\{H : H \in D\} = H_0 \cap H_\infty.$$

In other words, if $P \in H_0 \cap H_\infty$, then $P$ is in every hyperplane in the pencil.

Assume that $k$ is algebraically closed. Let $X$ be a nonsingular projective variety of dimension $d \geq 2$, and embed $X$ in $k\mathbb{P}^m$. A pencil $D$ is *Lefschetz* if there exists an open dense subset $U$ of $D$ such that the following conditions are met:

(a) The axis of $D$ cuts $X$ transversally.

(b) If $H \in U$ then the hyperplane section $X_H = X \cap H$ is nonsingular.

(c) If $H \in D \setminus U$, then $X_H$ has precisely one singularity, and it is a node.

**Joe Chan** *Thursday 24 May 2012*

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**4.2**

We generalise the story from (4.1), replacing $D$ with $S$.

Let $S = \mathrm{Spec}(A)$, where $A$ is a strictly Henselian DVR, and let $f : X \to \mathrm{Spec}(A)$ be a proper morphism of analytic spaces. Let $\eta$ be the generic point of $S$ (spectrum of the field of fractions of $A$), and let $s$ be the closed (i.e. special) point (spectrum of the residue field). The role of $t$ is played by a generic geometric point $\bar{\eta}$ (spectrum of an algebraic closure of the field of fractions of $A$).

Assume that $X$ is regular, of pure dimension $n+1$, and smooth but for an ordinary quadratic point in the special fibre $X_s$. Regular: every localisation $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$ has $\mathfrak{m}_x$ generated by $n+1$ elements. Let $l$ be a prime different to the characteristic $p$ of $\frac{A}{\mathfrak{m}}$, and let $X_{\bar{\eta}}$ be the generic geometric fibre. There is a specialisation morphism

$$\mathrm{sp}(H^i(X_s, \mathbb{Q}_l)) \xleftarrow{\sim} H^i(X, \mathbb{Q}_l) \to H^i(X_{\bar{\eta}}, \mathbb{Q}_l). \tag{181}$$

The role of $T$ is played by the action of the inertia group $I = \mathrm{Gal}(\bar{\eta}/\eta)$,[83] acting on $H^i(X_{\bar{\eta}}, \mathbb{Q}_l)$ by transport of structure (cf 1.15):

$$I = \mathrm{Gal}(\bar{\eta}/\eta) \to GL(H^i(X_{\bar{\eta}}, \mathbb{Q}_l)). \tag{182}$$

The data (181) and (182) entirely desribe the sheaves $R^i f_* \mathbb{Q}_l$ on $S$.

**4.3**

Put $m = \lfloor n/2 \rfloor$. We can still define sp and local monodromy in terms of the vanishing cycle

$$\delta \in H^n(X_{\bar{\eta}}, \mathbb{Q}_l)(m). \tag{183}$$

This cycle is well defined up to sign. For $i \neq n, n+1$, we have

$$H^i(X_s, \mathbb{Q}_l) \xrightarrow{\sim} H^i(X_{\bar{\eta}}, \mathbb{Q}_l). \tag{184}$$

For $i = n, n+1$, we have an exact sequence

$$\begin{aligned} 0 \to H^n(X_s, \mathbb{Q}_l) \to H^n(X_{\bar{\eta}}, \mathbb{Q}_l) &\xrightarrow{x \mapsto \mathrm{Tr}(x \cup \delta)} \mathbb{Q}_l(m-n) \\ \to H^{n+1}(X_s, \mathbb{Q}_l) \to H^{n+1}(X_{\bar{\eta}}, \mathbb{Q}_l) &\to 0. \end{aligned} \tag{185}$$

The action of $I$ (local monodromy) is trival if $i \neq n$. For $i = n$, it is as follows.

A) $n$ odd. We have a canonical isomorphism[84]

$$t_l : I \to \mathbb{Z}_l(1),$$

and the action of $\sigma \in I$ is

$$x \mapsto x \pm t_l(\sigma)(x, \delta)\delta.$$

B) $n$ even. We don't use this case. We just say that if $p \neq 2$ then there exists a unique order two character

$$\varepsilon : I \to \{\pm 1\}$$

such that

$$\sigma(x) = \begin{cases} x, & \text{if } \varepsilon(\sigma) = 1 \\ x \pm (x, \delta)\delta), & \text{if } \varepsilon(\sigma) = -1. \end{cases}$$

The signs $\pm$ in A) and B) are the same as in (4.1).

**4.4**

These results imply the following information about the $R^i f_* \mathbb{Q}_l$.

a) $\delta \neq 0$.

1) For $i \neq n$, the sheaf $R^i f_* \mathbb{Q}_l$ is constant.

[83] Note that $I$ is the étale fundamental group $\pi_1(X, \bar{\eta})$. As a geometric point, $\bar{\eta}$ is induced by $A \hookrightarrow k^{\mathrm{sep}}$, where $k$ is the field of fractions of $A$ (the algebraic closure is given by $\bar{\eta} = \mathrm{Spec}(\bar{k})$, and this contains a unique separable closure). See example 63.

[84] Let $k$ and $k^{\mathrm{sep}}$ be as in the previous footnote. Then $\mathrm{Gal}(\bar{\eta}/\eta) = \mathrm{Gal}(k^{\mathrm{sep}}/k)$. This acts on $\frac{\mathbb{Z}}{l^n}(1)$ be permuting the $l^n$th roots of unity, and this induces an action on $\mathbb{Z}_l(1)$.

2) Let $j$ be the inclusion of $\eta$ in $S$. We have

$$R^n f_* \mathbb{Q}_l = j_* j^* R^n f_* \mathbb{Q}_l.$$

b) $\delta = 0$. This is an exceptional case. As $(\delta, \delta) = \pm 2$ for $n$ even, this case can only occur when $n$ is odd.

1) For $i \neq n+1$, the sheaf $R^i f_* \mathbb{Q}_l$ is constant.

2) Let $\mathbb{Q}_l(m-n)_s$ be the sheaf $\mathbb{Q}_l(m-n)$ on $\{s\}$,[85] extended by 0 on $S$. We have an exact sequence

$$0 \to \mathbb{Q}_l(m-n)_s \to R^{n+1} f_* \mathbb{Q}_l \to j_* j^* R^{n+1} f_* \mathbb{Q}_l \to 0,$$

where $j_* j^* R^{n+1} f_* \mathbb{Q}_l$ is a constant sheaf.

**Yi Huang** *Thursday 31 May 2012*

The Weil conjectures have four parts. The main ingredient for three of them is $l$-adic cohomology, so let's recap how that works.

We can count $\#X_m$ by regarding $X_m$ as the set of fixed points of the Frobenius and using the Lefschetz fixed point theorem:

$$Z(X,t) = \prod_{i=0}^{2n} \exp\Big(\sum_{r=1}^{\infty} \mathrm{Tr}(F^{*r}, H^i(X, \mathbb{Q}))\frac{t^r}{r}\Big)^{(-1)^i}.$$

Recall this fact from linear algebra:

$$\exp\Big(\mathrm{Tr}\Big(\sum_{r=1}^{\infty} A^r \frac{t^r}{r}\Big)\Big) = \det(1 - At)^{-1}.$$

Now

$$Z(X,t) = \prod_{i=0}^{2n} \det(1 - F^* t)^{(-1)^{i+1}}.$$

This gives us rationality, Betti numbers, and the easier part of the Riemann hypothesis. The duality comes from Poincaré duality, with a bit of algebra. We proved most of the Riemann hypothesis for curves in section 3. In general, we need the machinery of Lefschetz pencils.

A Lefschetz pencil is a fibration over $\mathbb{C}P^1 = S^2$, nice at all but finitely many points. In other words, if we remove the bad points and their fibres, we get a really nice fibration. If we cut between the bad points and remove the corresponding fibres, we get a fibration over a contractible space. We can get the homology above from the homology of the fibres. The bad fibres correspond to vanishing cycles.

Let's talk sheaves.

**Eg1**

Consider the topological space $\{*\}$, and assign to it the constant sheaf $\mathbb{R}$ (which is an

[85] The étale morphism $s \to S$ is induced by $A \to A/\mathfrak{m}$.

abelian group).

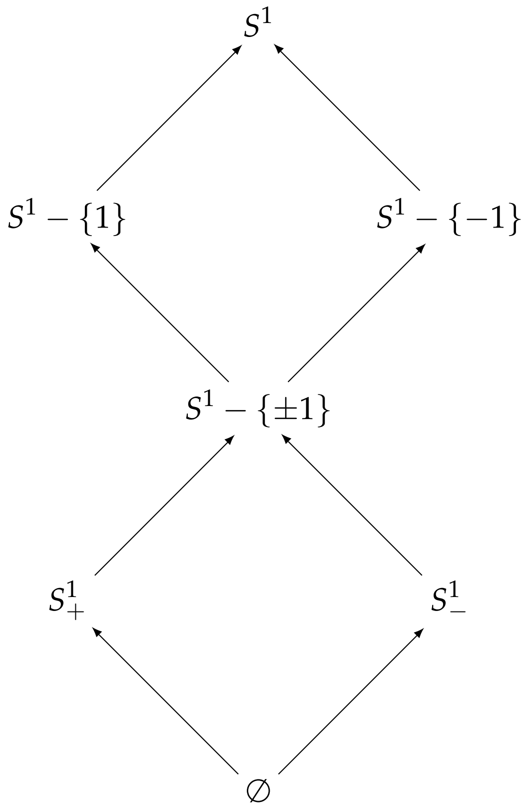
**Eg2**

Consider $S^1 \subseteq \mathbb{C}$ with topology

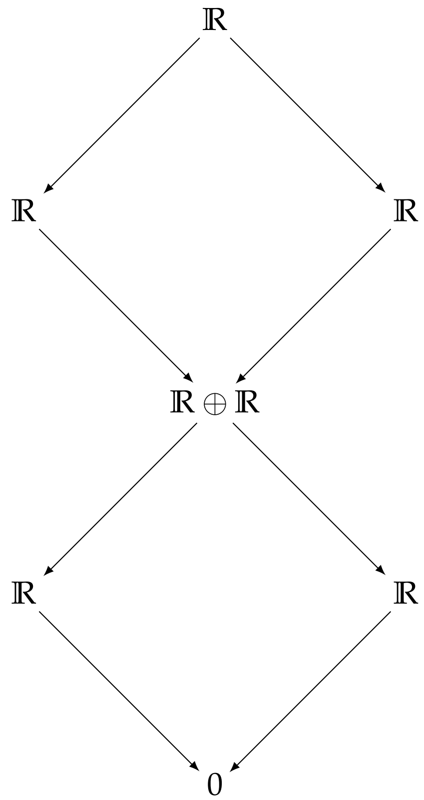
$$T = \{\emptyset, S^1, S^1 - \{1\}, S^1 - \{-1\}, S^1 - \{\pm 1\}, S^1_+ = S^1 \cap \mathbb{H}^+, S^1_- = S^1 \cap \mathbb{H}^-\}, \quad (186)$$

where $H^+ = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ and $H^- = \{z \in \mathbb{C} : \text{Im}(z) < 0\}$.

We get the following 'irreducible' inclusions.




We get a presheaf by reversing the arrows and going to abelian groups. We again take the constant sheaf $\mathbb{R}$.

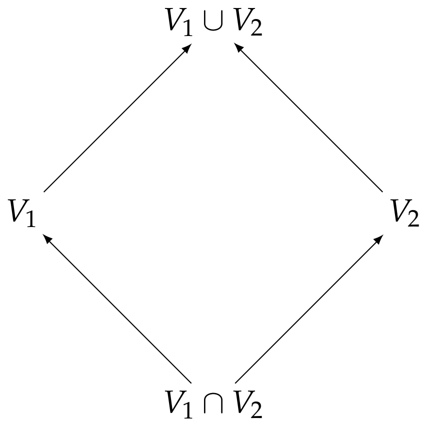



In the above commutative diagram, $a \mapsto (a, a) : \mathbb{R} \to \mathbb{R} \oplus \mathbb{R}$ and $(a, b) \mapsto a + b : \mathbb{R} \oplus \mathbb{R} \to \mathbb{R}$.

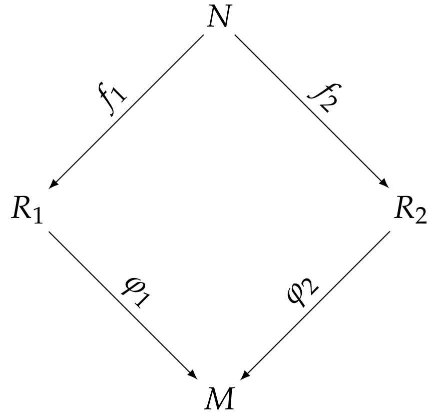
What does an arbitrary sheaf on $S^1$ look like? The following is a criterion for a presheaf to be a sheaf.

**Lemma 83** (Yisy lemma)**:** *Let $F$ be a presheaf on a topological space $X$.*

*(a) Let $V_1, V_2$ be open subsets of $X$ such that the following is a commutative diagram of inclusions that does not factor.*



*Applying F gives*



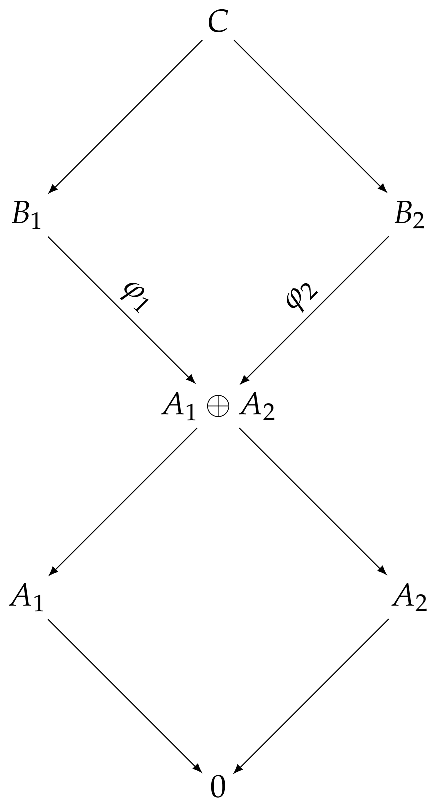

*Then $(N, f_1, f_2)$ is isomorphic to $\{(v_1, v_2) \in R_1 \oplus R_2 : \varphi_1(v_1) = \varphi_2(v_2)\}$ with projection maps.*

(b) *Assume that X is compact and that the above holds everywhere. Then F is a sheaf.*

*Proof.* Think of the assigned abelian groups as functions. The sheaf axioms say that functions that agree on overlaps glue to a unique function. The union corresponds to $N$, the overlap to $M$, and the agreement is the commutativity of the diagram. □

**Eg3**

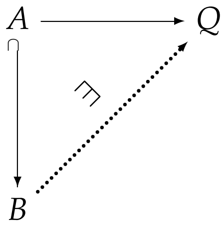
We learn that an arbitrary sheaf on $S^1$ looks like this.



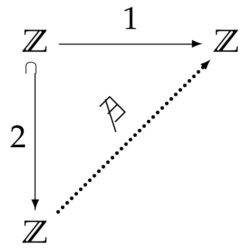

Here $C = \{(b_1, b_2) \in B_1 \oplus B_2 : \varphi_1(b_1) = \varphi(b_2)\}$.

Now let's talk sheaf cohomology.

Recall that an object $Q$ is *injective* if morphisms to $Q$ can be extended. More precisely, given $A \hookrightarrow B$ and $A \to Q$, there exists $B \to Q$ making the diagram commute.




In the category of abelian groups, injective objects look like quotients of $\mathbb{Q}$-vector spaces. Note that $\mathbb{Z}$ is not injective, as seen from the following diagram.




Prüfer groups are also injective. For a prime $p$, the Prüfer $p$-group is the Sylow $p$-subgroup of $\frac{\mathbb{Q}}{\mathbb{Z}}$,

$$\mathbb{Z}[p^\infty] = \frac{\mathbb{Z}[1/p]}{\mathbb{Z}}. \tag{187}$$

It has presentation

$$\mathbb{Z}[p^\infty] = \langle x_1, x_2, x_3, \ldots \mid x_1^p = 1, x_2^p = x_1, x_3^p = x_2, \ldots \rangle. \tag{188}$$

So how to we compute sheaf cohomology?

1. Get a sheaf $F$ on $X$.
2. Find an injective resolution
$$0 \to F \to I^0 \to I^1 \to \ldots$$
3. Apply global sections ($\Gamma$), which is evaluation (what does $X$ map to?), to get a cochain complex of abelian groups.
4. Apply the right derived functor. In other words, replace the head with a $\{0\}$ and compute cohomology. As the global sections functor is left-exact, we would never get any nonzero cohomology from the head anyway.

**Eg1**

Cohomology of a point.

1. Sheaf: $\{*\} \mapsto \mathbb{R}$.
2. Injective resolution:
$$(\{*\} \mapsto \{0\}) \to (\{*\} \mapsto \mathbb{R}) \to (\{*\} \mapsto \mathbb{R}) \to (\{*\} \mapsto \{0\}). \tag{189}$$

3. Global sections: $0 \to \mathbb{R} \to \mathbb{R} \to 0$.
4. Replace the head with $\{0\}$:

$$0 \to \mathbb{R} \to 0. \tag{190}$$

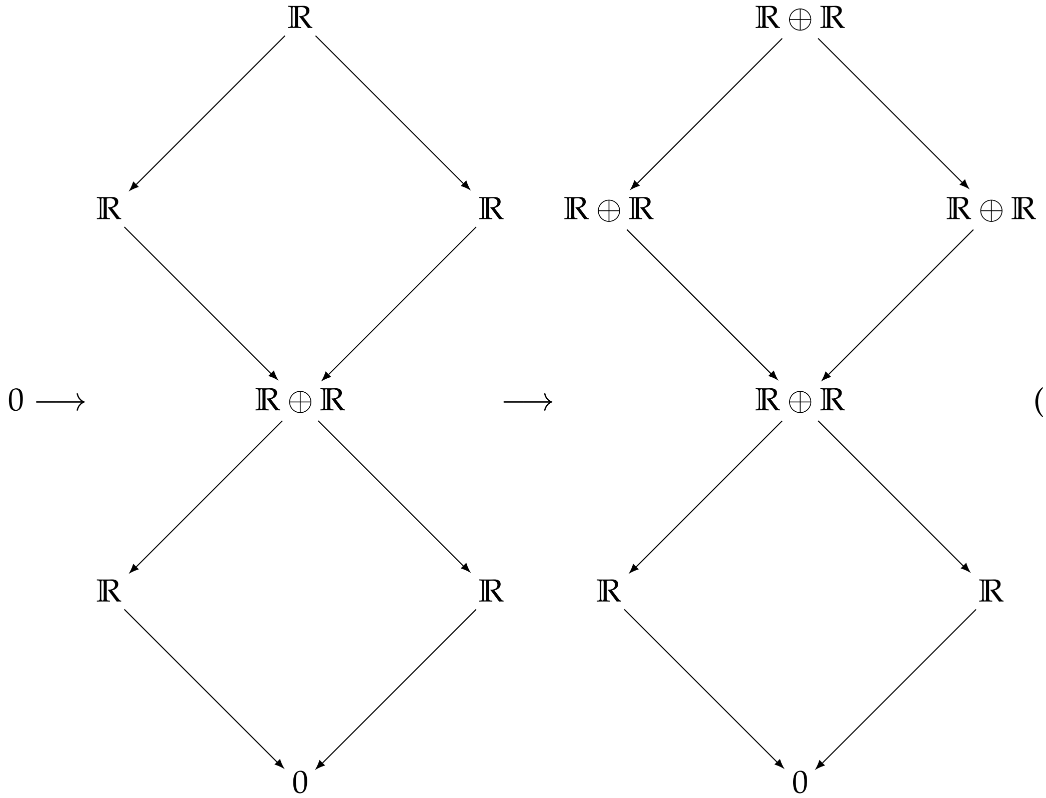
This gives

$$H^i(\{*\}, \mathbb{R}) = \begin{cases} \mathbb{R}, & \text{if } i = 0, \\ 0, & \text{if } i \neq 0. \end{cases} \tag{191}$$

**Eg2**

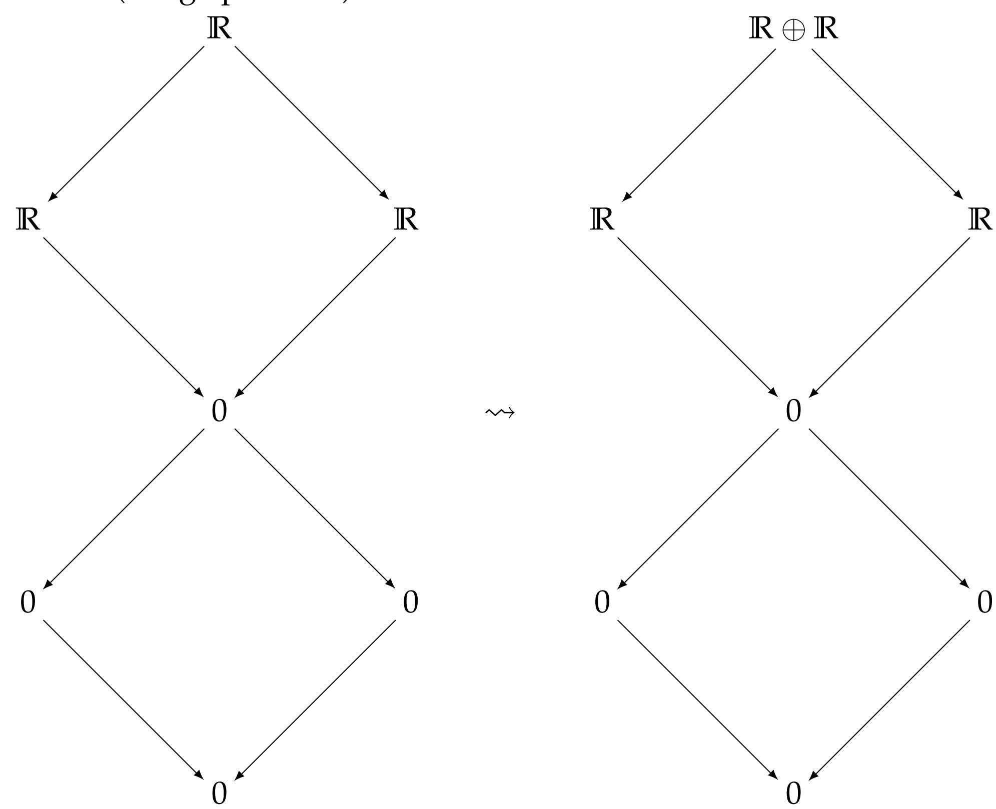
Cohomology of $S^1$ with topology $T$.

1. Sheaf: constant sheaf $\mathbb{R}$.
2. Injective resolution. Intuitively, for an injective sheaf, the dimension doesn't get smaller as you go up. The injective resolution begins with




(192)

Then sheafify the cokernel presheaf, since the image sheaf in the category of sheaves is (image presheaf)$^{sh}$:




(193)

It was clear from the Yisy lemma what the answer would be, but how did the sheafification actually work? From the presheaf, we get the stalk being $\mathbb{R}$ at $\pm 1$ and $\{0\}$ everywhere else. What does the stalk do intuitively? Functions are the same if they're the same in some small neighbourhood. For an open set $V$, we therefore get $\mathbb{R}$ worth of functions for each point in $V \cap \{\pm 1\}$.

3. Global sections: $0 \to \mathbb{R} \to \mathbb{R}^2 \to \mathbb{R}^2 \to 0$. The map $\mathbb{R}^2 \to \mathbb{R}^2$ factors through $\mathbb{R}$ before sheafification, so its image is one-dimensional, so its kernal is also isomorphic to $\mathbb{R}$.

4. Replace the head with $\{0\}$:

$$0 \to \mathbb{R}^2 \xrightarrow{\ker = \mathbb{R}} \mathbb{R}^2 \to 0. \tag{194}$$

This gives

$$H^i(\{*\}, \mathbb{R}) = \begin{cases} \mathbb{R}, & \text{if } i = 0, 1, \\ 0, & \text{otherwise.} \end{cases} \tag{195}$$

**Étale cohomology**

In Zariski topology, the open set $S^1 - \{\pm 1\}$ (this is a scheme with $S^1 = \mathbb{C}P^1$) cannot be split into $S^1_+$ and $S^1_-$, so we would get trivial cohomology. In étale topology, we still can't add these sets to the topology, so what do we do?

Regard $S^1$ as

$$\frac{\mathbb{C} - \{0\}}{\operatorname{Spec}(\mathbb{C}[t, t^{-1}])}.$$

The denominator is $\mathbb{C}^\times$, as a set at least. We can interpret the "polynomials" more geometrically:

$$\mathbb{C}[t, t^{-1}] = \frac{\mathbb{C}[x, y]}{xy - 1}.$$

Let $V = (\mathrm{Spec}\frac{\mathbb{C}[x,y]}{xy-1}, \mathcal{O})$. Consider the following family of morphisms.

$$\begin{aligned} \varphi_i : V &\to V \\ x &\mapsto x^i \\ y &\mapsto y^i. \end{aligned}$$

To be continued (Yi: although this is continued at some point, I never did successfully manage to do this computation).

**Alex Ghitza** *Thursday 7 June 2012*

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Previously, in [3], Deligne showed that the Weil conjectures implied the Ramanujan-Petersson conjecture:

**Theorem 84** (Theorem 8.2, [3])**:** *Let $N \geq 1$ and $k \geq 2$ be integers. Let $f \in S_k^{new}(\Gamma_0(N))$ be an anemic Hecke eigenform with Fourier expansion $(a_n)$. Then*

$$|a_p| \leq 2p^{(k-1)/2}$$

*for primes $p$ that do not divide $N$.*

We aim to show that the roots of $X^2 - a_p X + p^{k-1}$ have absolute value $p^{(k-1)/2}$. The strategy is to find a variety $X$ over $\mathbb{F}_p$ such that these roots are eigenvalues of the Frobenius acting on $H^{k-1}(X, \mathbb{Q}_l)$ (then use the Riemann hypothesis).

**Modular forms - classical definition**

Let $\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. Let $N \in \mathbb{Z}_{>0}$ and $k \in \mathbb{Z}$. A *modular form* is a holomorphism $f : \mathcal{H} \to \mathbb{C}$ such that

- If $z \in \mathcal{H}$ and
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\}$$
then
$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z), \tag{196}$$
and
- a particular growth condition is satisfied as $z \to i\infty$.

How do we motivate this definition?

Suppose firstly that $k = 0$. Then equation (196) becomes

$$f\left(\frac{az+b}{cz+d}\right) = f(z). \tag{197}$$

In other words, $f$ is invariant under $z \mapsto \frac{az+b}{cz+d}$, so it gives an action of $\Gamma_0(N)$ on $\mathcal{H}$. In particular, $f$ defines a function

$$\frac{\mathcal{H}}{\Gamma_0(N)} \to \mathbb{C}. \tag{198}$$

For larger $k$, roughly speaking we get higher differential forms. The growth condition allows us to compactify $\frac{\mathcal{H}}{\Gamma_0(N)}$ without losing the function in (198).

**Elliptic curves**

Let $K$ be a field. An *elliptic curve* over $K$ is a smooth curve of genus 1 with a chosen $K$-rational point.[86] Any elliptic curve has an abelian group structure, where the chosen rational point is the identity.

If $K = \mathbb{C}$, there is a geometrization theorem that tells us that every such curve takes the form

$$E = \frac{\mathbb{C}}{\Lambda}, \tag{199}$$

where $\Lambda$ is a full $\mathbb{Z}$-lattice in $\mathbb{C} = \mathbb{R}^2$. More explicitly, this means that there exist $\mathbb{R}$-independent $\omega_1$ and $\omega_2$ such that

$$\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2. \tag{200}$$

We see that such vectors describe $E$ as a parallelogram with opposite sides identified, i.e. a torus. A notion of isomorphism can be defined on elliptic curves over $\mathbb{C}$. Up to isomorphism, we can take

$$\Gamma = \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot \tau. \tag{201}$$

More generally, there is a bijection

$$\{\text{isomorphism classes of } E/\mathbb{C}\} \leftrightarrow \frac{\mathcal{H}}{\Gamma_0(1)}. \tag{202}$$

Some elliptic curves have nontrivial automorphisms, for instance multiplication by $i$ in

$$E = \frac{\mathbb{C}}{\mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot i}.$$

This prevents $E$ from being a moduli space. Here are three ways in which we can introduce a level structure on $E$:

1. Fix a cyclic subgroup of order $N$ in $E$.
2. Fix a point of order $N$ in $E$.
3. Fix an isomorphism

   $$\alpha : E[N] \xrightarrow{\cong} \left(\frac{\mathbb{Z}}{n\mathbb{Z}}\right)^2,$$

   where $E[N] = \{P \in E : NP = 0\}$.

[86] For instance, $\frac{\mathbb{R}[x,y]}{x^2+y^2+1}$ has no $\mathbb{R}$-rational points.

These give a bijection each:

$$\{\text{isomorphism classes of } (E,C)\} \leftrightarrow \frac{\mathcal{H}}{\Gamma_0(N)}$$
$$\{\text{isomorphism classes of } (E,P)\} \leftrightarrow \frac{\mathcal{H}}{\Gamma_1(N)}$$
$$\{\text{isomorphism classes of } (E,\alpha)\} \leftrightarrow \frac{\mathcal{H}}{\Gamma(N)}.$$

This corresponds to the hierarchy $\Gamma_0(N) \supseteq \Gamma_1(N) \supseteq \Gamma(N)$, where

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N \right\},$$
$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\}, \text{ and}$$
$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}.$$

At this stage, the reader is encouraged to reflect momentarily on this hierarchy. It implies that there are fewer isomorphism classes of $(E,C)$ than isomorphisms of $(E,P)$ or $(E,\alpha)$. Indeed, given an abelian group, there are at least as many elements of order $N$ as there are cyclic subgroups of order $N$ (since each cyclic subgroup of order $N$ corresponds to at least one generator).

Note that we are indeed asserting that the group of $N$-torsion points in $E$ is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^2$. For elliptic curves over $\mathbb{C}$, the $N$-torsion points are those of the form $a + b\tau$, where $a, b \in \mathbb{Z}/N\mathbb{Z}$.

If $N \geq 5$, then there are no nontrivial automorphisms of $(E,P)$, $(E,C)$, or $(E,\alpha)$. In this case, for elliptic curves over $\mathbb{C}$ with some extra structure, we get a moduli space

$$\frac{\mathcal{H}}{\Gamma_0(N)}.$$

Given a $\mathbb{Z}$-scheme and a ring $R$, an $R$-scheme is induced by the action of $\mathbb{Z}$ on $R$. It is the pullback / fibred product:

$$\begin{array}{ccc} S \otimes_{\mathbb{Z}} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathrm{Spec}(R) & \longrightarrow & \mathrm{Spec}(\mathbb{Z}). \end{array}$$

If $S$ is an elliptic curve $S = E : y^2 = x^3 + ax + b$ and $R = \mathbb{F}_p = \frac{\mathbb{Z}}{p\mathbb{Z}}$ for some prime number $p$, then this is reduction modulo $p$. We get $E_p : y^2 = x^3 + \bar{a}x + \bar{b}$ with $\bar{a}, \bar{b} \in \mathbb{F}_p$.[87]

There is a moduli space $Y_0(N)$ of elliptic curves (with extra structure depending on $N$)

[87] There is a subtlety here, in that we need to see if the non-singularity condition is preserved. This condition is equivalent to the cubic in $x$ having no repeated roots. It is easy to construct examples in which it is not preserved, for instance reduce $x^3 + 3x + 3$ modulo 3, or take any similar Eisenstein-type example.

over $\mathbb{Z}[1/N]$. It is a smooth curve over $\mathbb{Z}[1/N]$. Moreover, there is a compactification $X_0(N)$ such that $Y_0(N)$ is a dense open subset of $X_0(N)$. Note that we're using the Zariski topology pretty much exclusively. The fact that $Y_0(N)$ is a moduli space implies the existence of the *universal elliptic curve*. This is a covering

$$\begin{array}{c} \varepsilon \\ \Big\downarrow{\scriptstyle \pi} \\ Y_0(N) \end{array}$$

such that the fibres are elliptic curves. Note that

$$\frac{\mathcal{H}}{\Gamma_0(N)} = Y_0(N)(\mathbb{C}) = \{\sigma : \mathrm{Spec}(\mathbb{C}) \to Y_0(N)\}. \tag{203}$$

Given $\sigma$ corresponding to $\tau \in \mathcal{H}$, we get an elliptic curve

$$\varepsilon \times_{Y_0(N)} \mathrm{Spec}(\mathbb{C}) = \frac{\mathbb{C}}{\mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot \tau} \tag{204}$$

from the following diagram:

$$\begin{array}{ccc} \varepsilon \times_{Y_0(N)} \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \varepsilon \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \pi} \\ \mathrm{Spec}(\mathbb{C}) & \xrightarrow{\ \sigma\ } & Y_0(N). \end{array}$$

In fact all elliptic curves arise in this fashion (choose $\sigma$), so we can regard $\varepsilon$ as the family of elliptic curves. Consider the sheaf of relative differential 1-forms on $\pi : \varepsilon \to Y_0(N)$,

$$\Omega^1_{\varepsilon/Y_0(N)}.$$

This is a line bundle on $\varepsilon$, i.e. an invertible (locally free[88]) sheaf on $\varepsilon$ with one-dimensional fibres. Now

$$\underline{\omega} = \pi_* \Omega^1_{\varepsilon/Y_0(N)}$$

is a line bundle on $Y_0(N)$. Extend $\underline{\omega}$ to $X_0(N)$:

$$\begin{array}{c} \underline{\omega} \\ \Big\downarrow \\ X_0(N) \end{array}$$

Note that $\underline{\omega}^{\otimes k}$ is a rank $k$ vector bundle over $\mathbb{Z}[1/N]$, so its elements are $(x, \alpha)$ in local coordinates, where $x \in X_0(N)$ and $\alpha \in \mathbb{Z}[1/N]^k$. With sheaf cohomology,

$$M_k(N) = H^0(X_0(N), \underline{\omega}^{\otimes k}) \tag{205}$$

[88] Let $\mathcal{L}$ be a sheaf on $X$. Cover $X$ with affine opens $\mathrm{Spec}(A)$. If we can do this in such a way that $\mathcal{L}_{\mathrm{Spec}(A)}$ is a free $A$-module for all $A$ then $\mathcal{L}$ is *locally free*.

is a $\mathbb{Z}[1/N]$-module. Now
$$M_k(N, \mathbb{C}) = H^0(X_0(N) \otimes_{\mathbb{Z}[1/n]} \mathbb{C}, \underline{\omega}^{\otimes k}) \tag{206}$$
turns out to be the complex vector space of weight $k$ modular forms of level $\Gamma_0(N)$.

An $f \in M_k(N)$ is a rule that assigns to $(E, C)$ a fibre $f(E, C)$,
$$\begin{array}{ccc} f(E,C) & & \underline{\omega}^{\otimes k} \\ \uparrow & & \downarrow \\ & (E,C) \in X_0(N). & \end{array}$$
i.e. $f$ is a section of $\underline{\omega}^{\otimes k} \to X_0(N)$, so $f : X_0(N) \to \underline{\omega}^{\otimes k}$. In other words, if $\eta$ is a differential $k$-form on $E$, then[89]
$$(E, C, \eta) \rightsquigarrow f(E, C, \eta) \in \mathbb{Z}[1/N], \tag{207}$$
such that
$$f(E, C, \lambda\eta) = \lambda^{-k} f(E, C, \eta), \qquad \lambda \neq 0. \tag{208}$$
The point is that we can evaluate a modular form at an elliptic curve.

**Hecke operators**

Let $p$ be a prime that does not divide $N$, and let $E[p]$ be the $p$-torsion subgroup of $E$. Then
$$E[p] \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^2.$$
It is easy to see that $\left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^2$ has $p+1$ subgroups of order $p$, since there are $p^2 - 1$ elements of order $p$. Define $T_p : M_k(N) \to M_k(N)$ by
$$(T_p f)(E, P, \eta) = \sum_{C \leq E, |C| = p} f(E', P', \eta'), \tag{209}$$
where $E' = E/C$ and $P', N'$ come from $E \to E'$.[90]

These operators commute: if $p$ and $l$ are prime then $T_p \circ T_l = T_l \circ T_p$. A modular form $f \in M_k(\Gamma_0(N))$ is an *anemic Hecke eigenform* if it is an eigenvector for all primes $p \not| N$.

**$q$-expansions (Fourier expansions)**

Let $f \in M_k(N, \mathbb{C})$. As $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)$,
$$f(z+1) = f(z), \qquad z \in \mathcal{H}. \tag{210}$$
This horizontal periodicity gives $f$ a Fourier expansion on any horizontal line in the

[89] Given $x \in X_0(N)$, we get $\eta_x : T_x^k \to \mathbb{Z}[1/N]$.
[90] Any group of prime order is obviously cyclic.

complex plane, and these all have to agree with $f$ on $\mathbb{C}$.[91] Now

$$f(z) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n z}. \tag{211}$$

The growth condition as $z \to i\infty$ implies that $a_n = 0$ for $n < 0$. As a function of the *nome* $q = e^{2\pi i z}$,

$$f(z) = \sum_{n=0}^{\infty} a_n q^n. \tag{212}$$

A modular form $f$ is a *cuspform* (or $f$ is *cuspidal*) if $a_0 = 0$. The cuspforms form a complex subspace $S_k(N, \mathbb{C})$. One can check that if $f$ is a cuspidal Hecke eigenform then $a_1 \neq 0$. Consequently, we can *normalise* $f$, i.e. multiply it by a constant such that $a_1 = 1$. Then

$$T_p f = a_p \cdot f. \tag{213}$$

In words, the Fourier coefficients are precisely the eigenvalues! In the case of normalised Hecke eigenforms, it can be shown that the $a_p$ are algebraic integers.

**Newforms and oldforms**

From the definition, if $N|M$ then $\Gamma_0(N) \supseteq \Gamma_0(M)$. We get a covering of Riemann surfaces over $\mathbb{C}$:

$$\begin{array}{c} Y_0(M) = \dfrac{\mathcal{H}}{\Gamma_0(M)} \\ \Big\downarrow \pi \\ Y_0(N) = \dfrac{\mathcal{H}}{\Gamma_0(N)}. \end{array}$$

Moreover, the deck transformation group is the quotient $\Gamma_0(N)/\Gamma_0(M)$. In particular, given $p$ prime and $N \in \mathbb{Z}_{>0}$, there are embeddings

$$i_p : M_k(\Gamma_0(N), \mathbb{C}) \hookrightarrow M_k(\Gamma_0(Np), \mathbb{C})$$

and

$$j_p : M_k(\Gamma_0(N), \mathbb{C}) \hookrightarrow M_k(\Gamma_0(Np), \mathbb{C}).$$

At level $N$, the direct sum of the images of $i_p$ and $j_p$, over all primes $p|N$, is called *oldspace*, and is denoted $M_k^{old}(N)$. *Newspace* is the orthogonal complement of oldspace with respect to the Petersson inner product:

$$S_k^{new} \oplus S_k^{old} = S_k. \tag{214}$$

This is just motivation, and we will come back to this. Intuitively, newforms are the modular forms that really belong at that level. This notion is particularly useful when analysing the growth of coefficients as the level grows.

**James Withers** *Thursday 14 June 2012*

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We recall what we said at the start of last time, about how Deligne used the Weil conjectures to prove the Ramanujan-Petersson conjecture.

[91] Any one of them is equal to the function $f$ on a set with a limit point, and $\mathbb{C}$ is connected.

**Theorem 85:** *Let $f \in S_k(\Gamma_0(N), \mathbb{C})$, with $k \geq 2$, and suppose that $T_p f = a_p f$ for primes $p$ that do not divide $N$. Then*

$$|a_p| \leq 2p^{\frac{k-1}{2}}$$

*for all such $p$.*

We aim to show that the roots of $X^2 - a_p X + p^{k-1}$ have absolute value $p^{(k-1)/2}$. The strategy is to find a variety $X$ over $\mathbb{F}_p$ such that these roots are eigenvalues of the Frobenius acting on $H^{k-1}(X, \mathbb{Q}_l)$ (then use the Riemann hypothesis).

Recall theorem 49:

**Theorem 86:** *Let $X_0$ be a smooth projective variety over $\mathbb{F}_q$, and let $p > 0$ be the characteristic of $\mathbb{F}_q$. Let $X = X_0 \times_{\mathbb{F}_p} \overline{\mathbb{F}_p}$ (extension of scalars). For each $i$, the characteristic polynomial $\det(t - F^*, H^i(X, \mathbb{Q}_l)) \in \mathbb{Z}[t]$ has coefficients independent of $l$ (here $l \neq p$). Moreover, the conjugates $\alpha$ of a roots[92] of this polynomial have absolute value $|\alpha| = q^{i/2}$.*

We specialize $q = p$, and fix $(X, X_0)$.

For any sheaf $\mathcal{F}$ on $X$, let $\tilde{H}^i(X, \mathcal{F})$ denote the image of $H^i_c(X, \mathcal{F}) \to H^i(X, \mathcal{F})$. Note that

$$H^i_c(X, \mathbb{Q}_l) = H^i(Y, j_! \mathbb{Q}_l), \tag{215}$$

where $j : X \hookrightarrow Y$ is an open immersion into a compact $\mathbb{F}_p$-scheme.[93]

For each prime number $l$, let $K_l$ be the largest extension of $\mathbb{Q}$ that is unramified away from $l$ and, for $p \neq l$, let $F_p$ denote the inverse, in $\mathrm{Gal}(K_l/\mathbb{Q})$, of the relative Frobenius $\varphi_p$ at $p$. This is well defined up to inner automorphism.

In the context of the Ramanujan-Petersson conjecture, let

$${}^{k-2}_{N}W_l = \tilde{H}^1(Y(N) \otimes \mathbb{Q}, \mathrm{Sym}^{k-2}(R^1 f_{N^*}(\mathbb{Q}_l))),$$

where $Y(N)$ is the moduli space corresponding to $\mathcal{H}/\Gamma_0(N)$.[94] We have

$$\begin{array}{c} E \\ \Big\downarrow f_N \\ Y(N) \otimes \mathbb{Q}, \end{array}$$

where $E$ is the universal elliptic curve.[95] Note that $\mathrm{Gal}(K_l/\mathbb{Q})$ acts on ${}^{k-2}_{N}W_l$.

Let $f : X \to Y$, and let $\mathcal{F}$ be a sheaf on $X$. Then $R^i f_*$ is the sheaf on $Y$ associated to the presheaf

$$U \mapsto H^i(f^{-1}(U), \mathcal{F}|_{f^{-1}(U)}).$$

[92] These are precisely the eigenvalues of $F^*$ acting on $H^i(X, \mathbb{Q}_l)$.

[93] For étale sheaves, this is well defined insofar as it does not depend on the chosen compactification.

[94] This was denoted $Y_0(N)$ in the previous talk.

[95] This was denoted $\varepsilon$ in the previous talk.

If $f : X \to \mathrm{Spec}(A)$ is quasi-coherent,[96] then $R^i f_*(\mathcal{F}) \simeq H^i(X, \mathcal{F})$.

We aim to show the following theorem:

**Theorem 87:** *The eigenvalues of $F$ acting on ${}^{k-2}_{N}W_l$ are algebraic integers with absolute value $p^{\frac{k-1}{2}}$.*

We show this using a series of lemmata.

**Lemma 88:** *Let $X_0$ be a smooth open subscheme of a scheme $X_0^*$ over $\mathbb{F}_p$, and let $X = X_0 \times_{\mathbb{F}_p} \overline{\mathbb{F}_p}$. Then the eigenvalues of $F^*$ acting on $\tilde{H}^i(X, \mathbb{Q}_l)$ are algebraic integers with absolute value $p^{i/2}$.*

*Proof.* The map $H^i_c(X, \mathbb{Q}_l) \to H^i(X, \mathbb{Q}_l)$ factors through

$$\begin{array}{ccc} H^i_c(X,\mathbb{Q}_l) & \xrightarrow{\quad\iota\quad} & H^i(X,\mathbb{Q}_l) \\ & {\scriptstyle j^*}\searrow \quad \nearrow{\scriptstyle \pi} & \\ & H^i(X^*,\mathbb{Q}_l). & \end{array} \tag{216}$$

The image of $\iota$ is $\pi(\mathrm{Im} j^*)$, so the first isomorphism theorem yields

$$\tilde{H}^i(X, \mathbb{Q}_l) = \frac{\mathrm{Im} j^*}{\ker \pi}, \tag{217}$$

which is a quotient space of the $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$-module $H^i(X^*, \mathbb{Q}_l)$, so the eigenvalues of $F^* \curvearrowright \tilde{H}^i(X, \mathbb{Q}_l)$ are a subset of the eigenvalues of $F^* \curvearrowright H^i(X^*, \mathbb{Q}_l)$. The result now follows from theorem 49. □

**Lemma 89:** *Let $S_0$ be a smooth scheme over $\mathbb{F}_p$, and let $S = S_0 \times_{\mathbb{F}_p} \overline{\mathbb{F}_p}$. Let $A_0 \xrightarrow{f} S_0$ be an abelian scheme (fibres are abelian varieties) over $S_0$, where $A_0$ is an open subscheme of a smooth projective scheme $A_0^*$. Then the eigenvalues of*

$$F^* \curvearrowright \tilde{H}^i(S, R^j f_* \underline{\mathbb{Q}_l})$$

*are algebraic integers $\alpha$ with $|\alpha| = p^{\frac{i+j}{2}}$.*

The proof uses Leray spectral sequences. Fix $r_0 \in \mathbb{Z}_{\geq 0}$, and consider $r \geq r_0$. A *spectral sequence* is a bigraded object

$$E_r = \oplus_{i,j} E_r^{i,j} \tag{218}$$

with arrows $d_r : E_r \to E_r$ such that $d_r \circ d_r = 0$,

$$d_r(E_r^{i,j}) \subseteq E_r^{i+r, j-r+1} \tag{219}$$

and $H(E_r) \simeq E_{r+1}$, i.e.

$$E_{r+1}^{i,j} = \frac{\ker(d_r)}{\mathrm{Im}(d_r)}. \tag{220}$$

[96] ?

Given $p, q$, the group $E_r^{p,q}$ eventually stabilizes, since there are only a finite number of nonzero arrows, so let $E_\infty^{p,q}$ denote the final group. The spectral sequence *converges to groups $H^n$*. written

$$E_r^{p,q} \implies H^{p+q}, \tag{221}$$

if there exists a filtration

$$0 = H_{n+1}^n \subseteq H_n^n \subseteq \ldots \subseteq H_1^n \subseteq H_0^n = H^n \tag{222}$$

such that

$$E_\infty^{p,n-p} \simeq \frac{H_p^n}{H_{p+1}^n} \tag{223}$$

for all $n, p$.

*Proof of 89.* We have Leray spectral sequences

$$E : E_2^{i,j} = H^i(S, R^j f_* \mathbb{Q}_l) \implies H^{i+j}(A, \mathbb{Q}_l)$$

and

$${}_cE :_c E_2^{i,j} = H_c^i(S, R^j f_* \mathbb{Q}_l) \implies H_c^{i+j}(A, \mathbb{Q}_l).$$

Let $m \in \mathbb{Z}_{>0}$. Then the endomorphism

$$\begin{aligned} \psi_m : A &\to A \\ a &\mapsto ma \end{aligned}$$

induces maps $\psi_m^*$ such that

$$\begin{array}{ccc} {}_cE & \longrightarrow & E \\ \Big\downarrow \psi_m^* & & \Big\downarrow \psi_m^* \\ {}_cE & \longrightarrow & E. \end{array} \tag{224}$$

The action of $\psi_m^*$ on $R^j f_* \mathbb{Q}_l$ is multiplication by $m^j$, so this is also the action on the ${}_cE_r^{i,j}$ and the $E_r^{i,j}$. The arrows $d_r$ (for $r \geq 2$) commute with $\psi_m^*$, and take $E_r^{i,j}$ to $E_r^{i',j'}$ with $j' \neq j$:

$$\begin{array}{ccc} E_r^{i,j} & \xrightarrow{d_r} & E_r^{i+r, j'=j-r+1} \\ \Big\downarrow \times m^j & & \Big\downarrow \times m^{j'} \\ E_r^{i,j} & \xrightarrow{d_r} & E_r^{i+r, j'=j-r+1}. \end{array} \tag{225}$$

This implies that $d_r = 0$ for $r \geq 2$, so

$$E_2^{i,j} = E_\infty^{i,j} = \frac{H_i^{i+j}}{H_{i+1}^{i+j}}. \tag{226}$$

As $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules, $H^i(S, R^j f_* \mathbb{Q}_l)$ is therefore a quotient of a subspace of $H^{i+j}(A, \mathbb{Q}_l)$, so the result follows from 88. □

Let $f_N : E \to Y(N) \otimes \mathbb{F}_p$ be the universal elliptic curve modulo $p$, and let

$$f_{N,k-2} : E_{k-2} \to Y(N) \otimes \mathbb{F}_p$$

be the $(k-2)$-fold fibre product gotten by iterating

$$\begin{array}{ccc} E_2 = E \times_{Y(N)\otimes \mathbb{F}_p} E & \longrightarrow & E \\ \downarrow & \overset{f_{N2}}{\searrow} & \downarrow f_N \\ E & \xrightarrow{f_N} & Y(N) \otimes \mathbb{F}_p. \end{array} \tag{227}$$

**Lemma 90:** *There exists a smooth projective scheme $E^*_{k-2}$ that has $E_{k-2}$ as an open subscheme.*

*Proof.* Get $E^*_{k-2}$ from $E_{k-2}$ by resolving its singularities. □

Now we prove theorem 87.

*Proof.* There are inclusions

$$\mathrm{Sym}^{k-2}(R^1 f_{N^*} \mathbb{Q}_l) \hookrightarrow (R^1 f_{N^*} \mathbb{Q}_l)^{\otimes(k-2)} \hookrightarrow R^{k-2} f_{(N,k-2)^*} \mathbb{Q}_l, \tag{228}$$

where the latter follows from the Künneth formula. The result now follows from the previous two lemmata. □

To be continued. The end is near.

**Arun Ram** *Thursday 21 June 2012*

Let's have a broad discussion of Deligne's Weil I ([1]).

The main theorem of the paper is theorem (1.6). In section 1, it is shown that theorem (1.6) follows from lemma (1.7), which is below.

**Lemma 91** (1.7)**:** *For each $i$ and each prime $l \neq p$, the eigenvalues of the endomorphism $F^*$ on $H^i(X, \mathbb{Q}_l)$ are algebraic numbers, all of whose $\mathbb{C}$-conjugates $\alpha$ have absolute value $|\alpha| = q^{i/2}$.*

The rest of the paper is spent proving this lemma, so let's look at the end to see how it all fits together.

**Section 7. End of the proof of (1.7)**

**Lemma 92** (7.1)**:** *Let $X_0$ be a nonsingular, absolutely irreducible projective variety of even dimension $d$ over $\mathbb{F}_q$. Let $X$ over $\overline{\mathbb{F}_q}$ be obtained from $X_0$ by extension of scalars. Then the eigenvalues of*

$$F^* \curvearrowright H^d(X, \mathbb{Q}_l)$$

*are algebraic numbers whose* $\mathbb{C}$*-conjugates* $\alpha$ *satisfy*

$$q^{\frac{d}{2}-\frac{1}{2}} \le |\alpha| \le q^{\frac{d}{2}+\frac{1}{2}}.$$

**Lemma 93** (7.2)**:** *Let* $X_0$ *be an absolutely irreducible projective varitey of dimension* $d$ *over* $\mathbb{F}_q$. *Let* $X$ *over* $\overline{\mathbb{F}_q}$ *be obtained from* $X_0$ *by extension of scalars. Then the eigenvalues of*

$$F^* \curvearrowright H^d(X, \mathbb{Q}_l)$$

*are algebraic numbers whose* $\mathbb{C}$*-conjugates* $\alpha$ *have absolute value*

$$|\alpha| = q^{\frac{d}{2}}.$$

*Proof.* Let $k$ be an even integer. By the Künneth formula, $\alpha^k$ is an eigenvalue of

$$F^* \curvearrowright H^{kd}(H^k, \mathbb{Q}_l).$$

By lemma 92,

$$q^{\frac{kd}{2}-\frac{1}{2}} \le |\alpha^k| \le q^{\frac{kd}{2}+\frac{1}{2}}, \tag{229}$$

so

$$q^{\frac{d}{2}-\frac{1}{2k}} \le |\alpha| \le q^{\frac{d}{2}+\frac{1}{2k}}. \tag{230}$$

The sandwich rule now completes the proof (let $k \to \infty$). □

In fact lemma (1.7) follows from lemma (7.2); we just need to work out how to get from the top-level homology to all of the levels below. We'll talk about that another time, but (7.1) is very interesting because it looks like Section 3: *The fundamental bound*.

**Corollary 94** (3.8)**:** *The eigenvalues of*

$$F^* \curvearrowright H^1_c(U, \mathcal{F})$$

*are algebraic numbers, all of whose* $\mathbb{C}$*-conjugates* $\alpha$ *satisfy*

$$|\alpha| \le q^{\frac{\beta+1}{2}+\frac{1}{2}}.$$

The context is that $U$ is a nice curve, where the subscript 0 indicates the finite field version. Chapters 4-6 use Lefschetz theory to jack up the curve result to a higher-dimensional one.

Let's briefly recall some key aspects of Section 3. The above is a corollary of theorem (3.2), which concludes that $\mathcal{F}_0$ is of weight $\beta$.

### 3.1

Let $\beta \in \mathbb{Q}$. We say that $\mathcal{F}_0$ is *weight* $\beta$ if for all $x \in |U_0|$ the eigenvalues of

$$F_x \curvearrowright \mathcal{F}_0$$

are algebraic numbers, all of whose $\mathbb{C}$-conjugates $\alpha$ satisfy

$$|\alpha| = q_x^{\beta/2}.$$

**Theorem 95** (3.2)**:** *Make the following assumptions:*

*(i) $\mathcal{F}$ has a nondegenerate alternating bilinear form*

$$\mathcal{F}_0 \otimes \mathcal{F}_0 \to \mathbb{Q}_l(-\beta).$$

*(ii) The image of $\pi_1(U, u)$ in $GL(\mathcal{F}_u)$ is an open subgroup of the symplectic group $Sp(\mathcal{F}_u, \psi_u)$.*

*(iii) For all $x \in |U_0|$, the polynomial $\det(1 - F_x t, \mathcal{F}_0)$ has coefficients in $\mathbb{Q}$.*

*Then $\mathcal{F}_0$ has weight $\beta$.*

Here open implies Zariski-dense, so it's worth reflecting on the close relationship between the geometric fundamental group and the symplectic group. Also, recall that $\mathbb{Q}_l(-\beta)$ is isomorphic to $\mathbb{Q}_l$ in some canonical way, so the bilinear form really does go to the underlying field. Finally, remember that the third condition – that a determinant (which is a formal power series) has rational coefficients – is crucial in order to assert that the eigenvalues are algebraic numbers. This gives meaning to the very notion of $\mathbb{C}$-conjugates, as the roots in $\mathbb{C}$ of the minimal polynomial.

We also state corollary 3.9, as Deligne seems to think that it's important.

**Corollary 96** (3.9)**:** *Let $j_0$ be the inclusion of $U_0$ into $\mathbb{P}^1_{\mathbb{F}_q}$ and $j$ the inclusion of $U$ into $\mathbb{P}^1$. Then the eigenvalues of*

$$F^* \curvearrowright H^1(\mathbb{P}^1, j_*\mathcal{F})$$

*are algebraic numbers whose conjugates $\alpha$ satisfy*

$$q^{\frac{\beta+1}{2}-\frac{1}{2}} \leq |\alpha| \leq q^{\frac{\beta+1}{2}+\frac{1}{2}}.$$

The only theorem in chapter 6 is below, so it must be important!

**Theorem 97** (6.2)**:** *For every $x \in |U_0|$, the polynomial*

$$\det\Big(1 - F_x^* t, \frac{\varepsilon_0}{\varepsilon_0 \cap \varepsilon_0^{\perp}}\Big)$$

*has coefficients in $\mathbb{Q}$.*

**Corollary 98** (6.3)**:** *Let $j_0$ be the inclusion of $U_0$ into $D_0$ and $j$ the inclusion of $U$ into $D$. Then the eigenvalues of*

$$F^* \curvearrowright H^1\Big(D, \frac{j_*\varepsilon}{\varepsilon \cap \varepsilon^{\perp}}\Big)$$

*are algebraic numbers, all of whose $\mathbb{C}$-conjugates $\alpha$ satisfy*

$$q^{\frac{n+1}{2}-\frac{1}{2}} \leq |\alpha| \leq q^{\frac{n+1}{2}+\frac{1}{2}}.$$

Let's go back and try to get a feel for what made chapter 3 'work'. The following two lemmata address the rationality of the coefficients of the determinant, which is a formal power series.

**Lemma 99** (3.3)**:** *...*

$$t\frac{d}{dt}\log\left(\det\Big(1 - F_x t^{\deg(x)}, \mathcal{F}_0^{\otimes 2k}\Big)^{-1}\right)$$

*is a formal power series with coefficients in $\mathbb{Q}$.*

**Lemma 100** (3.4): *The local factors*

$$\det\left(1 - F_x t^{\deg(x)}, \mathcal{F}_0^{\otimes 2k}\right)^{-1}$$

*are formal power series with coefficients in* $\mathbb{Q}$.

The characteristic polynomial of the Frobenius acting on cohomology is getten by multiplying and dividing these, so Deligne talks us through some of the elementary real and complex analysis that we need to bound the poles of a products of such power series.

**Lemma 101** (3.5): *For* $i \in \mathbb{Z}_{>0}$*, let* $f_i = \sum_n a_{i,n} t^n$ *be a formal power series with constant term 1 and with coefficients in* $\mathbb{R}_{\geq 0}$*. Assume that the order (i.e. the power of the first nonzero coefficient) of* $f_i - 1$ *goes to infinity as* $i \to \infty$*, and put*

$$f = \prod_i f_i = f_1 f_2 f_3 \cdots .$$

*Then the radius of absolute convergence of* $f_i$ *is at least equal to that of* $f$.

In other words, if any $f_i$ diverges then the product $f$ diverges. Let's see how this translates when we put on our complex glasses.

**Lemma 102** (3.6): *Under the hypotheses of (3.5), if* $f$ *and* $f_i$ *are the Taylor series of meromorphic functions, then*

$$\inf\{|z| : f(z) = \infty\} \leq \inf\{|z| : f_i(z) = \infty\}.$$

*These numbers are in fact the radii of absolute convergence.*

You could take this as a definition, but there is some content in this statement if you have some other preconceived notion of 'radius of absolute convergence'.

Interestingly, the following subsection appears late in section 6.

**(6.10) Preliminaries**

Let $u \in U$, and let $\mathcal{F}_u$ be the fibre of $\mathcal{F}$ at $u$. The arithmetic fundamental group $\pi_1(U_0, u)$, which is the extension of $\hat{\mathbb{Z}} = \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ (generator $\varphi$) by the geometric fundamental group $\pi_1(U, u)$, acts on $\mathcal{F}_u$ by symplectic similitudes:

$$\rho : \pi_1(U_0, u) \to CSp(\mathcal{F}_u, \psi) = \left\{g \in GL(\mathcal{F}_u, \psi) : \psi(gx, gy) = \alpha\psi(x, y) \text{ for some } \alpha \in \overline{\mathbb{Q}_l}\right\}. \tag{231}$$

The extension is the exact sequence

$$1 \to \pi_1(U, u) \xrightarrow{p_*} \pi_1(U_0, u) \xrightarrow{s_*} \hat{\mathbb{Z}} \to 1, \tag{232}$$

where $p$ is the projection

$$p : U = U_0 \times_{\mathbb{F}_q} \overline{\mathbb{F}_q} \to U_0$$

and $s$ is the structural morphism

$$s : U_0 \to \mathrm{Spec}\mathbb{F}_q.$$

**3.7**

The hypothesis (ii) guarantees that the coinvariants of $\pi_1(U,u)$ in $\mathcal{F}_u^{\otimes 2k}$ are the coinvariants of the symplectic group in $\mathcal{F}_u^{\otimes 2k}$ ($\pi_1$ is open and therefore Zariski dense in Sp).

This reinforces the understanding that $\pi_1(U,u)$ is pretty much all of the symplectic group. We see it again in chapter 5:

**5.9** ... The monodromy representation thus induces

$$\rho : \pi_1(U,u) \to Sp(\frac{E}{E \cap E^\perp}, \psi).$$

**Theorem 103** (5.10, Kazhdan-Margulis)**:** *The image of $\rho$ is open.*

**Arun Ram** *Thursday 28 June 2012*

**5.9**

The subspace $E \cap E^\perp$ of $E$ is the kernel of the restriction to $E$ of the intersection form $\mathrm{Tr}(x \cup y)$. This form then induces a nondegenerate bilinear form

$$\psi : \frac{E}{E \cap E^\perp} \otimes \frac{E}{E \cap E^\perp} \to \mathbb{Q}_l(-n),$$

alternating for $n$ odd and symmetric for $n$ even.[97] This form is respected by the monodromy; for $n$ odd, the induced monodromy representation is thus

$$\rho : \pi_1(U,u) \to \mathrm{Sp}\Big(\frac{E}{E \cap E^\perp}, \psi\Big).$$

**Theorem 104** (5.10, Kazhdan-Margulis)**:** *The image of $\rho$ is open.*

So $\pi_1(U,u)$ is essentially the symmetric group. This theorem follows from the lemma below.

**Lemma 105** (5.11)**:** *Let $V$ be a finite-dimensional vector space over a field $k$ of characteristic 0, $\psi$ a nondegenerate alternating bilinear form, and $\mathcal{L}$ a Lie subalgebra of $\mathfrak{sp}(V,\psi)$. Assume that:*

1. *$V$ is a simple $\mathcal{L}$-representation ($\mathcal{L}$-module).*
2. *$\mathcal{L}$ is generated by a family of endomorphisms of $V$ of the form*

$$x \mapsto \psi(X,\delta)\delta.$$

*Then $\mathcal{L} = \mathfrak{sp}(V,\psi)$.*

Point: take the log and reduce to the Lie algebra (exponentiation goes from the Lie algebra to the Lie group). Monodromy is a matrix, so it has a logarithm as long as it

[97]For a comprehensive discussion of bilinear forms, see Bourbaki, *Algebra*, chapter 9.

is invertible.[98]

**5.8**

In the rest of this discussion, we study a Lefschetz pencil of hyperplane sections of $X$, excluding the case $p = 2, n$ even. The case where $n$ is odd will suffice for the following. Put $U = D - S$. Let $u \in U$, and $l$ prime ($l \neq p$). The local results of section 4 show that $R^n f_* \mathbb{Q}_l$ is tamely ramified at each $s \in S$. The tame fundamental group of $U$ is a quotient of the profinite completion of the analagous transcendental fundamental group (obtained in characteristic 0 by tame covers, and the existence theorem of Riemann). The algebraic situation is completely parallel to the transcendental situation, and the results orresponding to Lefschetz results are obtained by standard arguments. In the proof of (5.4) the Lefschetz theorem on $\pi_1$ becomes the theorem of Bertini, and one must invoke Abhyankar's lemma to control the ramification of $R^\bullet g_* \mathbb{Q}_l$ along the smooth region of $\check{X}$ of codimension 1. The results are as follows.

(a) Case: the vanishing cycles are nonzero.

1. For $i \neq n$, the sheaf $R^i f_* \mathbb{Q}_l$ on $D$ is constant.
2. Let $j$ be the inclusion of $U$ in $D$,
$$j : U \hookrightarrow D.$$
We have
$$R^n f_* \mathbb{Q}_l = j_* j^* R^n f_* \mathbb{Q}_l.$$
In other words, pulling back and pushing forward via $j$ has no effect on the sheaf we're interested in.
3. Let $E \subseteq H^n(X_u, \mathbb{Q}_l)$ be the subspace generated by the vanishing cycles. This subspace is stable under $\pi_1(U, u)$, and
$$E^\perp = H^n(X_u, \mathbb{Q}_l)^{\pi_1(U,u)}.$$
The representation of $\pi_1(U, u)$ is absolutely irreducible,[99] and the image of $\pi_1$ in $GL\left(\frac{E}{E \cap E^\perp}\right)$ is generated (topologically) by the
$$x \mapsto x \pm (x, \delta_s)\delta_s, \qquad s \in S.$$
The sign $\pm$ is determined as in (4.1).

The point is that the hypotheses of lemma (5.11) are satisfied.

(b) Case: the vanishing cycles are 0. (This is an exceptional case. Since $(\delta, \delta) = \pm 2$ for $n$ even, we cannot be in this case unless $n = 2m + 1$. We note that if a vanishing cycle is zero then they all are, since they are conjugates.)

1. For $i \neq n + 1$, the sheaf $R^i f_* \mathbb{Q}_l$ is constant.
2. We have an exact sequence
$$0 \to \oplus_{s \in S} \mathbb{Q}_l(m - n)_s \to R^{n+1} f_* \mathbb{Q}_l \to \mathcal{F} \to 0, \tag{233}$$
with $\mathcal{F}$ constant.
3. $E = 0$.

**Theorem 106** (5.4)**:** *The vanishing cycles $\pm\delta_s$ (taken up to sign) are conjugate under the action of $\pi_1(U, u)$.*

[98] Certainly over $\mathbb{C}$, any invertible matrix has a logarithm, though it isn't necessarily unique.
[99] Irreducible even over an algebraic closure.

**Corollary 107** (5.5)**:** *The action of* $\pi_1$ *on* $\frac{E}{E\cap E^{\perp}}$ *is absolutely irreducible.*

We have a lot of the ingredients now. The key step is to go from the symplectic group to the Frobenius eigenvalues: what is the relationship?

**Richard Hughes** *Thursday 5 July 2012*

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### 6. A rationality theorem

#### 6.1

Let $\mathbb{P}_0$ be a projective space of dimension $\geq 1$ over $\mathbb{F}_q$, let $X_0 \subseteq \mathbb{P}_0$ be a nonsingular projective variety, let $A_0 \subseteq \mathbb{P}_0$ be a linear subspace of codimension two, let $D_0 \subseteq \check{\mathbb{P}}_0$ be its right dual, and let $\mathbb{P}, X, A, D$ over $\overline{\mathbb{F}_q}$ be gotten from $\mathbb{P}_0, X_0, A_0, D_0$ by extension of scalars. The theory of Lefschetz pencils gives a diagram

$$\begin{array}{ccc} X_0 & \xleftarrow{\ \pi\ } & \tilde{X}_0 \\ & & \Big\downarrow f \\ & & D, \end{array} \tag{234}$$

where $\tilde{X}_0$ is obtained from $X_0$ by resolving a finite number of singularities.

We assume that $X$ is connected and of even dimension $n+1 = 2m+2$, and that the pencil $(X_t)_{t\in D}$ of hyperplane sections of $X$ defined by $X$ is Lefschetz. The finite set

$$S = \{t \in D : X_t \text{ is singular}\}$$

is defined over $\mathbb{F}_q$, i.e. derived from some $S_0 \subseteq D_0$. Put $U_0 = D_0 - S_0$ and $U = D - S$.

Let $u \in U$. The vanishing part of the cohomology, $E \subseteq H^n(X_u, \mathbb{Q}_l)$, is stable under $\pi_1(U,u)$, so it defines on $U$ a local subsystem $\mathcal{E}$ of $R^n f_* \mathbb{Q}_l$. The latter is defined on $\mathbb{F}_q$, being the inverse image of the $\mathbb{Q}_l$-sheaf $R^n f_{0*}\mathbb{Q}_l$ on $D_0$. On $U$, the local system $\mathcal{E}$ is the inverse image of the local subsystem

$$\mathcal{E} \subseteq R^n f_{0*}\mathbb{Q}_l.$$

The cup product is an alternating form

$$\psi : R^n f_{0*}\mathbb{Q}_l \otimes R^n f_{0*}\mathbb{Q}_l \to \mathbb{Q}_l(-n).$$

Writing $\mathcal{E}_0^{\perp}$ as the orthogonal complement of $\mathcal{E}_0$ relative to $\psi$, on $R^n f_{0*}\mathbb{Q}_l|_{U_0}$ we see that $\psi$ induces a perfect pairing

$$\psi : \frac{\mathcal{E}_0}{\mathcal{E}_0 \cap \mathcal{E}_0^{\perp}} \otimes \frac{\mathcal{E}_0}{\mathcal{E}_0 \cap \mathcal{E}_0^{\perp}} \to \mathbb{Q}_l(-n). \tag{235}$$

#### 6.3

This is a crucial result in the context of the paper.

**Corollary 108** (6.3)**:** *Let $j_0$ be the inclusion of $U_0$ in $D_0$, and $j$ that of $U$ in $D$. Then the eigenvalues of*

$$F^* \curvearrowright H^1\Big(D, j_* \frac{\mathcal{E}}{\mathcal{E} \cap \mathcal{E}^\perp}\Big)$$

*are algebraic numbers, all of whose $\mathbb{C}$-conjugates $\alpha$ satisfy*

$$q^{\frac{n+1}{2} - \frac{1}{2}} \leq |\alpha| \leq q^{\frac{n+1}{2} + \frac{1}{2}}. \tag{236}$$

*Proof.* (5.10) and (6.2) ensure that the hypotheses of (3.2) are satisfied. We then apply (3.9). □

We have seen all of these ingredients aside from (6.2).

**6.2**

**Theorem 109** (6.2)**:** *For al $x \in |U_0|$, the polynomial*

$$\det\Big(1 - F_x^* t, \frac{\mathcal{E}_0}{\mathcal{E}_0 \cap \mathcal{E}_0^\perp}\Big)$$

*has rational coefficients.*

For the rest of today, we focus on proving (6.2).

**6.4**

An $l$-adic unit is a unit in $\mathbb{Z}_l$.

**Lemma 110** (6.4)**:** *Let $\mathcal{G}_0$ be a twisted constant $\mathbb{Q}_l$-sheaf on $U_0$ such that the inverse image sheaf $\mathcal{G}$ on $U$ is a constant sheaf. Then there exist l-adic units $\alpha_i$ such that if $x \in |U_0|$ then*

$$\det(1 - F_x^* t, \mathcal{G}_0) = \prod_i (1 - \alpha_i^{\deg(x)} t).$$

We can apply this lemma to:

- $R^i f_{0*}\mathbb{Q}_l$, for $i \neq n$,
- $\frac{R^n f_{0*}\mathbb{Q}_l}{\mathcal{E}_0}$, and
- $\mathcal{E}_0 \cap \mathcal{E}_0^\perp$.

We begin the proof of (6.2).

From (96), for $x \in |U_0|$ with $X_x = f_0^{-1}(x)$,

$$Z(X_x, t) = \prod_i \det(1 - F_x^* t, R^i f_{0*}, \mathbb{Q}_l)^{(-1)^{i+1}}. \tag{237}$$

As

$$R^n f_{0*}\mathbb{Q}_l \cong \frac{R^n f_{0*}\mathbb{Q}_l}{\mathcal{E}_0} \oplus \frac{\mathcal{E}_0}{\mathcal{E}_0 \cap \mathcal{E}_0^\perp} \oplus (\mathcal{E}_0 \cap \mathcal{E}_0^\perp), \tag{238}$$

we see that $Z(X_x, t)$ is the product of

$$Z^f = \det\Big(1 - F_x^* t, \frac{R^n f_{0*}\mathbb{Q}_l}{\mathcal{E}_0}\Big) \cdot \det(1 - F_x^* t, \mathcal{E}_0 \cap \mathcal{E}_0^{\perp}) \cdot \prod_{i \neq n} \det(1 - F_x^* t, R^i f_{0*}\mathbb{Q}_l)^{(-1)^{i+1}}$$

and

$$Z^m = \det\Big(1 - F_x^* t, \frac{\mathcal{E}_0}{\mathcal{E}_0 \cap \mathcal{E}_0^{\perp}}\Big).$$

Recall that the notation is shorthand: we actually mean at the stalks. The point is that we're working with bona fide vector spaces, so determinants and direct sums make sense. Applying the lemma to the terms in $Z^f$, we find that there exist $l$-adic units $\alpha_i, \beta_j$ (for $1 \leq i \leq N$ and $1 \leq j \leq M$) such that if $x \in |U_0|$ then

$$Z(X_x, t) = \frac{\prod_i (1 - \alpha_i^{\deg(x)} t)}{\prod_j (1 - \beta_j^{\deg(x)} t)} \cdot \det(1 - F_x^* t, \mathcal{F}_0), \tag{239}$$

where $\mathcal{F}_0 = \frac{\mathcal{E}_0}{\mathcal{E}_0 \cap \mathcal{E}_0^{\perp}}$.

**6.5**

It suffices to prove that the polynomials $\prod_i (1 - \alpha_i t)$ and $\prod_j (1 - \beta_j t)$ have coefficients in $\mathbb{Q}$. We derive this from (6.6), (6.7), and (6.8).

**6.9**[Proving (6.5) and so (6.2) modulo (6.6)]

To show: $\prod_i (1 - \alpha_i t)$ and $\prod_j (1 - \beta_j t)$ have rational coefficients, i.e. $\{\alpha_i\}$ and $\{\beta_j\}$ are defined over $\mathbb{Q}$.

Lemma (6.6) gives us an intrinsic characterization of the $\beta_j$ in terms of the coefficients of $Z(X_x, t) \in \mathbb{Q}(t)$, from which it follows that $\{\beta_j\}$ is defined over $\mathbb{Q}$. Proposition (6.8) now tells us that $\prod_j (1 - \beta_j t)$ divides $\prod_i (1 - \alpha_i t)$, so $\{\alpha_i\}$ is defined over $\mathbb{Q}$.[100]

It remains to prove (6.6)...and (6.8) I suppose, but maybe Dougal or someone will do that.

**6.6**

**Proposition 111** (6.6): *Let $(\gamma_i)$ (for $1 \leq i \leq P$) and $(\delta_j)$ (for $1 \leq j \leq Q$ be two families of l-adic units. Assume that $\gamma_i \neq \delta_j$ for all $i, j$. If $K$ is a 'large enough' finite set of integers $\neq 1$ and $L$ is a 'large enough' subset of $|U_0|$ then, if $x \in |U_0| - L$ satisfies $k | \deg(x)$ for all $k \in K$, then the denominator of*

$$\frac{\det(1 - F_x^* t, \mathcal{F}_0) \prod_i (1 - \gamma_i^{\deg(x)} t)}{\prod_j (1 - \delta_j^{\deg(x)} t)}$$

*written in simplest form is $\prod_j (1 - \delta_j^{\deg(x)} t)$.*

[100] The paragraph after ((96) seems to imply that $Z(X_x, t) \in \mathbb{Z}[[t]]$, but that doesn't seem to be the case here. What's up with that? Also, *divides* means in $\mathbb{Q}[t]$ or what?

**6.13**

*of (6.6).* For each choice of $i$ and $j$, the set

$$\{n \in \mathbb{Z} \mid \gamma_i^n = \delta_j^n\} \tag{240}$$

is an ideal $n_{ij}\mathbb{Z}$, and by hypothesis $n_{ij} \neq 1$. Let $K = \{n_{ij}\}$, and let $L$ be the set of $x \in |U_0|$ such that a $\delta_j^{\deg(x)}$ is an eigenvalue of $F_x^* \curvearrowright \mathcal{F}_0$. By lemma (6.12) and Čebotarev's theorem, $L$ has density 0. □

It remains to prove (6.8), as well as to go through (6.10) to (6.12).

**Dougal Davis** *Thursday 12 July 2012*

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We were going over the proof of this theorem.

**Theorem 112** (6.2)**:** *For $x \in |U_0|$, the polynomial* $\det(1 - F_x^* t, \mathcal{F}_0)$ *has rational coefficients.*

We established that there exist $\alpha_i, \beta_j \in \overline{\mathbb{Q}_l} - \{0\}$ such that $\alpha_i \neq \beta_j$ for all $i, j$ and

$$Z(X_x, t) = \frac{\prod_i (1 - \alpha_i^{\deg(x)} t)}{\prod_j (1 - \beta_j^{\deg(x)} t)} \deg(1 - F_x^* t, \mathcal{F}_0)$$

for all $x \in |U_0|$. Since $Z(X_x, t)$ has rational coefficients, it suffices to show that $\prod_i (1 - \alpha_i^{\deg(x)} t)$ and $\prod_j (1 - \beta_j^{\deg(x)} t)$ have rational coefficients for all $x \in |U_0|$. In fact, it suffices to prove that $\prod_i (1 - \alpha_i t)$ and $\prod_j (1 - \beta_j t)$ have rational coefficients.

We also have, from (6.6), that there exists a finite set $K$ of integers $\neq 1$ and a set $L \subseteq |U_0|$ of density 0 such that: if $x \in |U_0| - L$ and $k \not| \deg(x)$ for $k \in K$, then the denominator of (in simplest form) $Z(X_x, t)$ is

$$\prod_j (1 - \beta_j^{\deg(x)} t).$$

This implies that $\prod_j (1 - \beta_j^{\deg(x)} t) \in \mathbb{Q}[t]$ for 'most' valkues of $\deg(x)$.

To establish the $\prod_j (1 - \beta_j t) \in \mathbb{Q}[t]$, let $f_j \in \mathbb{Q}[t]$ be the minimal polynomial of $\beta_j$ over $\mathbb{Q}$, and let $E \supseteq \mathbb{Q}$ be the splitting field of $\prod_j f_j$. Then, for all $g \in \mathrm{Gal}(E/\mathbb{Q})$,

$$\prod_j (1 - g(\beta_j)^{\deg(x)} t) = g(\prod_j (1 - \beta_j^{\deg(x)} t)) = \prod_j (1 - \beta_j^{\deg(x)} t),$$

since $\prod_j (1 - \beta_j^{\deg(x)} t) \in \mathbb{Q}[t]$. Thus, the family $(\beta_j^{\deg(x)})$ coincides with the family $(g(\beta_j)^{\deg(x)})$ in some order. The idea from here is that if this holds for enough values of $\deg(x)$ then we can conclude that

$$(\beta_j) = (g(\beta_j)) \tag{241}$$

up to reordering. Then

$$g\Big(\prod_j (1-\beta_j t)\Big) = \prod_j (1-\beta_j t)$$

for all $g \in \mathrm{Gal}(E/\mathbb{Q})$, so

$$\prod_j (1-\beta_j t) \in \mathbb{Q}[t].$$

Thus,

$$\prod_i (1-\alpha_i^{\deg(x)} t)\det(1-F_x^* t, \mathcal{F}_0) \in \mathbb{Q}[t]. \tag{242}$$

**Proposition 113** (6.8)**:** *Let*

$$(\gamma_i)_{1\le i\le P} \qquad \text{and} \qquad (\delta_j)_{1\le j\le Q}$$

*be two families of nonzero elements of $\overline{\mathbb{Q}_l}$. Let*

$$R(t) = \prod_i (1-\gamma_i t) \qquad \text{and} \qquad S(t) = \prod_j (1-\delta_j t).$$

*Assume that if $x \in |U_0|$ then*

$$\prod_i (1-\delta_i^{\deg(x)} t) \qquad \text{divides} \qquad \prod_i (1-\gamma_i^{\deg(x)} t)\det(1-F_x^* t, \mathcal{F}_0).$$

*Then $S(t)$ divides $R(t)$.*

*Proof.* Delete pairs of common elements between $(\gamma_i)$ and $(\delta_j)$ until none are left. Then, by (6.6), there exists $x \in |U_0|$ such that the denominator of

$$p(t) = \frac{\det(1-F_x^* t, \mathcal{F}_0)\prod_i (1-\gamma_i^{\deg(x)} t)}{\prod_j (1-\delta_j^{\deg(x)} t)}$$

in simplest form is $\prod_j (1-\delta_j^{\deg(x)} t)$. But $p(t)$ is a polynomial by hypothesis, so none of the $\delta_j$ can remain. □

Setting $\gamma_i = \alpha_i$, this tells us that

$$\begin{aligned} R(t) &= \prod_i (1-\alpha_i t) \\ &= \mathrm{lcm}\{S(t) : \text{the conditions hold}\}, \end{aligned}$$

since $S(t) = R(t)$ satisfies the conditions. As

$$\prod_i (1-\alpha_i^{\deg(x)} t)\det(1-F_x^* t, \mathcal{F}_0) \in \mathbb{Q}[t],$$

it follows that $R(t)$ also has rational coefficients. Thus,

$$\det(1-F_x^* t, \mathcal{F}_0) \in \mathbb{Q}[t]$$

for all $x \in |U_0|$.

**Section 7: end of the proof of (1.7)**

**Lemma 114** (7.1)**:** *Let $X_0$ be a nonsingular absolutely irreducible projective variety of even dimension $d$ over $\mathbb{F}_q$. Let $X$ be obtained by extension of scalars to $\overline{\mathbb{F}_q}$, and let $\alpha$ be an* *eigenvalue of*

$$F^* \curvearrowright H^d(X, \mathbb{Q}_l).$$

*Then $\alpha$ is an algebraic number, all of whose conjugates in $\mathbb{C}$, also denoted $\alpha$, satisfy*

$$q^{\frac{d}{2}-\frac{1}{2}} \leq |\alpha| \leq q^{\frac{d}{2}+\frac{1}{2}}. \tag{243}$$

*Proof.* We proceed by induction on $d$ (always even). The case $d = 0$ is trivial even without assuming that $X_0$ is absolutely irreducible, so we assume henceforth that $d \geq 2$. Let $d = n + 1 = 2m + 2$. If $\mathbb{F}_{q^r}$ is a degree $r$ extension of $\mathbb{F}_q$ and $X_0'/\mathbb{F}_{q^r}$ is obtained from $X_0/\mathbb{F}_q$ by extension of scalars, then the assertion (7.1) for $X_0/\mathbb{F}_q$ is equivalent to (7.1) for $X_0'/\mathbb{F}_{q^r}$ (replace $q$ by $q^r$ and $\alpha$ by $\alpha^r$).

By (5.7), with a convenient projective embedding $i : X \hookrightarrow \mathbb{P}$, we know that $X$ admits a Lefschetz pencile of hyperplane sections. The preceding remark allows us to assume that this pencil is defined over $\mathbb{F}_q$. We may suppose therefore, that there exists a Lefschetz pencil defined by

- a projective embedding $X_0 \to \mathbb{P}_0$ over $\mathbb{F}_q$ and
- a codimension-2 subspace $A_0$ of $\mathbb{P}_0$.

Recall the notation of (6.1) and (6.3):

- $D_0 \subseteq \check{\mathbb{P}}_0$ is the projective line dual to $A_0$.
- $A = A_0 \otimes \overline{\mathbb{F}_q}$, $D = D_0 \otimes \overline{\mathbb{F}_q}$ (tensors are over $\mathbb{F}_q$).
- $S$ is the set of $t \in D$ such that $X_t$ is singular, and $S_0$ is the corresponding set over $\mathbb{F}_q$.
- $U_0 = D_0 - S_0, U = D - S$.
- $$\begin{array}{ccccccc} X_0 & \xleftarrow{\pi_0} & \tilde{X}_0 & \quad & X & \xleftarrow{\pi} & \tilde{X} \\ & & \downarrow{\scriptstyle f_0} & & & & \downarrow{\scriptstyle f} \\ & & D_0, & & & & D, \end{array}$$
  $\tilde{X}$ is the blowup of $X_0$ along $X_0 \cap A_0$.
- Inclusions $j_0 : U_0 \to D_0$ and $j : U \to D$.

□

By a new extension of scalars, we may assume the following.

(a) The points of $S$ are defined over $\mathbb{F}_q$.

(b) The vanishing cycles in $X_s$ (for $s \in S$) are defined over $\mathbb{F}_q$.

(c) There exists a rational point (i.e. an $\mathbb{F}_q$ point) $u_0 \in U_0$. We take a corresponding point $u \in U$ as a base point.

(d) $X_{u_0} = f_0^{-1}(u_0)$ admits a smooth hyperplane section $Y_9$ defined over $\mathbb{F}_q$. We set

$$Y = Y_0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}.$$

As $\tilde{X}$ is obtained from $X$ by blowup of the smooth subvariety (of codimension 2) $A \cap X$, we have

$$H^i(X, \mathbb{Q}_l) \hookrightarrow H^i(\tilde{X}, \mathbb{Q}_l). \tag{244}$$

It suffices, therefore, to prove (243) for the eigenvalues of

$$F^* \curvearrowright H^d(\tilde{X}, \mathbb{Q}_l).$$

The Leray spectral sequence for $f$ is

$$E_2^{pq} = H^p(D, R^q f_* \mathbb{Q}_l) \implies H^{p+q}(\tilde{X}, \mathbb{Q}_l), \tag{245}$$

so it suffices to prove (243) for the eigenvalues of $F^* \curvearrowright E_2^{pq}$ for $p + q = d$. Deligne remarks that the vanishing cycles are either all zero or all nonzero.

(A) $E_2^{2,n-1}$.

By (5.8), $R^{n-1} f_* \mathbb{Q}_l$ is constant. Recall (2.10):

> Let $X$ be a smooth connected curve over an algebraically closed field $k$, let $x \in |X|$, and let $\mathcal{F}$ be a cosntant $\mathbb{Q}_l$-sheaf. Then
> $$H_c^2(X, \mathcal{F}) = (\mathcal{F}_x)(-1).$$

Applying this with $D$ in place of $X$ and $\mathcal{F} = R^{n-1} f_* \mathbb{Q}_l$ gives

$$\begin{aligned} E_2^{2,n-1} &= H^2(D, R^{n-1} f_* \mathbb{Q}_l) \\ &= (R^{n-1} f_* \mathbb{Q}_l)_u(-1) \\ &= H^{n-1}(X_u, \mathbb{Q}_l)(-1). \end{aligned}$$

By the weak Lefschetz theorem, we have

$$H^{n-1}(X_u, \mathbb{Q}_l)(-1) \hookrightarrow H^{n-1}(Y, \mathbb{Q}_l)(-1), \tag{246}$$

and we apply the inductive hypothesis to $Y_0$, since $n - 1 = d - 2$.

(B) $E_2^{0,n+1}$.

If the vanishing cycles are nonzero then $R^{n+1} f_* \mathbb{Q}_l$ is constant and

$$E_2^{0,n+1} = H^{n+1}(X_u, \mathbb{Q}_l).$$

Using Poincaré duality and (246), we get

$$H^{n+1}(Y, \mathbb{Q}_l)(-1) \twoheadrightarrow H^{n+1}(X_u, \mathbb{Q}_l),$$

and we apply the inductive hypothesis to $Y_0$.

If the vanishing cycles are zero, then the exact sequence in (5.8) gives an exact sequence

$$\oplus_{s \in S} \mathbb{Q}_l(m-n)_s \to E_2^{0,n+1} \to H^{n+1}(X_u, \mathbb{Q}_l). \tag{247}$$

As $m - n = -d/2$, the eigenvalues of $F^*$ acting on $\mathbb{Q}_l(m-n)$ are $q^{d/2}$, so (7.1.1) holds (on $H^{n+1}$) by the above exact sequence, so (7.1.1) holds on $E_2^{0,n+1}$>

(C) $E_2^{1,n} = H^1(D, \mathbb{R}^n f_* \mathbb{Q}_l)$. Recall that $D = \mathbb{P}^1$.

If the vanishing cycles are zero, then $R^n f_* \mathbb{Q}_l$ is constant (by (5.8)b) and $E_2^{1,n} = 0$. We will therefore assume that the vanishing cycles are nonzero. By (5.8),

$$R^n f_* \mathbb{Q}_l = j_* j^* R^n f_* \mathbb{Q}_l.$$

Filter this by the subsheaves $j_* \mathcal{E}$ and $j_*(\mathcal{E} \cap \mathcal{E}^\perp)$.

If the vanishing cycles are not in $\mathcal{E} \cap \mathcal{E}^\perp$, we have exact sequences

$$0 \to j_* \mathcal{E} \to R^n f_* \mathbb{Q}_l \to \text{ constant sheaf } \to 0 \tag{248}$$

and

$$0 \to j_*(\mathcal{E} \cap \mathcal{E}^\perp) \to j_* \mathcal{E} \to j_* \Big( \frac{\mathcal{E}}{\mathcal{E} \cap \mathcal{E}^\perp} \Big). \tag{249}$$

Note that $j_*(\mathcal{E} \cap \mathcal{E}^\perp)$ is constant, as there are no vanishing cycles in $\mathcal{E} \cap \mathcal{E}^\perp$.

The long exact sequences of cohomology give

$$H^1(D, j_* \mathcal{E}) \to H^1(D, R^n f_* \mathbb{Q}_l) \to 0 \tag{250}$$

and

$$0 \to H^1(D, j_* \mathcal{E}) \to H^1 \Big( D, j_* \frac{\mathcal{E}}{\mathcal{E} \cap \mathcal{E}^\perp} \Big), \tag{251}$$

and we apply (6.3).

If, God forbid, some of the $\delta$s are in $\mathcal{E} \cap \mathcal{E}^\perp$, then $\mathcal{E} \subseteq \mathcal{E}^\perp$, and exact sequences

$$0 \to j_* \mathcal{E}^\perp \to R^n f_* \mathbb{Q}_l \to \mathcal{F} \to 0 \tag{252}$$

(where $j_* \mathcal{E}^\perp$ is constant and $\mathcal{F}$ is some sheaf) and

$$0 \to \mathcal{F} \to j_* j^* \mathcal{F} \to \oplus_{s \in S} \mathbb{Q}_l(n-m)_s \to 0 \tag{253}$$

(where $j_* j^* \mathcal{F}$ is constant). The long exact sequences of cohomology give

$$0 \to H^1(D, R^n f_* \mathbb{Q}_l) \to H^1(D, \mathcal{F}) \tag{254}$$

and

$$\oplus_{s \in S} \mathbb{Q}_l(n-m)_s \to H^1(D, \mathcal{F}) \to 0, \tag{255}$$

and we remark that $F^*$ acts on $\mathbb{Q}_l(n-m)$ by multiplication by $q^{d/2}$.

**Dougal Davis** *Thursday 19 July 2012*

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**Lemma 115** (7.2): *Let $X_0$ be a nonsingular, absolutely irreducible projective variety of dimension $d$ over $\mathbb{F}_q$. Let $X = X_0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ and $\alpha$ an eigenvalue of*

$$F^* \curvearrowright H^d(X, \mathbb{Q}_l).$$

*Then $\alpha$ is an algebraic number, all of whose conjugates in $\mathbb{C}$ (also denoted $\alpha$) satisfy*

$$|\alpha| = q^{d/2}.$$

*Proof.* For all $k \in \mathbb{Z}_{>0}$, $\alpha^k$ is an eigenvalue of

$$F^* \curvearrowright H^{kd}(X^k, \mathbb{Q}_l),$$

by the Künneth formula. For $k$ even, $X^k$ satisfies the conditions of (7.1), so

$$q^{\frac{kd}{2}-\frac{1}{2}} \leq |\alpha^k| \leq q^{\frac{kd}{2}+\frac{1}{2}},$$

so

$$q^{\frac{d}{2}-\frac{1}{2k}} \leq |\alpha| \leq q^{\frac{d}{2}+\frac{1}{2}}.$$

Letting $k \to \infty$ gives (7.2). □

*Proof of (1.7).* For $X_0$ projective and nonsingular over $\mathbb{F}_q$, and $i \in \mathbb{Z}_{>0}$, we need to prove the statement $W(X_0, i)$:

> Let $X = X_0 \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$. If $\alpha$ is an eigenvalue of $F^* \curvearrowright H^i(X, \mathbb{Q}_l)$, then $\alpha$ is algebraic, and all of its conjugates $\alpha \in \mathbb{C}$ satisfy $|\alpha| = q^{i/2}$.

Note the following.

(a) If $\mathbb{F}_{q^n}$ is a degree $n$ extension of $\mathbb{F}_q$ and $X_0' = X_0 \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$, then $W(X_0, i)$ is equivalent to $W(X_0', i)$.

(b) If $X_0$ is purely of dimension $n$, then $W(X_0, i)$ is equivalent to $W(X_0, 2n - i)$, by Poincaré duality.

(c) If $X_0$ is a disjoint union of varieties $X_0^\alpha$, then $W(X_0, i)$ is equivalent to the conjunction of $W(X_0^\alpha, i)$.

(d) If $X_0$ is purely of dimension $n$, $Y_0$ a smooth hyperplane section of $X_0$, and $i < n$, then
$$W(Y_0, i) \implies W(X_0, i).$$
This follows from the weak Lefschetz theorem.

To prove the assertions $W(X_0, i)$, we refine them successively. By (c), we may suppose that $X_0$ is purely of dimension $n$. By (a), and (d), we may suppose that $i = n$. For if $i < n$, we can extend scalars until $X_0$ has a smooth hyperplane section $Y_0$ of dimension $n - 1$. By (d),

$$W(Y_0, i) \implies W(X_0, i).$$

We repeat this until we get a $Y_0'$ of dimension $i$. Then

$$W(Y_0', i) \implies \dots \implies W(Y_0, i) \implies W(X_0, i). \tag{256}$$

By (a) and (c), we may assume that $X_0$ is absolutely irreducible. Now apply (7.2). □

**James Withers** *Thursday 19 July 2012*

We were proving the Ramanujan-Petersson conjecture.

**Theorem 116:** *Let $f \in S_k = S_k(\Gamma_0(N))$ be a normalised cuspidal newform, $p \not| N$. Then*
$$|a_p(f)| \leq 2p^{\frac{k-1}{2}}.$$

To show: the roots of $X^2 - a_p X + p^{k-1}$ have absolute value $p^{\frac{k-1}{2}}$.
To show: the roots of $\det(X^2 - T_p X + p^{k-1}, S_k)$ have absolute value $p^{\frac{k-1}{2}}$.

We proved the following.

**Theorem 117:** *The eigenvalues of*
$$F \curvearrowright^{k-2}_N W_l$$
*have absolute value $p^{\frac{k-1}{2}}$.*

Recall that
$$^{k-2}_N W_l = \tilde{h}^1\Big(Y(N) \otimes \overline{\mathbb{Q}}, \mathrm{Sym}^{k-2}(R^1 f_{N*}(\mathbb{Q}_l))\Big).$$

**Proposition 118:** *Some of these could be definitions.*

*(a)* $T_p = F + V$.

*(b)* $F = \varphi_p^{-1} \curvearrowright^k_N W_l$.

*(c)* $V$ *is the transpose of* $F$ *relative to the Petersson inner product.*

*(d)* $FV = p^{k-1}$.

Note that
$$1 - T_p X + p^{k-1} X^2 = (1 - FX)(1 - VX).$$
As $F = V^T$,
$$\det(1 - FX, ^{k-2}_1 W_l) = \det(1 - VX, ^{k-2}_1 W_l).$$
The action $T_p \curvearrowright^{k-2}_1 W_l$ is induced by $T_p \curvearrowright^{k-2}_1 W$, and is compatible with the Eichler-Shimura isomorphism
$$^{k-2}_1 W \otimes \mathbb{C} \cong S_k \oplus \overline{S_k},$$
where $\overline{S_k} = \{\bar{f} : f \in S_k\}$, since $T_p$ is a Hermitian operator for the Petersson inner product. Now
$$\begin{aligned}\det(1 - T_p X + p^{k-1} X^2, S_k)^2 &= \det(1 - T_p X + p^{k-1} X^2, ^{k-2}_1 W_l) \\ &= \det(1 - FX, ^{k-2}_1 W_l)\det(1 - VX, ^{k-2}_1 W_l),\end{aligned}$$
so
$$\det(1 - FX, ^{k-2}_1 W_l) = \det(1 - T_p X + p^{k-1} X^2, S_k).$$
The final equality follows by subsitutting $X = 0$, since formal equality of polynomials dictates that $LHS = RHS$ or $LHS = -RHS$.

Let $\alpha$ be a root of $\det(X^2 - T_p X + p^{k-1}, S_k)$. Then $1/\alpha$ is a root of
$$\det(1 - T_p X + p^{k-1} X^2, S_k) = \det(1 - FX, ^{k-2}_1 W_l),$$

so $\alpha$ is an eigenvalue of

$$F \curvearrowright_1^{k-2} W_l,$$

which is a $G_{\mathbb{Q}}$-submodule of ${}^{k-2}_{N}W_l$. Now theorem 117 completes the proof.

**Yi Huang** *Thursday 26 July 2012*

> "It was my lot to plant the harpoon of algebraic topology into the body of the whale of algebraic geometry."
>
> Lefschetz

**Weil II, section 6: Lefschetz pencils**

**Theorem 119** (4.1.1)**:** *Let $X$ be a smooth projective variety over $k$, pure of dimension $n$. Let $\mathcal{L}$ be an invertible and ample sheaf over $X$ (ample line bundle over $X$), and let $\eta = c_1(\mathcal{L}) \in H^2(X)$ be the first Chern class. Then, for $i \geq 0$, the cup product by $\eta^i$*

$$\eta^i \cup : H^{n-i}(X, \mathbb{Q}_l) \to H^{n+1}(X, \mathbb{Q}_l)$$

*is an isomorphism.*

We prove this by induction on $n$. The case $n = 0$ is trivial. The inductive hypothesis is equivalent to the following.

**Lemma 120** (4.1.2)**:** *The intersection form (trace) $H^{n-1}(Y)$ is nondegenerate over the image*

$$H^{n-1}(X) \xrightarrow{\iota^*} H^{n-1}(Y).$$

We note that $c_1(\mathcal{L}^{\otimes m}) = m \cdot c_1(\mathcal{L})$. The point is to prove (4.1.1) using $\mathcal{L}^{\otimes m}$ for some large $m$, instead of $\mathcal{L}$. An *ample line bundle* is a line bundle $\mathcal{L}$ such that there exists $m \in \mathbb{Z}_{>0}$ such that $\mathcal{L}^{\otimes m}$ is very ample. The point is that we may assume that $\mathcal{L}$ is *very ample*, i.e. there is a very nice embedding

$$X \hookrightarrow \mathbb{P}^N$$

for some $N \in \mathbb{Z}_{>0}$. Let $Y$ be a smooth hyperplane section of $X$ with respect to this embedding.

For $i = 0$, (4.1.1) is trivial.

For $i \geq 1$,

$$\begin{array}{ccccc} Y & & H^{n-i}(X) & \xrightarrow{\eta^i \cup} & H^{n+1}(X) \\ \iota \big\downarrow & & \big\downarrow \iota^* & & \big\uparrow PD \\ X & & H^{n-i}(Y) & \xrightarrow{\eta^{i-1} \cup} & H^{(n-1)+(i-1)}(Y), \end{array} \tag{257}$$

where $H^*(Y) \xrightarrow{PD} H^{*+2}(X)$ comes from applying Poincaré duality to

$$H_{2n-2-*}(Y) \xrightarrow{\iota_*} H_{2n-2-*}(X).$$

For $i \geq 2$, the vertical arrows are isomorphisms (by the weak Lefschetz theorem), and the induction hypothesis ensures that the bottom arrow is an isomorphism.

Thus, we may assume that $i = 1$. The weak Lefschetz theorem gives

$$\iota^* : H^{n-1}(X) \hookrightarrow H^{n-1}(Y)$$

and

$$PD : H^{n-1}(Y) \twoheadrightarrow H^{n+1}(X).$$

For these to compose to get a bijection $H^{n-1}(X) \to H^{n+1}(X)$, it is necessary and sufficient to show that the inductive hypothesis implies (4.1.2). So let's see why this is the case.

Let $Y = Y_{t_0}$ be a hyperplane section, for some $t_0 \in \mathbb{P}^1$.[101] Let

$$S = \{t \in \mathbb{P}^1 : Y_t \text{ is singular}\}.$$

Then the $H^i(Y_t)$, for $t \in \mathbb{P}^1 - S$, are the stalks of a smooth $\mathbb{Q}_l$-sheaf over $\mathbb{P}^1 - S$:

$$\begin{array}{ccc} X & & X - \cup_{t \in \mathbb{P}^1 - S} Y_t \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & & \mathbb{P}^1 - S. \end{array} \tag{258}$$

The image of

$$\iota^* : H^{n-1}(X) \hookrightarrow H^{n-1}(Y)$$

is the subspace of $H^{n-1}(Y)$ of invariants of the monodromy action. In other words,

$$\iota^*(H^{n-1}(X)) = H^{n-1}(Y)^{\pi_1(\mathbb{P}_1 - S, t_0)}.$$

See SGA7 for the proof.

Thus,

$$\begin{array}{ccccc} H^{n-1}(X) & \xrightarrow{\cong} & H^{n-1}(Y)^{\pi} & \subseteq & H^{n-1}(Y) \\ & \searrow & \downarrow & \swarrow & \\ & & H^{n+1}(X), & & \end{array} \tag{259}$$

so it suffices to show that $H^{n-1}(Y)^{\pi} \to H^{n+1}(X)$ is an isomorphism.

Fact: $H^{n-1}(Y)$ is a semisimple $\pi_1(\mathbb{P}^1 - S, t_0)$-module.

We will apply the following.

[101] Why can we do this?

**Lemma 121** (4.1.4): *Let $\pi$ be a group algebra, and let $V$ be a semisimple $\pi$-module equipped with a nondegenerate bilinear form $\Phi$ that's invariant under $\pi$. Then the restriction of $\Phi$ to $V^\pi$ is nondegenerate.*

*Proof.* By semisimplicity, $V = V^\pi \oplus W$, where $W$ does not contain any trivial representation. The subspace $W$ is $\Phi$-orthogonal to $V^\pi$, i.e.

$$\Phi = (\Phi|_{V^\pi}) \oplus (\Phi|_W), \tag{260}$$

so $\Phi|_{V^\pi}$ is nondegenerate. □

From (259), it therefore suffices to show that $\ker(H^{n-1}(Y) \to H^{n+1}(X)) = W$. This map decomposes as

$$\begin{array}{ccc} H_{n-1}(Y) & \xleftarrow{PD} & H^{n-1}(Y) \\ \downarrow \iota_* & & \downarrow \\ H_{n-1}(X) & \xrightarrow{PD} & H^{n+1}(X). \end{array} \tag{261}$$

Let $y \in H^{n-1}(Y)$. Then

$$\begin{aligned} PD \circ \iota_* \circ PD(y) &= 0 \\ \Leftrightarrow \iota_* \circ PD(Y) &= 0 \\ \Leftrightarrow (\iota_* \circ PD(y))(x) &= 0 \qquad \text{for all } x \in H^{n-1}(X) \\ \Leftrightarrow \mathrm{Tr}(y \cup \iota^* x) &= 0 \qquad \text{for all } x \in H^{n-1}(X) \\ \Leftrightarrow \mathrm{Tr}(y \cup z) &= 0 \qquad \text{for all } z \in H^{n-1}(Y)^{\pi_1(\mathbb{P}-S)} \\ \Leftrightarrow y \in W&, \end{aligned}$$

as $\mathrm{Tr}(\cdot \cup \cdot) = \mathrm{Tr}(\cdot \cup \cdot)|_{H^{n-1}(Y)^{\pi_1(\mathbb{P}^1-S)}}$ is nondegenerate. This shows that $W$ is indeed the kernel, completing the proof of (4.1.1).

**Dougal Davis** *Thursday 2 August 2012*

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To prove the hard Lefschetz theorem, we needed:

"By (3.4.3), $H^{n-1}(Y)$ is a semi-simple representation of $\pi_1(\mathbb{P}^1 - S, t_0)$."

**Corollary 122** (3.4.13): *Let $S$ be a normal connected scheme over an algebraically closed field $K$, and $f : X \to S$ a smooth proper morphism. Then the sheaves $R^i f_* \mathbb{Q}_l$ are semi-simple.*

Specialising $f : U \to \mathbb{P}^1 - S$ and $Y = f^{-1}(t_0)$ gives that $H^{n-1}(Y) = (R^{n-1} f_* \mathbb{Q}_l)_{t_0}$ is semi-simple.

(3.4.13) is a corollary of

**Theorem 123** (3.4.1)**:** *Let $\mathcal{F}_0$ be an $\iota$-mixed sheaf on a scheme of finite type over $\mathbb{F}_q$. Then*

*(i)*

*(ii)*

*(iii) Assume that $\mathcal{F}_0$ is smooth and pointwise $\iota$-pure. Assume that $X_0$ is normal. Then the sheaf $\mathcal{F}$ on $X$ is semisimple.*

**Notation and terminology**

$\mathbb{F} = \overline{\mathbb{F}}, \iota : \overline{\mathbb{Q}_l} \xrightarrow{\cong} \mathbb{C}$. A $\mathbb{Q}_l$-sheaf $\mathcal{F}_0$ on $X_0$ is *pointwise $\iota$-pure of weight $n$* if for all $x_0 \in |X_0|$ the eigenvalues $\alpha$ of $F_{x_0} \curvearrowright \mathcal{F}_0$ satisfy

$$|\iota\alpha| = N(x_0)^{n/2},$$

where $N(x_0) = q^{\deg(x_0)}$. A sheaf $\mathcal{F}_0$ is *$\iota$-mixed* if it is the iterated extension of pointwise $\iota$-pure sheaves. Then weights of these are the weights of $\mathcal{F}_0$. A sheaf $\mathcal{F}_0$ is *smooth* if it is twisted constant. A scheme $X_0$ is *normal* if every stalk of $\mathcal{O}_{X_0}$ is an integrally closed integral domain. We aim to prove (3.4.1)

We need

**Lemma 124** (3.4.3)**:** *Assume that $X_0$ is smooth. Let $\mathcal{F}_0$ and $\mathcal{G}_0$ be smooth sheaves on $X_0$, pointwise $\iota$-pure of weights $\beta$ and $\gamma$ respectively. If there exists a geometrically nontrivial extension $\mathcal{E}_0$ of $\mathcal{F}_0$ by $\mathcal{G}_0$ then $\beta - \gamma \in \mathbb{Z}_{>0}$.*

The extension

$$0 \to \mathcal{G}_0 \to \mathcal{E}_0 \to \mathcal{F}_0 \to 0$$

is *geometrically trivial* if

$$0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0$$

is a trivial extension (i.e. $\mathcal{E} = \mathcal{F} \oplus \mathcal{G}$).

*Proof of (3.4.1)(iii).* If $U_0 \subseteq X_0$ is open and $\overline{u}$ is a geometric point of $U_0$, then

$$\pi_1(U_0, \overline{u}) \twoheadrightarrow \pi_1(X_0, \overline{u}). \tag{262}$$

Replacing $X_0$ by $U_0$ allows us to assume that $X_0$ is smooth. Let $\mathcal{F}'$ be the largest semisimple subsheaf of $\mathcal{F}$. By transport of structures, $\mathcal{F}'$ is stable under the Frobenius (i.e. $F_*\mathcal{F}^1 = cF'$) and therefore provides us with a subsheaf $\mathcal{F}_0'$ of $\mathcal{F}_0$. Let

$$\mathcal{F}_0'' = \frac{\mathcal{F}_0}{\mathcal{F}_0'}.$$

By (3.4.3), since $\mathcal{F}_0'$ and $\mathcal{F}_0''$ are $\iota$-pure of the same weight, the extension $\mathcal{F}_0$ of $\mathcal{F}_0''$ by $\mathcal{F}_0'$ is geometrically trivial, i.e. $\mathcal{F}_0 = \mathcal{F}_0' \oplus \mathcal{F}_0''$. Now

$$\mathcal{F}_0 = \mathcal{F}' \oplus \mathcal{F}''.$$

If $\mathcal{F}_0'' \neq 0$ then we can find a larger semisimple subsheaf that $\mathcal{F}'$,[102] contradiction. So $\mathcal{F}'' = 0$ and $\mathcal{F} = \mathcal{F}'$ is semisimple. □

[102] $\mathcal{F}_0''$ contains a simple submodule $W$. Take $\mathcal{F}' \oplus W$.

To prove (3.4.3), we need

**Lemma 125** (3.4.2)**:** *If $\mathcal{F}_0$ and $\mathcal{G}_0$ are smooth sheaves on $X_0$, we have an exact sequence*

$$0 \to H^0(X, \mathrm{Hom}(\mathcal{F}, \mathcal{G})) \xrightarrow{F} \mathrm{Ext}^1(\mathcal{F}_0, \mathcal{G}_0) \to H^1(X, \mathrm{Hom}(\mathcal{F}, \mathcal{G}))^F,$$

*where*

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U).$$

Here $\mathrm{Ext}^1(\mathcal{F}_0, \mathcal{G}_0)$ is the group of extension classes, and the morphism on the right is the inverse image on $X$:

$$\mathrm{Ext}^1(\mathcal{F}_0, \mathcal{G}_0) \to \mathrm{Ext}^1(\mathcal{F}, \mathcal{G}) = H^1(X, \mathrm{Hom}(\mathcal{F}, \mathcal{G})).$$

The image of this is an invariant part of $H^1$.

*Proof.* If an extension $\mathcal{E}_0$ of $\mathcal{F}_0$ by $\mathcal{G}_0$ is geometrically trivial, it admits a splittin

$$\varphi : \mathcal{F} \to \mathcal{E} \qquad (\mathcal{E} = \mathcal{F} oplus \mathcal{G}).$$

The other splittings are of the form $\varphi - f$ with $f \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$. The extension on $X_0$ is trivial if and only if $\varphi - f$ can be chosen invariant under $F$, i.e.

$$F\varphi - \varphi \in \mathrm{Hom}(\mathcal{F}, \mathcal{G})$$

is of the form $Ff - f$, i.e. has zero image in $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$.

⋮ □

**Dougal Davis** *Wednesday 8 August 2012*

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Nobody prepared anything, so let's have a chat about the overall strategy for Weil1.

We start with the fundamental bound

$$q^{\frac{\beta+1}{2} - \frac{1}{2}} \leq |\alpha| \leq q^{\frac{\beta+1}{2} + \frac{1}{2}},$$

where $\beta$ is a weight. For $d$ even, we can put $\beta = d - 1$, to give

$$q^{\frac{d}{2} - \frac{1}{2}} \leq |\alpha| \leq q^{\frac{d}{2} + \frac{1}{2}}.$$

We may replace $X$ with $X^k$ for $k$ even, so

$$q^{\frac{kd}{2} - \frac{1}{2}} \leq |\alpha^k| \leq q^{\frac{kd}{2} + \frac{1}{2}}$$

$$q^{\frac{d}{2} - \frac{1}{2k}} \leq |\alpha| \leq q^{\frac{d}{2} + \frac{1}{2k}},$$

and $k \to \infty$ gives

$$|\alpha| = q^{\frac{d}{2}}.$$

So how did we go from curves to higher dimension?

We assumed $X$ had even dimension $d$, and stepped down inductively 2 at a time. We could assume that $X$ was very nice, so $X \hookrightarrow \mathbb{P}^n$. We took a Lefschetz pencil $(X_t)_{t \in \mathbb{P}^1}$, and let $Y$ be a hyperplane section (so codimension 2) of $X_{t_0}$. We used the

Leray spectral sequence

$$E_2^{p,q} \implies H^{p+q}(\tilde{X}, \mathbb{Q}_l),$$

where $\tilde{X}$ is the blowup of $X$ along a codimension 2 subspace.

**Trithang Tran** *Wednesday 8 August 2012*

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We started with another quick summary of Weil I. In particular, with the part about Lefschetz pencils, we used the fact that $X \subseteq \mathbb{P}^n$, since $X$ is very nice. We let $A \subseteq X$ be a codimension 2 subspace of $\mathbb{P}^n$. Somehow we mapped every point in $X - A$ to a hyperplane containing $A$. We let $\tilde{X}$ be the blowup of $X$ along $A$. Now every point in $\tilde{X}$ maps to a hyperplane containing $A$. These are parametrized by $\mathbb{P}^1$, so we get $f : \tilde{X} \to \mathbb{P}^1$. We then hit $f$ with a Leray SS.

This seems to be the point of Weil II:

**Theorem 126** (3.3.1)**:** *Let $f : X \to Y$ be a morphism of schemes of finite type over $\mathbb{Z}$, and let $\mathcal{F}$ be a sheaf on $X$. If $\mathcal{F}$ is mixed of weight $\leq n$ then for each $i$, $R^i f_! \mathcal{F}$ is mixed of weight $\leq n + 1$.*

We need some definitions to understand this.

**1.2**

**(1.2.1) Weights.**

Let $q$ be a prime power and $n \in \mathbb{Z}$. A number is *pure of weight $n$ rel $q$* if it is algebraic and all of its $\mathbb{C}$-conjugates have absolute value $q^{n/2}$.

**(1.2.2)**

Let $X$ be a scheme of finite type over $\mathbb{Z}$.

1. $\mathcal{F}$ is *pointwise pure* if there exists $n \in \mathbb{Z}$ (the *weight* of $\mathcal{F}$) such that if $x \in |X|$ then the eigenvalues of $F_x$ are pure of weight $N(x)$ (the size of the residue field).
2. $\mathcal{F}$ is *mixed* if it admits a finite filtration of successive quotients of pointwise pure sheaves. More precisely,
$$0 = \mathcal{F}^{(0)} \subseteq \mathcal{F}^{(1)} \subseteq \ldots \subseteq \mathcal{F}^{(r)} = \mathcal{F}, \tag{263}$$
and the quotient sheaves are pure. The *weight* of a mixed sheaf $\mathcal{F}$ is the collection of nonzero weights of the quotient sheaves for such a filtration.

**Dougal Davis** *Wednesday 22 August 2012*

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**Theorem 127** (3.3.1)**:** *Let $f : X \to Y$ be a morphism of schemes of finite fype over $\mathbb{Z}$, and* *let $\mathcal{F}$ be a sheaf on X. If $\mathcal{F}$ is mixed of weights $\leq n$ then, for each i, $R^i f_i \mathcal{F}$ is mixed of weight $\leq n + i$.*

We start by restating some definitions.

(i) A (constructible $\overline{\mathbb{Q}_l}$-) sheaf $\mathcal{F}$ on $X$ is *pointwise pure of weight n* if, for all $x \in |X|$, the eigenvalues of $F_x \curvearrowright \mathcal{F}_x$ are pure of weight $n$ rel. $N(x)$, i.e. for $X/\mathbb{F}_q$, the eigenvalues of $F_x$ are algebraic, all of whose $\mathbb{C}$-conjugates $\alpha$ satisfy

$$|\alpha| = q^{\frac{n\deg(x)}{2}}.$$

(ii) $\mathcal{F}$ is *mixed* if it admits a finite filtration where the successive quotients are pointwise pure. The (nonzero) weights of these are the *weights* of $\mathcal{F}$.

Let's do some examples of constructible $\overline{\mathbb{Q}_l}$-sheaves on the one-point space $X = \mathrm{Spec}(\mathbb{F}_q)$. There is an equivalence of categories between these and the category of $\overline{\mathbb{Q}_l}$-vector spaces with an action of $F$ (where $F$ acts by an automorphism whose eigenvalues are $l$-adic units). As $F$ generates (topologically) the étale fundamental group $\pi(X)$, there is also a category equivalence to continuous $\pi_1$-representations. The intuition comes from representations of the fundamental group of a manifold. Given

$$\rho : \pi_1(M) \to GL(V),$$

there's a vector bundle over $M$, namely

$$\frac{\tilde{M} \times V}{(m,v)\tilde{(}\gamma \cdot m, \rho(\gamma)v), \rho(\gamma)v)}, \tag{264}$$

where $\tilde{M}$ is the universal cover of $M$ and $\gamma \in \pi_1(M)$ acts on $\tilde{M}$ as a deck transformation.

Examples:

- $\overline{\mathbb{Q}_l}$ with $F$ acting by the identity is the trivially constant sheaf, pure of weight 0.
- $\overline{\mathbb{Q}_l}(1)$, where $F$ acts by $q^{-1}$ is pure of weight -2.
- $\overline{\mathbb{Q}_l} \oplus \overline{\mathbb{Q}_l}(1)$ is mixed of weights 0 and -2.
- $\overline{\mathbb{Q}_l}^2$ with $F$ acting by

  $$\begin{pmatrix} 1 & 1 \\ 0 & q^{-1} \end{pmatrix}$$

  is mixed of weights 0 and -2. A filtration is

  $$0 \subseteq \langle (1,0) \rangle \subseteq \overline{\mathbb{Q}_l}^2. \tag{265}$$

**Yi Huang** *Wednesday 29 August 2012*

**Theorem 128** (3.3.1)**:** *Let $f : X \to Y$ be a morphism of schemes of finite fype over $\mathbb{Z}$, and let $\mathcal{F}$ be a sheaf on $X$. If $\mathcal{F}$ is mixed of weights $\leq n$ then, for each $i$, $R^i f_! \mathcal{F}$ is mixed of weight $\leq n + i$.*

Reminder of notation (continuous $f : X \to Y$)

- direct image
$$\begin{aligned} f_* : Sh(X) &\to Sh(Y) \\ \mathcal{F} &\mapsto f_* \mathcal{F} \\ & \qquad U \mapsto F(f^{-1}(U)). \end{aligned}$$
- inverse image
$$\begin{aligned} f^* : Sh(Y) &\to Sh(X) \\ \mathcal{G} &\mapsto f^* \mathcal{G} \\ & \qquad U \mapsto sheafify(\lim_{V \supseteq f(U)} \mathcal{G}) \end{aligned}$$
- direct image with compage support
$$f_! : Sh(X) \to Sh(Y)$$
(only push forward compact bits)

(A) Devissage for the sheaf $\mathcal{F}$.
Given an exact sequence of sheaves
$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$
- if $R^i f_! \mathcal{F}'$ and $R^i f_! \mathcal{F}''$ are mixed of weights $\leq n + i$ then so too is $R^i f_! \mathcal{F}$.
- if $R^i f_! \mathcal{F}$ and $R^i f_! \mathcal{F}''$ are mixed of weights $\leq n + i$ then so too is $R^i f_! \mathcal{F}'$.
- if the sequence splits and $R^i f_! \mathcal{F}$ is mixed of weights $\leq n + i$ then so too are $R^i f_! \mathcal{F}''$ and $R^i f_! \mathcal{F}'$.

(B) Devissage for the scheme $X$.
Let $j : U \hookrightarrow X$ be open in $X$ and let $i : S \to X$ be the complement of $U$. Let $\mathcal{F}$ be a sheaf on $X$. If
$$R^i (f \circ j)_! j^* \mathcal{F} \qquad \text{and} \qquad R^i (f \circ i)_! i^* \mathcal{F}$$
are mixed of weights $\leq n + i$ then so too is $R^i f_! \mathcal{F}$.
Idea for proof: apply (A) to
$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0.$$

(C) Devissage for $Y$. Let $j : V \hookrightarrow Y$ be open with complement $i : T \hookrightarrow Y$. If the fundamental theorem holds for the change of bases given by $i$ and $j$ then it holds for $Y$:
$$\begin{array}{ccccc} X \times_Y V & \longrightarrow & X & \longleftarrow & X \times_Y T \\ \downarrow \alpha & & \downarrow f & & \downarrow \beta \\ V & \hookrightarrow & Y & \hookleftarrow & T. \end{array} \tag{266}$$

If (3.3.1) holds for $\alpha$ and $\beta$ then it holds for $f$.

(D) Transitivity

If $f = g \circ h$ and the sheaves $R^i g_! R^j h_! \mathcal{F}$ are mixed of weight $\leq n + i + j$ then the sheaves $R^k f_! \mathcal{F}$ are mixed of weight $\leq k$. How is this useful? Maybe we can make a sequence

$$X \to X_1 \to X_2 \ldots \to Y$$

of maps of relative dimension 1, in which case we may assume that $f : X \to Y$ has relative dimension 1.

(E) If $Y' \xrightarrow{G} Y$ is a universal homeomorphism, then it suffices to verify (3.3.1) after base change by $g$ (étale cohomology doesn't 'see' this). For example,

- $Y' = Y_{red}$ (kill off all nilpotents in the corresponding spectra)
- $Y'$ is the normaliser of $Y$ in an inseparable extension of the function field

(F) If $f$ is of relative dimension 0 and $\mathcal{F}$ is pointwise pure then $f_! \mathcal{F} = R^0 f_! \mathcal{F}$ is pointwise pure of the same weight, and $R^i f_! \mathcal{F} = 0$ for $i \neq 0$.

So what does this all allow us to do?

(B) and (C) break $X$ and $Y$ into nice pieces. On these pieces, (D) constructs a chain of maps of relative dimension 1. Thus, we may assume that $f : X \to Y$ is of relative dimension 1.

(B) breaks $\mathcal{F}$ up over pieces on which it is lisse. (A) takes the filtration and lets you deal with successive quotients using induction:

$$0 \to \mathcal{F}_1 \hookrightarrow \mathcal{F}_2 \twoheadrightarrow \frac{\mathcal{F}_2}{im \mathcal{F}_1} \to 0. \tag{267}$$

Pure lisse sheaves satisfy (3.3.1), implying that $\mathcal{F}_2$ does as well, etc.

We've now reduced to the case that $f : X \to Y$ is of relative dimension 1 and $\mathcal{F}$ is pure, so the fibres of $f$ are curves. Theorem 3.23 now reduces us to the curve case.

**Theorem 129** (3.2.3)**:** *Let $X$ be a smooth projective curve over $\mathbb{F}_q$, $j : U_0 \hookrightarrow X_0$ open dense, and $\mathcal{F}_0$ lisse and pointwise $\iota$-pure of weight $\beta$ over $U_0$. Then the eigenvalues of $F$ over $H^i(X, j_* \mathcal{F})$ weigh $\beta + i$.*

To prove this, we need to know how to calculate with vanishing cycles.

**Dougal Davis** *Wednesday 5 September 2012*

## Goal for Today

- Understand the vanishing cycle functors $\Phi^q$.

These are defined in terms of derived categories. The following is based on SGA section 13.

Let $\mathcal{A}$ be an abelian category, let $K^+(\mathcal{A})$ be the category where

- Objects are bounded below complexes in $\mathcal{A}$
- Morphisms are morphisms of complexes, modulo homotopy

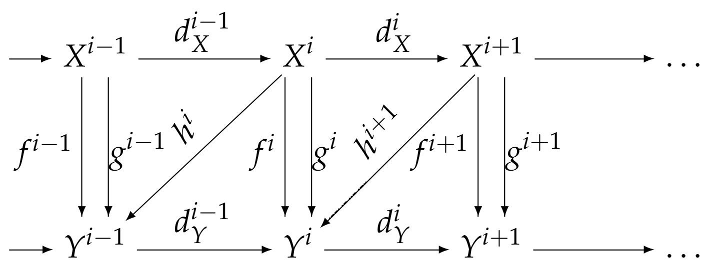
ie. An object in $K^+(\mathcal{A})$ looks like

$$0 \longrightarrow 0 \longrightarrow X^{i+1} \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} X^{i+1} \xrightarrow{d^{i+1}} \dots$$

And a morphism looks like

$$\begin{array}{ccccccc}
\to X^{i-1} & \xrightarrow{d_X^{i-1}} & X^i & \xrightarrow{d_X^i} & X^{i+1} & \longrightarrow & \dots \\
\downarrow f^{i-1} & & \downarrow f^i & & \downarrow f^{i+1} & & \\
\to Y^{i-1} & \xrightarrow{d_Y^{i-1}} & Y^i & \xrightarrow{d_Y^i} & Y^{i+1} & \longrightarrow & \dots
\end{array}$$

such that $d_Y^i \circ f^i = f^{i+1} \circ d_X^i$, ie the squares commute. However in addition two morphisms $f$ and $g$ are identified if there is a chain-homotopy between them. That is there is a family of morphisms $h : X^i \mapsto Y^{i-1}$ such that $f^i - g^i = h^{i+1} \circ d_X^i + d_Y^{i-1} \circ h^i$




**Definition 130:** *A quasi-isomorphism $f : X \mapsto Y$ is a morphism in $K^+(\mathcal{A})$ which is an isomorphism on cohomology.*

The bounded-below derived category of $\mathcal{A}$, denoted $D^+(\mathcal{A})$, is like $K^+(\mathcal{A})$ but every quasi-isomorphism is an isomorphism. There is a canonical functor $Q : K^+(\mathcal{A}) \mapsto D^+(\mathcal{A})$.

If $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $F : \mathcal{A} \mapsto \mathcal{B}$ is a left-exact functor there is a canonical functor

$$RF : D^+(\mathcal{A}) \mapsto D^+(\mathcal{B})$$

called the (total) right derived functor of $F$. The classical right derived functors are just cohomology of this.

$$R^iF(X) = H^i(RF(X))$$

## Vanishing Cycles

Our aim is to define $\Phi^q$.

We have the familiar setup:

Let $S$ be the spectrum of a Henselian discrete valuation ring $V$ (local PID such that Hensel's lemma holds), with closed point $s = \text{Spec}(k(s))$ and generic point $\eta = \text{Spec}(k(\eta))$[103].

Let $\overline{k(\eta)}$ be a separable closure of $k(\eta)$ and denote the corresponding geometric point by $\overline{\eta} = \text{Spec}(\overline{k(\eta)})$. This gives a corresponding geometric point $\bar{s} = \text{Spec}(k(\bar{s}))$ with image $s \in S$.

So in the world of rings, we have
$V \hookrightarrow k(\eta) =$ field of fractions of $\eta$.
$V \mapsto k(s) = V/\mathfrak{m}$, $\mathfrak{m} =$ maximal ideal in $V$.

V is integrally closed in $k(\eta)$ so it seems like a good idea to take the integral closure.

$\overline{V} =$ integral closure of $V$ in $\overline{k(\eta)}$. This will also be a local ring with maximal ideal $\overline{\mathfrak{m}} \supseteq \mathfrak{m}$.

So we get $V/\mathfrak{m} = k(s) \hookrightarrow \overline{V}/\overline{\mathfrak{m}}$.

This is not in general separable but it contains a unique separable closure $k(s) \supseteq k(s)$ which gives us the geometric point $\bar{s}$.

For convenience, write $\text{Gal}(\overline{\eta}/\eta) := \text{Gal}(k(\overline{\eta})/k(\eta))$ and $\text{Gal}(\bar{s}/s) := \text{Gal}(k(\bar{s})/k(s))$. Also write $\overline{S} = \text{Spec}(\overline{V})$.

The construction of $k(\bar{s})$ gives us an exact sequence

$$0 \longrightarrow I \longrightarrow \text{Gal}(\overline{\eta}/\eta) \longrightarrow \text{Gal}(\bar{s}/s) \longrightarrow 0$$

where $I$ is the inertia group and is defined such that the above sequence is exact.

Let $Y$ be over a field $k$ and let $\bar{k}$ be the separable closure of $k$ and $\overline{Y} = Y \otimes_k \bar{k}$.

Then $\text{Gal}(\bar{k}/k)$ acts on $\overline{Y}$ by transport of structure.

[103] For us $S$ is supposed to be the unit disk in the complex plane, $s$ the 0 point and $\eta$ S/{0}

If $G$ is a profinite group and $u : G \mapsto \mathrm{Gal}(\overline{k}/k)$ a continuous homomorphism, then $G$ acts on $\overline{Y}$ via $u$.

Let $\mathcal{F}$ be a sheaf on $\overline{Y}$. An action of $G$ on $\mathcal{F}$, compatible with the action of $G$ on $\overline{Y}$ is an action of $G$ (by automorphisms) on ($\overline{Y}$,$\mathcal{F}$) which induces on $\overline{Y}$ the action of $G$ on $\overline{Y}$.

ie it's a system of isomorphisms

$$\sigma(g) : u(g)_*\mathcal{F} \mapsto \mathcal{F}$$

satisfying $\sigma(gh) = \sigma(g)\sigma(h)$.

If $\mathcal{F}$ is a sheaf on $Y$, with inverse image $\overline{\mathcal{F}}$ on $\overline{Y}$, then the group $\mathrm{Gal}(\overline{k}/k)$ acts on $\overline{\mathcal{F}}$ compatibly, by transport of structure. The functor $\mathcal{F} \mapsto \overline{\mathcal{F}}$ with the action of $\mathrm{Gal}(\overline{k}/k)$ is an equivalence of categories between that of sheaves on $Y$ and sheaves on $\overline{Y}$ with a continuous, compatible action of $\mathrm{Gal}(\overline{k}/k)$.

This is how we'll think of sheaves.

Sheaves on $S$

Let $S$ be as before, write $\iota : s \hookrightarrow S$ , $j : \eta \hookrightarrow S$. A sheaf on $S$ defines sheaves $\mathcal{F}_s = \iota^*\mathcal{F}$ and $\mathcal{F}_\eta = j^*\mathcal{F}$ on $s$ and $\eta$, which we think of as sheaves on $\overline{s}$ and $\overline{\eta}$ with appropriate Galois actions. There is also a natural morphism $\mathcal{F} \mapsto j_*j^*\mathcal{F}$ (since $j_*$ and $j^*$ are adjoints). which induces $\varphi : \mathcal{F}_s = \iota^*\mathcal{F} \mapsto \iota^*j_*j^*\mathcal{F} = \iota^*j_*\mathcal{F}_\eta$.

The functor $\mathcal{F} \mapsto (\mathcal{F}_s, \mathcal{F}_\eta, \varphi)$ is an equivalence of categories between sheaves on $S$ and triples
($\mathcal{F}_s$ = sheaf on $s$,$\mathcal{F}_\eta$ = sheaf on $\eta$, $\varphi : \mathcal{F}_s \mapsto \iota^*j_*\mathcal{F}_\eta$)

Geometrically, we think of $\iota^*j_*$ as coming from a retraction of $D$ (the unit disk in the complex plane) to $\{0\}$, and $\varphi$ keeps track of what has happened to sheaves as they're retracted.

Sheaves on $Y \times_s S$

Let $Y$ be a scheme over $s$. There's a topos called $Y \times_s S$ which gives us some "sheaves". We have the following characterisation which Deligne says we can take as our definition.

Construction/Definition

The sheaves $\mathcal{F}$ on $Y \times_s S$ are identified with triples $(\mathcal{F}_s, \mathcal{F}_\eta, \varphi)$ where

(a) $\mathcal{F}_s$ is a sheaf on $Y$, ie a sheaf $\mathcal{F}_{\overline{s}}$ on $\overline{Y} = Y \times_s \overline{S}$ with a continuous compatible action of $\mathrm{Gal}(\overline{s}/s)$.

(b) $\mathcal{F}_\eta$ is a sheaf $\mathcal{F}_{\overline{\eta}}$ on $\overline{Y}$ with a continuous action of $\mathrm{Gal}(\overline{\eta}/\eta)$ compatible with the action of $\mathrm{Gal}(\overline{\eta}/\eta)$ (via $\mathrm{Gal}(\overline{s}/s)$ on $\overline{Y}$).

(c) $\varphi$ is an equivariant morphism $\varphi : \mathcal{F}_{\overline{s}} \mapsto \mathcal{F}_{\overline{\eta}}$ similarly, a sheaf $\mathcal{F}_\eta$ on $Y \times_s \eta$ is an object as above.

**Dougal Davis** *Wednesday 12 September 2012*

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The functor $\Phi$

We are interested in schemes $X$ over $S$. ie with a morphism $p : X \mapsto S$.

So far, we've set up a language for talking about sheaves on $X_s$ which know something about what happens over $\eta$.

$\Phi$ is a functor which takes a sheaf on $X$ and turns it into a sheaf on $X_s \times_s S$.

Let $X$ be a $S$-scheme and let $\overline{X} = X \times_s \overline{S}$. We have a diagram

$$\begin{array}{ccccc} X_{\overline{s}} & \xrightarrow{\overline{\iota}} & \overline{X} & \xleftarrow{\overline{j}} & X_{\overline{\eta}} \\ \downarrow & & \downarrow & & \downarrow \\ X_s & \xrightarrow{\iota} & X & \xleftarrow{j} & X_\eta \end{array}$$

Where $X_s = X \times_S s$, $X_\eta = X \times_S \eta$, $X_{\overline{s}} = \overline{X} \times_{\overline{S}} \overline{s}$ and $X_{\overline{\eta}} = \overline{X} \times_{\overline{S}} \overline{\eta}$.

Let $\mathcal{F}$ be a sheaf on $X_\eta$ with inverse image $\mathcal{F}_{\overline{\eta}}$ on $X_{\overline{\eta}}$. We set $\Psi_\eta(\mathcal{F}) = \overline{\iota}^* \overline{j}_* \mathcal{F}_{\overline{\eta}}$. By transport of structure, the sheaf $\Psi_\eta(\mathcal{F})$ comes with a compatible action of $\mathrm{Gal}(\overline{\eta}/\eta)$.

This gives us a left exact functor

$$\Psi_\eta(\mathcal{F}) : Sh(X_\eta) \mapsto Sh(X_s \times_s \eta)$$

Let $\mathcal{F}$ be a sheaf on $X$, $\mathcal{F}_\eta$ it's restriction to $X_\eta$, and $\mathcal{F}_s$ it's restriction to $X_s$.

Set

$$\begin{aligned} (\Psi(\mathcal{F}))_\eta &= \Psi(\mathcal{F}_\eta) \\ (\Psi(\mathcal{F}))_s &= \mathcal{F}_s \end{aligned}$$

Let $\overline{\mathcal{F}}$ be the inverse image of $\mathcal{F}$ on $\overline{X}$ and $\varphi'$ the adjunction morphism $\varphi : \overline{\mathcal{F}} \mapsto \overline{j}_* \overline{j}^* \overline{\mathcal{F}}$.

This induces:

$$\varphi : \mathcal{F}_{\overline{s}} = \overline{\iota}^* \overline{\mathcal{F}} \mapsto \overline{\iota}^* \overline{j}_* \overline{j}^* \overline{\mathcal{F}} = (\Psi_\eta(\mathcal{F}_\eta))_{\overline{\eta}}$$

The triple $\Psi(\mathcal{F}) = (\Psi(\mathcal{F})_s, \Psi(\mathcal{F})_\eta, \varphi)$ is a sheaf on $X_s \times_s S$. The functor $\Psi : Sh(X_t) \mapsto Sh(X_s \times_s S)$ is left exact.

The functor $\Phi$

We are interested in cohomology on $X$ which vanishes at $X_s$. To get this to work, we look at complexes of sheaves on $X$, we use derived categories and right derived functors to push the complex onto $X_s$ with $\Phi$, look at the part which vanished on $X_s$, then take cohomology.

$\Phi$ is the functor which extracts the vanishing part. We consider an $s$-scheme $Y$. Let $\Lambda$ be a ring (or sheaf of rings on $Y$). A complex $K$ of $\Lambda$-modules on $Y \times_s S$ is identified with a triple $(K_s, K_\eta, \varphi)$ where

a) $K_s$ (respectively $K_\eta$) is a complex of sheaves $K_{\bar{s}}(K_{\bar{\eta}})$ on $\overline{Y}$ with a continuous, compatible action of $\mathrm{Gal}(\bar{s}/s)(\mathrm{Gal}(\bar{\eta}/\eta))$.

b) $\varphi$ is an equivariant morphism $\varphi : K_{\bar{s}} \mapsto K_{\bar{\eta}}$

Any such complex $K$ is always homotopy equivalent to a complex $K' = (K'_s, K'_\eta, \varphi')$ such that $\varphi'$ is an injective and the exact sequence

$$0 \longrightarrow K'_{\bar{s}} \xrightarrow{\varphi'} K'_{\bar{\eta}} \longrightarrow \mathrm{coker}(\varphi') \longrightarrow 0$$

is split degree by degree.

Define $\Phi(K) = \mathrm{coker}(\varphi')$. This gives a well-defined functor

$$\Phi : D^+(Y \times_s S, \Lambda) \mapsto D^+(Y \times_s \eta, \Lambda)$$

Vanishing Cycles

Let $X$ be a $S$-scheme, as before we have the derived functor

$$R\Psi : D^+(X, \Lambda) \mapsto D^+(X_s \times_s S, \Lambda)$$

and $\Phi : D^+(X_s \times_s S, \Lambda) \mapsto D^+(X_s \times_s \eta, \Lambda)$

We define

$$R\Phi = \Phi \circ R\Psi$$

then $\Phi^q(\mathcal{F}) = H^q(R\Phi(0 \mapsto \mathcal{F} \mapsto 0))$. (ie $\Phi^q : Sh(X) \mapsto Sh(X_s \times_s \eta)$).

**Dougal Davis** *Tuesday 2 October 2012*

Back to Weil *II*

Section 3.1: A calculation of vanishing cycles.

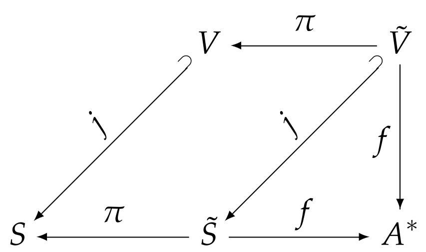
The results of this section will serve to calculate the weights of certain groups of vanishing cycles, modulo integers.

(3.1.1) Let $S$ be a smooth projective surface (dimension 2 variety) over an algebraically closed field $k$, $D$ a divisor with normal crossings on $S$, $V = S - D$, $j$ the inclusion of $V$ in $S$ and $\mathcal{F}$, a sheaf on $V$, moderately ramified along $D$.

We will study the cohomology groups $H^*_c(V, \mathcal{F})$ by the method of Lefschetz pencils (As in Weil 1, section 5), we embed $S$ in a projective space $\mathbb{P}$, and we have a pencil of hyperplanes $(H_t)_{t \in A^*}$.

Notation: $A^*$ is a 2-dimensional subspace of the dual projective space $\mathbb{P}$. The points of $A^*$ parametrize the hyperplanes in $\mathbb{P}$ which contain a particular codimension 2 subspace $A$ of $\mathbb{P}$.

For $t \in A^*$, the hyperplane section $S_t : S \cap H_t$ of $S$. $\tilde{S} \subseteq S \times A^*$ is the space of pairs $(x, t)$ such that $x \in H_t$, $\tilde{V}$ is the inverse image of $V$ in $\tilde{S}$ and the morphisms of projection to the first and second coordinate give the diagram:




The fibres of $f : \tilde{S} \mapsto A^*$ are the $S_t$.

We have the "general position hypothesis" below

A) The axis $A$ is transverse to $S$ and sidjoint from $D$. The space $\tilde{s}$ is therefore smooth, being derived from $S$ by blowup at $A^*$ a finite set of points. None of these points are on $D$, so we can identify $D$ with a divisor in $\tilde{s}$.

B) The only singularities of $f$ are ordinary quadratic points, none of these critical points are on $D$.

C) On the normalisation $D'$of $D$, the only singularities of $f$ are the ordinary quadratic points. None of which are above a singular point of $D$.

   A point of $S$ is called exceptional if it is one of these three types

   a) A critical point of $f$ on $S$
   b) A critical point of $f$ on $D'$
   c) A singular point of $D$

   A fibre $S_t$ is called exceptional if it contains an exceptional point. We also say $t$ is exceptional.

D) Each exceptional fibre contains only one exceptional point. The exceptional fibres are therefore one of these three types:

a) Curve having a double point with distinct tangents
b) $D$ tangent to $S_t$
c) Two branches of $D$ intersecting on $S_t$

The cohomologies of $V$ and $\tilde{V}$ are linked by a canonical isomorphism:

$$H_c^*(\tilde{V}, \pi^*\mathcal{F}) \cong H_c^*(V, \mathcal{F}) \oplus (H^0(V \cap A, \mathcal{F})(-1) \text{ placed in degree 2})$$

For the following, it suffices to know injectivity

$$\pi^* : H_c^*(V, \mathcal{F}) \mapsto H^*(\tilde{V}, \pi^*\mathcal{F})$$

is a retraction of (3.1.1.4).

To study the cohomology of $\tilde{V}$, we use the Leray spectral sequence

$$E_2^{pq} = H^p(A^*, R^2 f_! \pi^*\mathcal{F}) \Rightarrow H_c^{p+q}(\tilde{V}, \pi^*\mathcal{F})$$

The sheaves $R^q f_! \pi^*\mathcal{F} = R^q f_*(j_! \pi^*\mathcal{F})$ are lisse, except at the exceptional values of $f$.

Let $t$ be an exceptional value of $f$, $A^*(t)$ the henselisation of $A^*$ at $t$ (the spectrum of a strictly henselian discrete valuation ring) and $\overline{\eta}$ a generic geometric point of $A^*_{(t)}$. We apply the theory of vanishing cycles to

- the inverse image of $\tilde{S}$ on $A^*_{(t)}$.
- the inverse image of $j_! \pi^*\mathcal{F}$

The sheaves of vanishing cycles $\Phi^q = \Phi^q(j_! \pi^*\mathcal{F})$ are contracted at the exceptional point $x$ of $S_t$.

$$\begin{array}{ccc} \tilde{S} & \longleftarrow & \tilde{S}(x) \\ \downarrow & & \downarrow \\ A^* & \hookleftarrow & A^*_{(t)} \end{array}$$

$\Phi^q$ comes from a nice cokernal of complexes

$$0 \longrightarrow R\Psi(j_! \pi^*\mathcal{F})_t \longrightarrow R\Psi(j_! \pi^*\mathcal{F})_{\overline{\eta}} \longrightarrow R\Psi(j_! \pi^*\mathcal{F}) \longrightarrow 0$$

Since $\Phi^q$ is concentrated at $x$[104], some fiddling with the long exact sequence of cohomology gives a long exact sequence

$$\ldots \xrightarrow{\partial} (R^q f_! \pi^*\mathcal{F})_t \longrightarrow (R^q f_! \pi^*\mathcal{F})_{\overline{\eta}} \longrightarrow \Phi_x^q \xrightarrow{\partial}$$

[104] $\mathcal{G}$ is concentrated at $x$ means $\mathcal{G}(U) =$ something if $x \in U$ and 0 otherwise

We will calculate the $\Phi_x^q$ or really their gradation via a convenient filtration under the following additional hypothesis.

E) The local monodromy of $\mathcal{F}$ and $D$ is unipotent.

Let $d \in D$, $S(d)$ the strict localisation of $S$ at $d$ and $V(d)$ the inverse image of $V$ in $S(d)$. The hypothesis (E) ensures that the inverse image of $\mathcal{F}$ on $V(d)$ admits a finite filtration $\mathcal{F}$ (by lisse subsheaves) such that the sheaves $Gr_F^i(\mathcal{F})$[105] are constant on $V(d)$. Denote by $Gr_F^i(\mathcal{F})_d$ the stalk at $d$ of the constant continuation of $Gr_F^i(\mathcal{F})$ on $S(d)$: this is $H^0(V(d), Gr_F^i(\mathcal{F}))$.

**Dougal Davis** *Tuesday 9 October 2012*

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If $B$ is a set of two elements, $\epsilon(B)$ a group with two opposite isomorphisms with $\mathbb{Z}$. For example, $\wedge^2\mathbb{Z}^B$ or $\mathbb{Z}^B/(\mathbb{Z}$ diagonal).

We have three cases to consider, according to the nature of the exceptional point $x$.

(3.1.3)

a) Since $j_!\pi^*\mathcal{F}$ is lisse at $x$, we have

$$\Phi_c^*(j_!\pi^*\mathcal{F}) = \Phi_x^*(\overline{\mathbb{Q}}_l) \otimes \mathcal{F}_x$$

$\Phi_x^*(\overline{\mathbb{Q}}_l)$ is given by the theory of Picard-Lefshetz (in dimension 1): $\Phi_x^q(\overline{\mathbb{Q}}_l) = 0$ if $q \neq 1$, and if $B$ is the set of two elements of the branches of $\tilde{S}_t$ at $x$, we have $\Phi_x^1(\overline{\mathbb{Q}}_l) = \overline{\mathbb{Q}}_l(-1) \otimes \epsilon(B)$. In total:

$$\begin{aligned} \Phi_x^q(j_!\pi^*\mathcal{F}) &= 0 \text{ for } q \neq 1, \text{ and} \\ \Phi_x^1(j_!\pi^*\mathcal{F}) &\cong \mathcal{F}_x(-1) \otimes \epsilon(B) \end{aligned}$$

b) Assume $\mathcal{F} = \overline{\mathbb{Q}}_l$. We have the short exact sequence

$$0 \longrightarrow j_!\overline{\mathbb{Q}}_l \longrightarrow \overline{\mathbb{Q}}_l \longrightarrow \overline{\mathbb{Q}}_{lD} \longrightarrow 0$$

The groups of vanishing cycles $\Phi_x^*$ are zero for the constant sheaf $\overline{\mathbb{Q}}_l$, since $\mathcal{F}$ is lisse at $x$. For $\overline{\mathbb{Q}}_{lD}$, it coincides with the analogous group calculated on $D$. The long exact sequence for cohomology gives

$$\Phi_x^*(j_!\overline{\mathbb{Q}}_l) \cong \Phi_x^{q-1}(D, \overline{\mathbb{Q}}_l)$$

If $B$ is the set of two points at the hensalisation $D(x)$ above $\overline{\eta}$ we have

$$\begin{aligned} \Phi_x^q(j_!\overline{\mathbb{Q}}_l) &= 0 \text{ if } q \neq 1 \\ \Phi_x^1(j_!\overline{\mathbb{Q}}_l) &= \overline{\mathbb{Q}}_l \otimes \epsilon(B) \end{aligned}$$

In the general case, let $F$ be a filtration as in (3.1.2). We find by devissage that

$$\Phi_x^q(j_!\pi^*\mathcal{F}) = 0 \text{ for } q \neq 1$$

[105] for a filtration $F = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_n = \mathcal{F}$ then $Gr_F^i(\mathcal{F}) = \mathcal{F}_i/\mathcal{F}_{i-1}$.

and that $\Phi_x^1(j_!\pi^*\mathcal{F})$ admits a filtration $F$ for which

$$Gr_F^i \Phi_x^1(j_!\pi^*\mathcal{F}) = Gr_F^i(\mathcal{F}) \otimes \epsilon(B)$$

c) Let $B$ be the set of two elements of the branches of $D$ at $X$. Denote by $\iota$, the projection in $S$ of $\tilde{S}$ at the normilisation $D'$ of $D$. We have in a neighborhood of $x$ an exact sequence

$$0 \longrightarrow j_!\overline{\mathbb{Q}}_l \longrightarrow \overline{\mathbb{Q}}_l \longrightarrow \iota_*\overline{\mathbb{Q}}_l \longrightarrow \overline{\mathbb{Q}}_{lx} \otimes \epsilon(B) \longrightarrow 0$$

Since $f$ is smooth at $x$, and since $f_0\iota$ is smooth at the two points of $D'$ above $x$, the sheaves $\Phi_x^*$ are zero for $\overline{\mathbb{Q}}_l$ and $\iota_*\overline{\mathbb{Q}}_l$. We have the isomorphisms

$$\Phi_x^q(j_!\overline{\mathbb{Q}}_l) = \Phi_x^{q-2}(x, \overline{\mathbb{Q}}_l \otimes \epsilon(B))$$

The $\Phi_x^q(x, \overline{\mathbb{Q}}_l \otimes \epsilon(B))$ are zero for $q \neq -1$, and $\Phi_x^{-1}(x, \overline{\mathbb{Q}}_{lx} \otimes \epsilon(B)) = \overline{\mathbb{Q}}_l \otimes \epsilon(B)$.

This gives the value of $\Phi_x^q(j_!\overline{\mathbb{Q}}_l)$ and, by divissage, that of $\Phi_x^q(j_!\pi^*\mathcal{F})$:

$$\Phi_x^q(j_!\pi^*\mathcal{F}) = 0 \text{ for } q \neq 1$$

and for $F$ as in (b)

$$Gr_F^i \Phi_x^i(j_!\pi^*\mathcal{F}) = Gr_F^i(\mathcal{F})_x \otimes \epsilon(B)$$

**Martina Lanini** *Tuesday 16 October 2012*

Section (1.1) $l$-adic Sheaves.

a) Let $A$ be a Noetherian ring with torsion, $X$ a scheme. $\mathcal{F}$ a sheaf of $A$-modules on $X$ is said to be constructible if there exists a partition $X_i$ of $X$ such that

- $X_i$ is locally closed
- $\mathcal{F}|_{X_i}$ is locally constant

b) Let $R$ be a local Noetherian ring with character $l$, $\mathfrak{m}$ it's maximal ideal. $R$ is complete with respect to the $\mathfrak{m}$-adic topology.

A constructible $R$-sheaf is a projective system of sheaves of $R$-modules $((\mathcal{F}_I)_{I\subset R}, (\rho_{IJ}))$ where $I$ is an open ideal[106] such that

i) $I\mathcal{F}_I = (0)$

ii) for $J \subseteq I$, $\rho_{JI} : \mathcal{F}_J \mapsto \mathcal{F}_I$, which takes $\mathcal{F} \mapsto \mathcal{F} \otimes_{R/J} R/I$

We have a functor from $\mathbb{Z}_{\geq 0}$ to open ideals of $R$ which takes $n \mapsto \mathfrak{m}^n$ and a functor from constructible $R$-sheafs to projective limits of sheafs of $R$-modules.

c) Let $\mathbb{Q}_l \subseteq E \subseteq \overline{\mathbb{Q}}_l$ with $R$ the integral closure of $E$ in $\mathbb{Z}_l$

$\alpha$) Let $\mathcal{F}$ be a constructible $R$-sheaf and take $\mathcal{F} \otimes E$, a constructible $E$-sheaf

$\beta$) We have that $\mathrm{Hom}(\mathcal{F} \otimes E, \mathcal{G} \otimes E) = \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \otimes E$

[106] an ideal that is open with respect to the $\mathfrak{m}$-adic topology

Let $\mathcal{F}$ be an $E$-constructible sheaf, we call it lisse if locally it is $\mathcal{G} \otimes E$ with $\mathcal{G}$ a lisse $R$-sheaf[107]

d) Take $\mathbb{Q}_l = E \subset F \subseteq \overline{\mathbb{Q}}_l$, $\mathcal{F}$ an $R$-constructible sheaf. For iterated extensions we have a canonical isomorphism

$$(\mathcal{F} \otimes_R E) \otimes F \cong \mathcal{F} \otimes_R F$$

and for $\mathcal{F}'$ and $\mathcal{G}'$ $E$-constructible sheaves

$$\mathrm{Hom}(\mathcal{F}' \otimes F, \mathcal{G}' \otimes F) = \mathrm{Hom}(\mathcal{F}', \mathcal{G}') \otimes F$$

The Category of Constructible $\overline{\mathbb{Q}}_l$ Sheaves

For $\mathbb{Q}_l \subseteq E \subset \overline{\mathbb{Q}}_l$ consider the functor that takes $E$-constructible sheaves to $\overline{\mathbb{Q}}_l$-constructible sheaves which induces the isomorphism

$$\mathrm{Hom}(\mathcal{F} \otimes \overline{\mathbb{Q}}_l, \mathcal{G} \otimes \overline{\mathbb{Q}}_l) \cong \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \otimes \overline{\mathbb{Q}}_l$$

A $\overline{\mathbb{Q}}_l$-constructible sheaf is lisse if it is locally $\mathcal{F} \otimes_E \overline{\mathbb{Q}}_l$ with $\mathcal{F}$ lisse.

Here we will use derived categories. Let $\mathbb{Q}_l \subseteq E \subseteq \overline{\mathbb{Q}}_l$ with $E$ a finite extension of $\mathbb{Q}_l$. $R$, the integral closure of $E$ in $\mathbb{Z}_l$, is a local ring with maximal ideal $\mathfrak{m}$. $D^-(X, R)$ the derived category of bounded above complexes. An object $K$ of $D^-(X, R)$ is identified with a projective system $(K_n)_{n \in \mathbb{Z}_{\geq 0}}, (\rho_{n+1,n})$ where $K_n \in \mathrm{Ob}D^-(X, R/\mathfrak{m}^n)$ is a complex with isomorphisms

$$K_{n+1} \otimes_{R/\mathfrak{m}^{n+1}} R/\mathfrak{m}^n \tilde{\mapsto} K_n$$
$$\rho_{n+1,n}(K_n + 1) = K_{n+1} \otimes_{R/\mathfrak{m}^{n+1}} R/\mathfrak{m}^n$$

We have that

$$D^-(X, R) = \varinjlim_n D^-(X, R/\mathfrak{m}^n)$$

If $K \in \mathrm{Ob}D^b_c(X, R)$, an object in the derived category of bounded $R$-constructible sheaves, then

$$H^i K = \varinjlim_n H^i(K \otimes_{R/\mathfrak{m}^{n+1}} R/\mathfrak{m}^n)$$

is an $R$-constructible sheaf.

Also

$$D^b_c(X, \overline{\mathbb{Q}}_l) = \varinjlim D^b_c(X, E)$$

$l$-adic Representations

Let $\pi$ be a profinite group, An $l$-adic representation of $\pi$ is $(V, \sigma)$, where

- $V$ is a $\overline{\mathbb{Q}}_l$-vector space

[107] An $R$-sheaf $\mathcal{F}$ is lisse if $\mathcal{F}_I = \mathcal{F} \otimes R/I$ are locally constant

- $\sigma : \pi \mapsto \mathrm{GL}(V)$ is a group homomorphism such that
  - there exists a finite extension of $\mathbb{Q}_l$ denoted $E$
  - there exists an $E$ structure $V_E$ on $V$ such that the diagram

$$\begin{array}{ccc} \pi & \longrightarrow & \mathrm{GL}(V_E) \\ & \searrow^{\sigma} & \downarrow \\ & & \mathrm{GL}(V) \end{array}$$

    commutes

For $X$ connected, $\overline{x} \in X$ the geometric point, we have a functor from lisse sheaves on $X$ to continuous representations of $\pi_1(X, \overline{x})$ which sends $\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$ the $\pi_1$-module.

**Definition 131:** *A sheaf $\mathcal{F}$ on $X$ is simple/irreducible if*

- $\mathcal{F} \neq 0$
- *For every subsheaf $\mathcal{G}$ of $\mathcal{F}$, either $\mathcal{G} = 0$ or $\mathcal{G} = \mathcal{F}$*

We say $\mathcal{F}$ is semi-simple if $\mathcal{F} = \bigoplus_i \mathcal{F}_i$ with $\mathcal{F}$ simple.

**Definition 132:** *A Jordan-Holder series of $\mathcal{F}$ is a finite filtration*

$$\mathcal{F}_0 \subseteq \ldots \subseteq \mathcal{F}_{i-1} \subseteq \mathcal{F}_i \subseteq \ldots \mathcal{F}_r = \mathcal{F}$$

*such that*

- *$\mathcal{F}_i$ is lisse*
- *$Gr^i(\mathcal{F}) = \mathcal{F}_i / \mathcal{F}_{i-1}$ is simple*
- *$Gr^i(\mathcal{F}) \neq 0$ is constituant of $\mathcal{F}$*
- *$\bigoplus Gr^i(\mathcal{F})$ is semi-simplified of $\mathcal{F}$*

Let $K = \mathbb{F}_q$ with $\overline{K}$ it's algebraic closure, $\varphi \in \mathrm{Gal}(\overline{K}/K)$ the Frobenius automorphism $\varphi(x) = x^q$. Let $F$ be the geometric Frobenius[108], we define the Weil group $W(\overline{K}/K)$ as the subgroup generated by integer powers of $F^i$.

$$W(\overline{K}/K) := < F^i >_{i \in \mathbb{Z}} \subseteq \mathrm{Gal}(\overline{K}/K)$$

We think of $W(\overline{K}/K)$ as isomorphic to $\mathbb{Z}$ by sending $F^i \mapsto i$ and $\mathrm{Gal}(\overline{K}/K) \cong \hat{\mathbb{Z}}$ the profinite completion of $\mathbb{Z}$.

Take $X = \mathrm{Spec}(K)$ and $\mathcal{F}$ a sheaf on $X$. The pullback of $\mathcal{F}$ to $\mathrm{Spec}(\overline{K})$, denoted $\mathcal{F}_{\mathrm{Spec}(\overline{K})}$ is a $\overline{\mathbb{Q}}_l$-module on which $\mathrm{Gal}(\overline{K}, K)$ acts. This gives rise to an equivalence of the categories of sheaves on $\mathrm{Spec}(K)$ and $\overline{\mathbb{Q}}_l$-modules with an automorphism of $F$ with eigenvalues of $l$-adic units.

[108] that is such that $F \circ \varphi = \varphi \circ F = id$

i) A Weil sheaf $\mathcal{F}_0$ on $X_0$[109] is given by

- a sheaf $\mathcal{F}$ on $X$
- an action of $W(\mathbb{F}, \mathbb{F}_q)$ on $(X, \mathcal{F})$

ii) Let $\overline{x} \in X$ be a geometric point of $X$, $W(X_0, \overline{x}) = \pi_1(X_0, \overline{x}) \times_{\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)} W(\mathbb{F}/\mathbb{F}_q)$ with

$$\pi_1(X_0, \overline{x}) \mapsto \pi_1(\mathrm{Spec}(\mathbb{F}_q), \overline{x}) = \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q) \supset W(\mathbb{F}/\mathbb{F}_q)$$

iii) An automorphism of a Weil sheaf $(X, \mathcal{F})$ is given by $(f, s)$, where

- $f : X \mapsto X$ is an automorphism
- $g : f_* \mathcal{F} \tilde{\mapsto} \mathcal{F}$

Define: $\pi_1(X_0, \overline{x})$ as the arithmetic fundamental group

$$\ker(\pi_1(X_0, \overline{x}) \mapsto \pi_1(\mathrm{Spec}(\mathbb{F}_q), \overline{x}))$$

is the geometric fundamental group $\pi_1(X, \overline{x})$

**Martina Lanini** *Tuesday 23 October 2012*

Let's recall the set up from last time. We have $X_0$ a scheme over $\mathbb{F}_q$, $\overline{x} \in X_0$ the geometric point. $\mathbb{F} = \overline{\mathbb{F}}_q$ the algebraic closure of $\mathbb{F}_q$, $X = X_0 \otimes \mathbb{F}$, then the Weil group is

$$W(X_0, \overline{x}) = \pi_1(X_0, \overline{x}) \times_{\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)} W(\mathbb{F}/\mathbb{F}_q)$$

If $\tilde{X}$ is the universal cover of $X_0$, we can lift the action of $W(X_0, \overline{x})$[110] to $\tilde{X}$. We have an equivariant morphism from $\tilde{X} \to X$ with respect to the natural morphism $W(X_0, \overline{x}) \to W(\mathbb{F}/\mathbb{F}_q)$.

Let $\mathcal{F}_0$ be a lisse $\overline{\mathbb{Q}}_l$-constructible Weil sheaf on $X_0$. It is enough to look at the fibre at $\overline{x}$ then we have a continuous action of $W(X_0, \overline{x})$. So we have a functor from lisse Weil sheaves on $X_0$ to $\overline{\mathbb{Q}}_l$ representations of $W(X_0, \overline{x})$. In the case that $X_0$ is connected, we have an equivalence of categories.

Let $\mathcal{F}$ be a sheaf over $X$. If $F : X \to X$ is the Frobenius endomorphism of $X$ deduced by extension of scalars from the Frobenius endomorphism $F : X_0 \to X_0$, $x \mapsto x^q$, then

$$id_{X_0} \times F : X \to X$$

is the geometric Frobenius. By (SGA 5,XV) we have

$$(id_{X_0} \times F)^* \mathcal{F} \cong F^* \mathcal{F}$$

If $\mathcal{F}_0$ is a Weil sheaf on $X_0$[111] then we have $F^* \mathcal{F} \cong \mathcal{F}$.

So we have the functor $\mathcal{F}_0 \mapsto (\mathcal{F}, F^*)$ which is an equivalence of categories between Weil sheaves on $X_0$ and sheaves on $X$ with the isomorphism $F^* \mathcal{F} \cong \mathcal{F}$.

[109] $X_0 = X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$

[110] $W(X_0, \overline{X})$ acts trivially on $X_0$ as a subgroup of $\pi_1(X_0, \overline{x})$

[111] So $(id_{X_0} \times F)^* \mathcal{F} \cong \mathcal{F}$

Weights

Let $q = p^n$ for $p$ prime.

**Definition 133:** *A number $\alpha$ is pure of weight $n$ if it is algebraic and all it's complex conjugates have absolute value $q^{\frac{n}{2}}$.*

Let $X$ be a scheme of finite type over $\mathbb{Z}$, $\mathcal{F}$ a sheaf over $X$.

i) $\mathcal{F}$ is is pointwise pure of weight $m$ if for $x \in |X|$, the eigenvalues of $F_x$[112] are pure of weight $n$ with respect to $N(x)$

ii) $\mathcal{F}$ is mixte if it admits a finite filtration

$$0 = \mathcal{F}^{(0)} \subset \mathcal{F}^{(1)} \subset \ldots \subset \mathcal{F}^{(n)} = \mathcal{F}$$

where $\mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}$ is pointwise pure. The weights of the nonzero quotients are the weights of $\mathcal{F}$.

**Example 134:** *Take $\mathcal{F}$ the 0-sheaf, which is pointwise pure of weight $n \in \mathbb{Z}$. It is mixte, with weight set the empty set.*

Stability of Weights

i) The category of pointwise pure sheaves of weight $n$ is stable under
   - taking quotients
   - taking subsheaves
   - extension of scalars
   - under pullback $(f : Y \to X)$
   - under pushforward for $f$ finite[113]

ii) For $\mathcal{F}$, $\mathcal{G}$ pointwise pure of weight $n, m$ respectively, $\mathcal{F} \otimes \mathcal{G}$ is pointwise pure of weight $n + m$. If $\mathcal{F}$, pointwise pure of weight $n$ is lisse then it's dual $\check{\mathcal{F}}$ is pointwise pure of weight $-n$.

iii) The category of mixte sheaves is stable under the operations in (i), and under tensor products

iv) $\overline{\mathbb{Q}}_l(1)$ is pure of weight -2[114]

**Definition 135:** *An l-adic unit $\alpha \in \overline{\mathbb{Q}}_l^*$ is pure if weight $n$ if every embedding $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$, $\alpha \mapsto \iota\alpha$ has absolute value $|\iota\alpha| = q^{\frac{n}{2}}$*

More specifically, $\alpha \in \overline{\mathbb{Q}}_l^*$ is $\iota$-pure of weight $n$ with respect to $q$ if

[112] $F_x \in \mathrm{Gal}(\overline{k(x)}/k(x))$ is an endomorphism of $\mathcal{F}_x$

[113] $f : Y \to X$ is finite iff there is an open cover $V_i = \mathrm{Spec}(A_i)$ of $X$ such that $f^{-1}(V_i) = U_i = \mathrm{Spec}(B_i)$ is an open affine subscheme where the restriction of $f$ to $U_i$ makes $A_i$ a finitely generated $B_i$-module.

[114] $\overline{\mathbb{Q}}_l(1) \cong \overline{\mathbb{Q}}_l$ as a $\overline{\mathbb{Q}}_l$-vector space and $F$ acts by multiplication by $q^{-1}$

- $q = p^r$ is a power of a prime and
- the weight of $\alpha$, $w_q(\alpha) := 2\log_q |\iota\alpha| = n$

**Definition 136:** *For $\beta \in \mathbb{R}$, a sheaf $\mathcal{F}$ is pointwise $\iota$-pure of weight $\beta$ if for $x \in |X|$ every eigenvalue $\alpha$ of $F_x$ acting on $\mathcal{F}$ is $\iota$-pure of weight $\beta$.*

A sheaf $\mathcal{F}$ on $X$ is $\iota$-mixte if there is a finite filtration

$$0 \subset \mathcal{F}^{(1)} \subset \mathcal{F}^{(2)} \subset \ldots \mathcal{F}^{(m)} = \mathcal{F}$$

where $\mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}$ is pointwise $\iota$-pure and the weights of $\mathcal{F}$ are the weights of the non-zero $\mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}$.

Twisting

For $b \in \overline{\mathbb{Q}}_l^*$, let us twist by $\overline{\mathbb{Q}}_l^{(b)}$ any Weil sheaf of rank 1, on which $F$ acts by multiplication by $b$. If $\mathcal{F}$ is a Weil sheaf on $X$ is pointwise pure of weight $n$, then $\mathcal{F}^{(b)} := \mathcal{F} \otimes \overline{\mathbb{Q}}_l^{(b)}$ has weight $n + 2\log_q |\iota b|$, deduced via twisting from $\mathcal{F}$.

Any $\iota$-mixte sheaf is a direct sum of sheaves deduced via twisting which are mixte of integer weights.

Any lisse sheaf on $X$ (normal, connected, of finite type over $\mathbb{F}_q$) is deduced via twisting from a sheaf whose determinant is defined by a character of finite order of $\pi_1(X)$.

**Trithang Tran** *Tuesday 30 October 2012*

Autour de Jacobson-Morosov

Reminder: $\mathfrak{sl}_2(\mathbb{C})$ is the lie algebra generated by

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

with

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H$$

**Theorem 137:** *Given a nilpotent element $e \in \mathfrak{g}$ in a semi-simple lie algebra, then there exists elements $h, f \in \mathfrak{g}$ satisfying the relations above.*

Let $V$ be an object in an abelian category (such as a vector space)

**Proposition 138:** *If $N : V \to V$ is a nilpotent morphism, then there exists a unique finite increasing filtration $M$ of $V$ such that*

*1) $NM_i \subset M_{i-2}$*

*2) $N^k$ induces an isomorphism $Gr^k_M(V) \tilde{\to} Gr^{-k}_M(V)$*

*Proof.* induct on $d$, where $N^{d+1} = 0$.

Base cases: If $d = 0$ then $N = 0$, and

$$M_i = \begin{cases} V, & i \geq 0 \\ 0, & i < 0 \end{cases}$$

inductive step: Suppose $N^{d+1} = 0$, take $M_d = V$, $M_{d-1} = \ker N^d$, $M_{-d} = \operatorname{im} N^d$ and $M_{d-1} = 0$[115].

Now apply induction in $\ker N^d / \operatorname{im} N^d$[116]. We have a filtration of $\ker N^d / \operatorname{im} N^d$.

$$0 = L_{-(d-1)} \subset \ldots \subset L_{d-1} = \ker N^d / \operatorname{im} N^d$$

Then $M_i$ for $-d < i \leq d-1$ is the preimage in $\ker N^d$ of this filtration. □

**Definition 139:** *The primitive $P_i(V)$ of $Gr^i_M(V)$ is* $\ker(N : Gr^i_M \to Gr^{i-2}_M)$

Claim:

$$Gr^i_M \cong \bigoplus_{\substack{j \geq 1 \\ j \equiv i \mod 2}} P_{-j}$$

If $i > 0$, $N : Gr^i_M \to Gr^{i-2}_M$ is injective and $N^{i-1} \circ N$ is an isomorphism then $P_j = 0$.

If $i \geq 0$, write $N \circ N^{i+1} : Gr^{i+2}_M \to Gr^{-i}_M \to Gr^{-i-2}_M$. This composition is an isomorphism.

We get

$$\begin{aligned} Gr^{-i}_M &\cong P_j \oplus \operatorname{im}(Gr^{i+2}_M) \\ &\cong P_j \oplus Gr^{-i-2}_M \end{aligned}$$

Repeating gives

$$Gr^i_M \cong \bigoplus_{\substack{j \geq i \\ j \equiv i \mod 2}} P_{-j}$$

**Lemma 140:** $N : (V, M) \to (V, M$ *shifter by 2) is strictly compatible with the filtrations.*

*Proof.* To show: $N \cap M_i \subset NM_{i+2}$

two cases: If $i < 0$, then the morphism $N : M_{i+2} \to M_i$ is graded surjective, this implies $N$ is surjective, so $NM_{i+2} = M_i$.

If $i > 0$ the morphism $N : V/M_{i+2} \to V/M_i$ is graded surjective, so is injective and $N^{-1}M_i \subset M_{i+2}$.

[115] $Gr^d_M = V/\ker N^d$ and $Gr^{-d}_M = \operatorname{im} N^d$

[116] Since $N$ is nilpotent on $\ker N^d / \operatorname{im} N^d$ of order $d$

The grading on $V/M_{i+2}$ comes from the grading on $V$.

$$M_{i+4}(V/M_{i+2}) = M_{i+4}(V)/M_{i+2}$$

So

$$Gr_M^{i+4} = \frac{M_{i+4}(V/M_{i+2})}{M_{i+3}(V/M_{i+2})} = M_{i+4}/M_{i+3} = Gr_M^{i+4}(V)$$

□

**Corollary 141:** *The inclusion of* ker$N \to V$ *induces the isomorphisms.*

$$Gr_M^i(\ker N) \tilde{\to} P_i$$

Suppose now that $V$ is a finite dimensional vector space over $k$. We begin by describing $M$, when we have a basis $\underline{e}$.

$$N = \begin{pmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & 0 \\ 0 & \cdots & 1 \end{pmatrix}$$

Suppose dim$V = d+1$, label basis elements $e_d, e_{d-2}, \ldots, e_{-d}$ then $M_i = \langle e_j | j \leq i \rangle$.

In general, $V$ is a sum of subspaces $V_\alpha$ invariant under $N$.

When $N$ looks like $\begin{pmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & 0 \\ 0 & \cdots & 1 \end{pmatrix}$

1.6.8 (Jacobson-Morosov) If $k$ is of characteristic 0, we can interpret $M$ in terms of (Jacobson-Morosov).

Let $u : \mathrm{SL}(2) \to \mathrm{GL}(V)$

$$du \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = N$$

And $V_j$ is the subspace of $V$ formed from the vectors such that $u \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} v = \lambda^j v$.

Then $M_i$ is the sum of the $V_j$ for $j \leq i$.

<u>Rest of 1.6</u>

- Define tensor products of $V$ and $N$
- Define duals
- Say that the filtration we get behave nicely
- Suppose we have another filtration $W$ of $V$
- Construct a similar $M$
  With $NM_i \subset M_{i-2}$ such that $N^k = Gr_M^{i+k}(Gr_W^i V) \tilde{\to} Gr_M^{i-k}(Gr_W^i V)$

Define $(V, N) = (V', N') \otimes (V'', N'')$ with $V = V' \otimes V''$, $N = N' \otimes 1 + 1 \otimes N''$

The dual of $(V, N)$ is $(V^*, .^T N)$

1) If $k$ is of characteristic 0, the filtration $M$ of the tensor product is

$$M_i(V' \otimes V'') = \sum_{i'+i''=1} M_{i'}(V') \otimes M_{i''}(V'')$$

2) The filtration $M$ of a dual is the dual filtration $M$ of the space

$$M_i(V*) = M_{-i-1}(V)^{\perp}$$

3) $Gr_M$ is compatible with tensor products and dual.

**Arun Ram** *Tuesday 20 November 2012*

My job is to talk about §3.2 and §3.3. §3.3 has the main theorem and §3.2 has the main theorem leading to the *real* main theorem of §3.3.

**Proposition 142:** *(3.2.1) Let $X_0$ be a smooth absolutely irreducible curve over $\mathbb{F}_q$ and $\mathcal{F}_0$ a smooth pointwise ι-pure and ι-real sheaf on $X_0$.*

*Then the polynomials*

$$\iota\det(1 - Ft, H^i_c(X, \mathcal{F}))$$

*have real coefficients.*

**Remark:**: The above proposition is valid under slightly weaker hypotheses.

**Theorem 143:** *Let $X_0$ be a projective smooth curve over $\mathbb{F}_q$, $j : U_0 \hookrightarrow X_0$ an open dense subset of $X_0$, $\mathcal{F}_0$ a smooth pointwise ι-pure sheaf of weight $\beta$ on $U_0$. Then the eigenvalues $\alpha$ of $F$ on $H^i(X, j_*\mathcal{F})$ satisfy $w_q(\alpha) = \beta + i$.*

Deligne says "For a description of the main line of the proof, I refer to the introduction"

Main theorem of (3.3.1)

**Theorem 144:** [117]*(3.3.1) Let $f : X_0 \to S_0$ be a morphism of schemes of finite type on $\mathbb{F}_q$, $\mathcal{F}_0$ a mixed sheaf of weights $\leq n$ on $X$. For each $i$, the sheaf $R^i f_! \mathcal{F}_0$ on $S_0$ is mixed of weights $\leq n + i$.*

Introduction

In [1], we have proved the conjecture of Weil giving the complex absolute value of the eigenvalues of the Frobenius acting on the cohomology of a projective smooth

[117] In the proof of the Kazhdan-Lusztig conjectures by Beilinson-Bernstein they say "The proof follows the same lines as the proof of theorem (3.3.1) in [Deligne].

variety[118] defined over a finite field.

Here we study the sheaf valued cohomology; this amounts to the transfer of pointwise properties of a sheaf to properties of it's cohomology.

Let $X_0$ be a scheme of finite type on $\mathbb{F}_q$ and $\mathcal{F}_0$ a $\overline{\mathbb{Q}}_l$-sheaf on $X_0$. We assume a a fixed algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$ and we indicate by suppression of the index the extension of the base field from $\mathbb{F}_q$ to $\mathbb{F}$(cf. (0.7)).

For $x_0 \in |X_0|$ a closed point of $X_0$ and $x \in X(\mathbb{F})$ above it we have available the Frobenius automorphism $F^*_{x_0}$ of the fibre $\mathcal{F}_x$ of $\mathcal{F}$ at $x$.

We say that $\mathcal{F}_0$ is pointwise pure of weight $n$ if, for each $x_0 \in |X_0|$, the eigenvalues of $F_{x_0}$ on $\mathcal{F}_0$ are algebraic numbers all of whose complex conjugates have absolute value $N(x_0)^{\frac{n}{2}}$.

We say that $\mathcal{F}_0$ is mixed, if it is an iterated extension of pointwise pure sheaves.

The weights of these are the weights of $\mathcal{F}_0$.

For $S_0 = \mathrm{Spec}(\mathbb{F}_q)$, theorem (3.3.1) says that, for each eigenvalue $\alpha$ of the Frobenius on $H^i_c(X, \mathcal{F})$ there exists an integer $m \leq n + i$ (the weight of $\alpha$) such that the complex conjugates of $\alpha$ are all of absolute value $q^{\frac{n}{2}}$.

The Poincare duality then allows for the conversion of the upper bounds to lower bounds. (3.3.5) For example if $X_0$ is proper, smooth and the sheaf $\mathcal{F}_0$ is smooth[119] and pointwise pure[120] of weights $n$, then the eigenvalues of Frobenius on $H^i(X, \mathcal{F})$ are all of weights $n + i$, which we will indicate by saying that $H^i(X, \mathcal{F})$ is pointwise pure of weights $n + i$.

For $\mathcal{F}_0 = \mathbb{Q}_l$ (of weights 0) we recover the principal results of [1][121].

An easy introduction, parallel to the proof of the finiteness theorem for the $R^i f_!$(cf. SGA 4, XIV,1) converts theorem 1 to the following theorem, and to a local study at infinity of the smooth pointwise pure sheaves on a curve( C) below).

**Theorem 145:** *(cf. (3.2.3)) Let $X_0$ be a proper and smooth curve over $\mathbb{F}_q$. $j : U_0 \to X_0$ the inclusion of an open dense subset, $\mathcal{F}_0$ a pointwise pure sheaf of weight $n$ on $U_0$. Then $H^i(X, j_*\mathcal{F})$ is pure of weight $n + i$.*

The following are the outlines of the proof.

A) Nettoyage

[118] It helps to remember that by *projective* we mean compact, by *smooth* we mean the tangent space behaves how we expect it to and a variety is a set of solutions to equations

[119] locally constant, in alternate terminology

[120] same eigenvalues

[121] Weil 1

i) If $u : X_0' \to X_0$ is a finite surjective morphism from a proper smooth curve $X_0'$ and we designate by $'$ the change of base by $n$, the $H^i(X, j_*\mathcal{F})$ are the direct factors of the $H^i(X', j_*'\mathcal{F}')$. This argument allows one to reduce to the case where the local monodromy of $\mathcal{F}$ at the points $X - U$ is unipotent.

ii) A duality assures that $H^i(X, j_*\mathcal{F})$ and $H^{2-i}(X, j_*(\check{\mathcal{F}}))$ are in perfect duality with values in $\overline{\mathbb{Q}}_l(-1)$.

$$\langle , \rangle : H^i(X, j_*\mathcal{F}) \times H^{2-i}(X, j_*\check{\mathcal{F}}) \to \overline{\mathbb{Q}}_l$$

This reduces us to verifying that the complex conjugates $\alpha'$ of the eigenvalues $\alpha$ of the Frobenius on $H^i(X, j_*\mathcal{F})$ are of absolute value $|\alpha'| \leq q^{\frac{(n+i)}{2}}$. The difficult case is that of $H^1$.

**Arun Ram** *Tuesday 27 November 2012*

B) Complex embeddings.

Let $\alpha \in \overline{\mathbb{Q}}_l$ be an eigenvalue of the Frobenius on $H^i(X, j_*\mathcal{F})$. It is convenient to reformulate the estimates to be verified: for each isomorphism $\iota : \overline{\mathbb{Q}}_l \to \mathbb{C}$ we have $|\alpha| \leq q^{\frac{n+i}{2}}$.

In the proof each isomorphism $\iota$ will be treated separately; this motivates the introduction of notions of pointwise $\iota$-pure and $\iota$-mixed sheaves. It is also convenient to talk about mixed weights that are $\iota$-real. We will prove theorem 2 with pure replaced by $\iota$-pure. We refer to (1.2.11) the reader who, like the author, avoids the axiom of choice implicit in the usage of the isomorphisms between $\overline{\mathbb{Q}}_l$ and $\mathbb{C}$.

C) Local monodromy of $\iota$-pure sheaves:

Put $S_0 = X_0 - U_0$. We begin by showing that if $\mathcal{F}_0$ is smooth and pointwise $\iota$-pure of weight $\beta \in \mathbb{R}$, the weights $w_{N(x_0)}(\alpha) = 2\log_{N(x_0)} |\iota\alpha|$ of an eigenvalue $\alpha$ of $F_{x_0}$ on $j_*\mathcal{F}_0$ for $x_0 \in S_0$, is of the form $\beta - m$, with $m$ integral, positive and we determine $m$ in terms of the local monodromy of $\mathcal{F}_0$ at $x_0$ (1.8.4). More generally we determine $w_{N(x_0)}(\alpha)$ for an eigenvalue $\alpha$ of $F_{x_0}$ on $\mathcal{F}_0$ in the sense of (1.10.2).

First step: show that $w_{N(x_0)}(\alpha) \leq \beta + 2$.

We do this by exploiting the Grothendieck formula[122] for the function $Z(U_0, \mathcal{F}_0, t)$: applying $\iota$ we find that the left hand side is an infinite product converging to $w_q(t) < -\beta - 2$, the right hand side is a rational fraction with numerator $\iota\det(1 - Ft, H_c^1(U, \mathcal{F}))$, and we use the fact that the $(j_*\mathcal{F})_x$ for $x \in S$ contribute to $H_c^1(U, \mathcal{F})$.

Second step: Apply this result to tensor powers of $\mathcal{F}_0$ and their dual, keeping track of the local monodromy. Having obtained this, it is advisable and convenient, to study $j_!\mathcal{F}_0$ and $j_*\mathcal{F}_0$: if $\mathcal{F}_0$ is smooth and pointwise pure of weight $\beta \in \mathbb{R}$ then we show that the eigenvalues $\alpha$ of Frobenius on $H_c^1(U, \mathcal{F}) = H^1(X, j_!\mathcal{F})$ are of weights $w_q(\alpha) \leq \beta + 1$. We will assume, to simplify the exposition that $\beta = 0$. We arrive at this case by torsion (1.2.7).

[122] number of $F$ fixed points on $X$ is $|X|^F = \sum_i (-1)^i \mathrm{Tr}(F, H^i(X))$

D) Passage to $X_0 \times X_0$. The principal geometric idea is to pass from $(U_0, \mathcal{F}_0)$ to $(U_0 \times U_0, \mathcal{F}_0 \boxtimes \mathcal{F}_0)$[123] and to analyze

$$H^1_c(U \times U, \mathcal{F} \boxtimes \mathcal{F}) = H^*_c(X \times X, j_1\mathcal{F} \boxtimes j_!\mathcal{F})$$

with the help of a pencil of hyperplanes section of $X_0 \times X_0$, assumed to be in general position. This is via a convenient projective embedding of $X_0 \times X_0$.

Eigenvalues of Frobenius on $H^2(X \times X, j_!\mathcal{F} \boxtimes j_!\mathcal{F})$: Show $w_q(\alpha) \leq 3 = 2 + 1$. The Kunneth formula assures that $\alpha^2$ is an eigenvalue of the Frobenius on $H^2(X \times X, j_!\mathcal{F} \boxtimes j_!\mathcal{F})$, which gives $w_q(\alpha) \leq 1 + \frac{1}{2}$.

In Weil 1 there was a section: "The fundamental bound".

We take a pencil of sections (very general) of hyperplanes of $X_0$, $\mathcal{G}_0$ is the inverse image of $j_!\mathcal{F} \boxtimes j_!\mathcal{F}_0$ in $Y_0$ where the hyperplane sections are fibres of a morphism $f : Y_0 \to \mathbb{P}^1$ where $Y_0$ is $X_0 \times X_0$ union a finite number of points. The Leray spectral sequence for $f$, reduces the study of $H^2(X \times X, j_!\mathcal{F} \otimes j_!\mathcal{F})$ to the study of $H^1(\mathbb{P}^1, R^12f_*\mathcal{G})$.

How should one calculate the weights of $R^1f_*\mathcal{G}_0|_{V_0}$? ($V_0$ is a certain open of $\mathbb{P}^1_0$)

In the application of (1.6.3) and (1.3.2) it's hypotheses come from the theory of Lefschetz pencils, particularly from the theorem of conjugation of vanishing cycles. Here $R^1f_*\mathcal{G}_0$ admits ramification points of 3 distinct geometric types.

**Dougal Davis and Yi Huang** *Tuesday 11 December 2012*

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Examples

Examples of Lisse constructible $\mathbb{Z}_l/\mathbb{Q}_l/\overline{\mathbb{Q}}_l$-sheafs:

Recall: A lisse constructible $\mathbb{Z}_l$-sheaf $\mathcal{F}$ on a scheme $X$ is a projective system

$$(\mathcal{F}_n)_{n \in \mathbb{Z}_{>0}}$$

such that $\mathcal{F}_n$ is a locally constant etale sheaf of $\mathbb{Z}/l^n\mathbb{Z}$- modules on $X$, and the maps

$$\mathcal{F}_n \to \mathcal{F}_m, \quad n \geq m$$

induce an isomorphism

$$\mathcal{F}_n \otimes_{\mathbb{Z}/l^n\mathbb{Z}} \mathbb{Z}/l^n\mathbb{Z} \tilde{\to} \mathcal{F}_m$$

Examples:

- $\mathcal{F}_n =$ constant $\mathbb{Z}/l^n\mathbb{Z}$- sheaf Then $\mathcal{F}$ is called the constant $\mathbb{Z}_l$-sheaf on $X$.
$$\mathcal{F} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l = \underline{\mathbb{Q}_l}$$
is the constant $\mathbb{Q}_l$-sheaf.
- $X = \mathrm{Spec}(\mathbb{F}_q)$
$$\mathcal{F}_n^{(U)} = \mathbb{Z}/l^n\mathbb{Z}\text{-span } (\mathrm{Hom}_X(U, \mathrm{Spec}(\mathbb{F}_{q^2})))$$

[123] where $\mathcal{F}_0 \boxtimes \mathcal{F}_0 = pr_1^*\mathcal{F}_0 \otimes pr_2^*\mathcal{F}_0$

$$\begin{array}{ccc} U & \longrightarrow & \mathrm{Spec}(\mathbb{F}_{q^2}) \\ \downarrow & \swarrow & \\ X = \mathrm{Spec}(\mathbb{F}_q) & & \end{array}$$

In general, $U \in \mathrm{Ob}(\text{Ét}(X))$ is of the form

$$U = \mathrm{Spec}(\mathbb{F}_{q^{k_1}}) \sqcup \mathrm{Spec}(\mathbb{F}_{q^{k_2}}) \sqcup \ldots \sqcup \mathrm{Spec}(\mathbb{F}_{q^{k_n}})$$

$$\mathrm{Spec}(B) \to \mathrm{Spec}(A)$$
$$A \to B$$
$$B = \frac{A[t_1, \ldots, t_n]}{(f_1, \ldots, f_n)}$$

Because of the gluing conditions for sheaves the behavior of $\mathcal{F}$ is captured by $\mathrm{Spec}(\mathbb{F}_{q^k})$.

In the case of $U = \mathrm{Spec}(\mathbb{F}_{q^k}) \to \mathrm{Spec}(\mathbb{F}_q)$

$$\begin{aligned} \mathcal{F}_n(\mathrm{Spec}(\mathbb{F}_{q^k})) &= \mathbb{Z}/l^n\mathbb{Z}\text{ -span}(\mathrm{Hom}_{\mathbb{F}_q}(\mathbb{F}_{q^2}, \mathbb{F}_{q^k})) \\ &= \begin{cases} (\mathbb{Z}/l^n\mathbb{Z})^2 \text{ if } 2 \mid k \\ 0 \text{ otherwise} \end{cases} \end{aligned}$$

$\mathcal{F}_n|_{\mathrm{Spec}(\mathbb{F}_{q^2})} = \mathbb{Z}/l^n\mathbb{Z}$

Algebraic/étale fundamental groups

Recall: The algebraic fundamental group of a scheme $X$ with a geometric point $\overline{x}$, denoted $\pi_1(X, \overline{x})$ is defined as follows.

Let FÉT/X be the category of finite étale covers of $X$.

The algebraic fundamental group is

$\pi_1(X, \overline{x}) = \mathrm{Aut}(F)$ where $F$ :FÉT/X$\to$ {finite sets} is the functor given by $F(Y) =$ { geometric points of $Y$ above $\overline{x}$}.

If we were working in topology, the universal cover $\tilde{X}$ of $X$ would represent $F$, ie $F(.) \cong \mathrm{Hom}_X(\tilde{X}, .)$.

In the algebraic category, the "universal cover" of $X$ is a projective system $\tilde{X} = (P_i)_{i \in I}$ in FÉT/X with $F(.) \cong \varinjlim_{i \in I} \mathrm{Hom}(P_i, .)$.

If we choose the $P_i$ to be "nice"[124] then we get a projective system of finite groups

[124] analogous to Galois extensions

$(\mathrm{Aut}_X(P_i))_{i\in I}$ with
$$\pi_1(X,\overline{x}) = \varprojlim_i \mathrm{Aut}_X(P_i)$$

Examples

$X = \mathrm{Spec}(\mathbb{F}_q)$, $\overline{x} : \mathrm{Spec}(\overline{\mathbb{F}_q}) \to \mathrm{Spec}(\mathbb{F}_q)$. Then finite étale covers of $X$ are disjoint unions of schemes of the form $\mathrm{Spec}(\mathbb{F}_{q^n})$. Take $P_n = \mathrm{Spec}(\mathbb{F}_{q^n})$, $n \in \mathbb{Z}_{>0}$ then $(P_n)$ is nice, so
$$\begin{aligned} \pi_1(X,\overline{x}) &= \varprojlim_n \mathrm{Aut}_X(P_n) \\ &= \varprojlim_n \mathrm{Aut}_{\mathbb{F}_q}(\overline{\mathbb{F}_{q^n}}) = \varprojlim_n \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \\ &= \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \cong \hat{\mathbb{Z}} \end{aligned}$$

This acts on the stalk of a lisse $\mathbb{Q}_l$-sheaf on $X$.

Eg $\mathbb{Z}_l$-sheaf $\mathcal{F} = (\mathcal{F}_n)$ with $\mathcal{F}_n(.) = \mathbb{Z}/l^n\mathbb{Z}$ -span( Hom(.,Spec($\mathbb{F}_q$)) )
$$\mathcal{F}_n(\mathrm{Spec}(\mathbb{F}_{q^m})) = \begin{cases} (\mathbb{Z}/l^n\mathbb{Z})^2 \text{ if } 2 \mid m \\ 0 \text{ otherwise} \end{cases}$$

stalk: $\mathcal{F}_{\overline{x}} = \mathbb{Q}_l^2$ with $F$ acting by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$X = \mathrm{Spec}(\mathbb{F}_q[t,t^{-1}]) = \mathbb{F}_q^*$

$$\begin{aligned} \overline{x} : \mathrm{Spec}(\overline{\mathbb{F}_q}) &\to X \\ 1 &\leftarrow\!\shortmid t \end{aligned}$$

Some connected finite étale covers $Y_{m,n} = \mathrm{Spec}(\mathbb{F}_{q^n}[s,s^{-1}])$, $X = \mathrm{Spec}(\mathbb{F}_q[t,t^{-1}])$
$$\begin{aligned} \varphi_{m,n} : Y_{m,n} &\to X \text{ where } p \nmid m \\ t^m &\leftarrow\!\shortmid t \\ \mathbb{F}_{q^n} \hookleftarrow \mathbb{F}_q & \end{aligned}$$

Guess: $(Y_{m,n})_{\substack{m,n\in\mathbb{Z}_{>0} \\ p\nmid m}}$ is good enough to compute $\pi_1(X,\overline{x})$.

$\mathrm{Aut}_X(Y_{m,n}) \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$

So
$$\pi_1(X,\overline{x}) = \varprojlim_{\substack{m,n \\ p\nmid m}} \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} = \Pi_{l\neq p}\mathbb{Z}_l \times \hat{\mathbb{Z}}$$

Etale sites

Instead of building spaces from open balls, we instead build from pre-assembled pieces.

Etale morphism: 2 ways of building étale morphisms

1) Extend your field $\mathbb{F}_{q^k} \hookrightarrow \mathbb{F}_{q^{km}}$ (fattening your points).

2) Finite covers of open set in your variety (scheme).

In cohomology, we ignore 1), and care about 2) because that's where topology is involved.

Example of étale morphism.

$$X = \mathrm{Spec}(\overline{\mathbb{F}}_q[t]) = \overline{\mathbb{F}}_q.$$

The inclusion morphism of a Zariski open set $\iota : U \to X$ is étale. Take $U = \overline{\mathbb{F}}_q^* = \overline{\mathbb{F}}_q - \{0\}$, $U = \mathrm{Spec}(\overline{\mathbb{F}}_q[s, s^{-1}])$. Then

$$\begin{array}{rcl} \bar{\iota} : \overline{\mathbb{F}}_q[t] & \to & \overline{\mathbb{F}}_q[s, s^{-1}] \\ t & \mapsto & s \end{array}$$

is an étale morphism, but not finite.

Heuristic: non-finite étale morphisms are from (finite covers) of open sets in $X$ that are missing points (subramifies).

$$\begin{array}{c} \overline{\mathbb{F}}_q^* \\ \big\downarrow \\ \mathbb{F}_q^* \end{array}$$

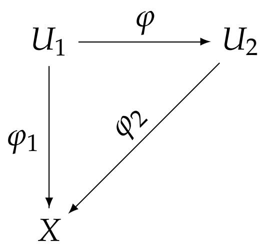
$$\begin{array}{rcl} \mathrm{Spec}(\overline{\rho_n}) = \rho_n : (\overline{\mathbb{F}}_q^*)_1 & \to & (\overline{\mathbb{F}}_q^*)_z \\ z & \mapsto & z^n \end{array}$$

(n-fold cover)

$$\begin{array}{rcl} \overline{\rho_n} : (\overline{\mathbb{F}}_q[s, s^{-1}])_z & \to & (\overline{\mathbb{F}}_q[t, t^{-1}])_1 \\ s & \mapsto & t^n \\ \langle s - z \rangle & \mapsto & \langle t^n - z \rangle = \Pi_{\alpha^n = z} \langle t - \alpha \rangle \end{array}$$

Etale topology:

open sets are these finite cover[125] of subsets of $X$. Morphisms are by inclusions



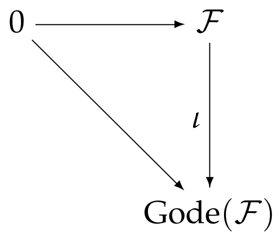

Sheaf cohomology - acyclic resolutions.

$X$, $\mathcal{F}$ a sheaf on $X$ (étale or otherwise).

Define: $\text{Gode}(\mathcal{F})$ to be the following sheaf:

$$\text{Gode}(\mathcal{F})(U) := \Pi_{u \in U}(\mathcal{F})_u$$

From the following diagram:

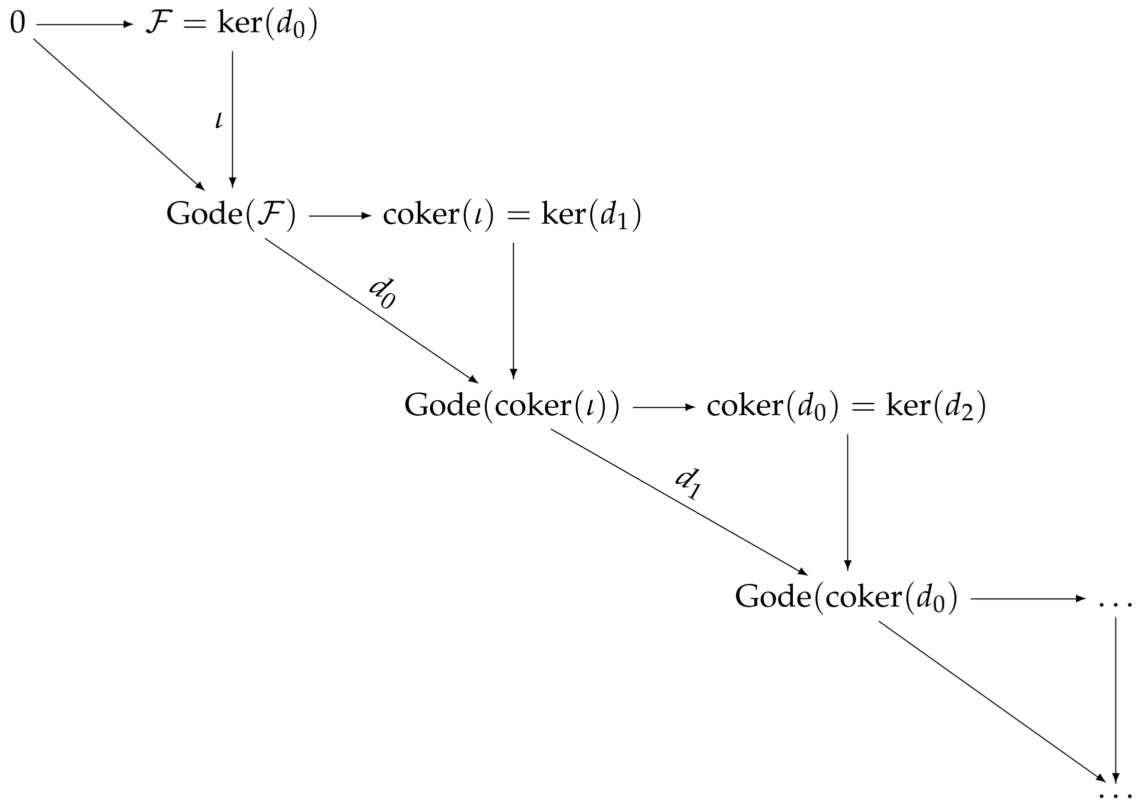



$$\begin{aligned} \iota : \mathcal{F}(U) &\rightarrow \text{Gode}(\mathcal{F})(U) \\ \sigma &\mapsto \Pi_{u \in U} \sigma|_u \end{aligned}$$

where $\sigma|_u$ is the germ of $\sigma$.

We have the following resolution

[125] Automorphisms, which means some chance of constructing fundamentals via a limiting procedure




this is the Godement resolution. Take $\Gamma$ of the diagonal - this complex gives cohomology. Cohomology measures what sheafification adds here.

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