Notes on Mirror Symmetry

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These are ongoing notes from a seminar series run by postgraduate students at the University of Melbourne.

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(Proposed) Speakers by Week:

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1 Introduction

1.1 A Physicsy rant

Many papers about mirror symmetry have a bit of physicsy jargon in it, and a part of this first section is to say a little about the sorts of mathematical objects physicists care about in *field theory* [1]:

An *electric field* is a *nice* vector field. That is: it's a map $E : \mathbb{R}^3 \to \mathbb{R}^3$ of the form:

$$(x, y, z) \mapsto a(x, y, z)\partial_x + b(x, y, z)\partial_y + c(x, y, z)\partial_z,$$

satisfying additional conditions (hence the word *nice*) imposed by physical considerations. Note that this electric field may be paraphrased as a slightly different map:

$$\hat{E} : \mathbb{R}^3 \to T\mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6,\ (x, y, z) \mapsto (x, y, z, a, b, c).$$

In physics, a *field*(P) [2] is a "nice" section of a bundle ξ_M over M. In particular, our electric field $\hat{E}$ is an example of a field. In maths, we use the notation $\Gamma(M, \xi_M)$ to mean the space of all sections of the bundle ξ_M over M. So, the fields that we care about in a given *field theory*(P) are elements of such spaces of sections. The physics governing a particular field theory then places additional constraints (i.e.: niceness) on the types of sections that we'd like to study.

The *energy*(P) of an electric field is

$$\text{Energy}(E) := c \int_{\mathbb{R}^3} |E|^2 \ \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

This is an example of an *action(P)*, which is a function from $\{$ fields $\} \to \mathbb{R}$ (you'll often hear mathematicians calling this a *functional*). Note that in order for the energy of an electric field to be finite, we would require that $|E|$ be square-integrable. This is an example of a niceness condition that we might impose on the space of sections/fields, albeit imposed due to mathsy reasons in this particular case.

Fields and actions are the cornerstones of a *Field Theory*. Adjectives placed in front of the words "Field Theory" such as "classical", "quantum" or "conformal" specify properties on the fields and actions of this particular field theory.

Mirror symmetry arises from a field theory called (type II) *superstring theory* (which should perhaps be called superstring *field* theory). Here's the rough idea:

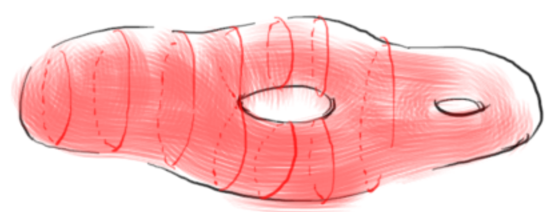
There are five different candidate superstring theories for describing our universe. They are each of $10 = 6 + 4$ dimensions. The last 4 are the usual

[1] possibly without realising that they care about these objects

[2] Look, let's try to put a (P) after physicsy jargon.

space-time dimensions — these are big in the sense that they're macroscopic and we can sense them (or at least Einstein could). The remaining 6 are probably very very small and hence we don't really experience them in real life. Of course, it's also possible that the remaining 6 dimensions are so big that they seem flat to us and hence we don't experience them in real life. Or possibly, it's just really flat near Earth, and it's actually a combination of these two extremes. Although Yau seems to believe that they should be small — probably due to Occam's razor-y reasons.

There is this picture that people often invoke to describe string theory. Imagine a "string" in the universe, so a circle. Now imagine it evolving in time, sometimes getting bigger, sometimes smaller, sometimes pinching and splitting into two or more strings and perhaps eventually coming back together into one string and vanishing altogether. This family of strings traces out a surface Σ in our 10-dimensional universe $M^{(6)} \times \mathbb{R}^4$.[3]



I (Yi) think that the idea is that manner in which Σ embeds in the 6 small dimensions denoted by $M = M^{(6)}$ governs the type of particle that this string represents (okay, this may or may not specifically be for Fermions and not Bosons, I don't understand the language well enough yet to discern if I'm completely wrong –Yi).

So, given that discussion, let's consider the map $f : \Sigma \to M$ obtained by projecting the embedding of Σ in our universe down to these 6 small dimensions. Such a map may be paraphrased as:

$$(\hat{f} : \Sigma \to \Sigma \times M) \in \Gamma(\Sigma, \Sigma \times M) \tag{1}$$
$$p \mapsto (p, f(p)) \tag{2}$$

which is a section of the trivial bundle $\Sigma \times M \to M$. [4] So f is possibly a field(P) in some field theory governing the "types" of particles that strings might be.

[3]Thara thought that this should have been 11-dimensions. This is true in some sense, in that the 11-dimensional *M-theory* is meant to be a unifying theory for all 5 of these superstring theories. So, you should be able to obtain each of these 5 superstring theories by taking some sort of limit of M-theory.

[4]Note that this is slightly inconsistent notation to before, since we're using the total space $\Sigma \times M$ to denote the actual trivial bundle.

To have a field theory, however, we need an action(P). And in this setup, the desired action is (very roughly) taking the area of $f(\Sigma) \subset M$.

The extra physics considerations that are imposed in type II superstring theories, means that M is *Calabi-Yau* (most of the time). This roughly means the following:

- M is (Ricci) flat.
- M has a Kähler structure. (Referred to as the A-model)
- M has a complex structure. (Physicists refer to this as the B-model)

Put all this together and (maybe) you get super(symmetric) conformal field theories! (SCFT)

Let's just talk about all this stuff as if we knew what we were talking about and see how things go. In future weeks, I hope that we will work out what all this means.

Consider a Calabi-Yau manfold (M, J, ω) (which we assume from now on is 6 dimensional), where J is the complex structure (it tells us how to rotate by $\frac{\pi}{2}$ on the tangent space) and ω is the Kähler structure (it tells us how to find areas). They have this additional property of admitting two types of structural deformations, so as to obtain families of different Calabi-Yau manifolds.

A-model / Kähler structure	B-model / $\mathbb{C}$-structure
We can change the Kähler structure ω by adding elements of $H^{1,1}(M)$	We can change the complex structure J by adding elements of $H^{2,1}(M)$

Calabi-Yau manifold (M, J, ω) would admit slightly different fields(P) and a slightly different action(P). That is, we get a super conformal field theory.

On the other hand, given a SCFT, does there exist a unique Calabi-Yau manifold M that induces this SCFT?

In general: *no.*

In fact, most of the time we instead get two intricately related Calabi-Yau manifolds M and $\tilde{M}$. *How* these two manifolds are related is given by *mirror symmetry.*

1.2 Philosophy

Here's a little diagram we drew on the board:

$$\begin{tikzcd}
M \arrow[dr, Rightarrow] & & \tilde{M} \arrow[dl, Rightarrow] & \text{geometric properties} \arrow[d, Rightarrow] \\
& SCFT \arrow[rr] & & \text{physical properties}
\end{tikzcd}$$

The idea is that the physical properties of a SCFT are determined by geometric properties of either one of these inducing Calabi-Yau manifolds. Therefore, the fact that M and $\tilde{M}$ impose the same SCFT means that their own geometric properties should closely correspond.

So how are M and $\tilde{M}$ related? One example is that the Euler characteristic $\chi(M)$ of M is the same as $-\chi(\tilde{M})$. Which is actually a corollary of our second example of a relationship:

$$H^{p,q}(M) \cong H^{3-p,q}(\tilde{M}).$$

The symbols $H^{p,q}$ denote the Dolbeault cohomology of a manifold.Roughly speaking, it lets us break up the cohomology of a manifold more finely by keeping track of holomorphic and antiholomorphic parts of the cohomology - note that this explanation assumes that we're thinking of cohomology in terms of differential forms *a la* DeRham cohomology. As an example, the relationship between the normal cohomology of M and $H^{p,q}$ is as follows:

$$H^3(M) = H^{3,0}(M) \oplus H^{2,1}(M) \oplus H^{1,2}(M) \oplus H^{0,3}(M)$$

Mirror symmetry then tells us these respective summands are the same as:

$$H^{0,0}(\tilde{M}) \oplus H^{1,1}(\tilde{M}) \oplus H^{2,2}(\tilde{M}) \oplus H^{3,3}(\tilde{M}).$$

We will see pictures that look like as follows:

For M:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 101 & & 0 & \\
1 & & 1 & & 1 & & 1 \\
 & 0 & & 101 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

For $\tilde{M}$:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 101 & & 101 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

These are called Hodge diamonds and those numbers denote the dimensions of the vector spaces $H^{p,q}$. The fact that these diagrams are the same up to rotation is the most commonly cited origins of the name "mirror symmetry".

Now, not only are these cohomologies isomorphic, but mirror symmetry predicts that triple products on these cohomologies are also preserved!

On the A-model side, if $\omega_i \in H^{1,1}(M)$ then we define the triple product

$$\langle \omega_1, \omega_2, \omega_3 \rangle = \int_M \omega_1 \wedge \omega_2 \wedge \omega_3 + \sum_{\beta \in H_2(M,\mathbb{Z}) \neq [0]} n_B \int_\beta \omega_1 \int_\beta \omega_2 \int_\beta \omega_3 \left(\frac{e^{2\pi i} \int_\beta \omega}{1 - e^{2\pi i} \int_\beta \omega} \right).$$

Here n_β is the number of genus 0 surfaces in M representing the cohomology class β.

On the B-model side, if $\theta_i \in H^{2,1}(M)$ then we define the triple product

$$\langle \theta_1, \theta_2, \theta_3 \rangle = \int_{\tilde{M}} \Omega \wedge \nabla_{\theta_1} \nabla_{\theta_2} \nabla_{\theta_3} \Omega.$$

The ∇'s are the Gauss-Manin connections[5]. It turns out that both of these triple products are equal!

By studying the deformation of a family of Calabi-Yau manifolds on these two models and the corresponding deforming triple products on these two sides, we can obtain these following correspondence of geometric structures of M and $\tilde{M}$:

For the A-model we get:

$$5 + n_1 q + 8(n_2 + \frac{n_1}{8} q^2 + 27(n_3 + \frac{n_1}{27})q^2 + 27(n_3 + \frac{n_1}{27} q^3 + 64(n_4 + \frac{n_2}{8} + \frac{n_1}{64})q^4 + \dots$$

where the n_i are the number of degree i spheres (rational curves) in M.

This is equal to the B-model terms:

$$-c_1 - 575 \frac{c_2}{c_2} q - \frac{1950750}{2} \frac{c_1}{c_2^2} q^2 - \frac{1027749000}{6} \frac{c_1}{c_3^3} q^3 - \frac{74486048625000}{24} \frac{c_1}{c_3^3} q^4 + \dots$$

We can then work out that $c_1 = -5$ and thus $n_1 = 2875$ which is a classical result but is still pretty cool! It tells us the number of $\mathbb{CP}^1$'s in M. For this example M is

$$\{[X_1 : X_2 : X_3 : X_4 : X_5] \in \mathbb{CP}^4 \mid X_1^5 + X_2^5 + X_3^5 + x_4^5 + X_5^5 = 0\}.$$

Its dual manifold, $\tilde{M}$ is (the resolution of) M quotiented by $\mathbb{Z}_5^3$ acting on the last three coordinates in $\mathbb{CP}^1$ by multiplying by the 5-th roots of unity.

We can work out that $c_2 = 1$ and $n_2 = 609250$. However, from n_3 and onwards, the numbers are new. For example $n_3 = 317206375$ and $n_4 = 24246753000$.

[5] Yi — I don't understand the notation yet, it was taken from another set of notes, let's change/edit this once we understand it.

1.3 Bits at the end

We will also learn a bit about homological mirror symmetry (due to Kontsevich). The strategy is to show that the Fukaya category of M (whatever that is) is the same as the derived category of coherent sheaves of $\tilde{M}$.

Example 1.1 (The mirror dual for T^2)**.** *Let $\tau, t \in \mathbb{C}$ such that $\mathrm{im}\tau, \mathrm{im}t > 0$. The Calabi-Yau manifold $(E_{\tau,t}, J_{\tau,t}\omega_{\tau,t})$ where the elliptic curve/torus*

$$E_{\tau,t} := \frac{\mathbb{C}}{\mathbb{Z} + \tau\mathbb{Z}}.$$

The complex structure $J_{\tau,t}$ comes from $\mathbb{C}$. The Kähler structure is $\frac{-\pi t}{\mathrm{im}\tau}\mathrm{d}z \wedge \mathrm{d}\overline{z}$. Then $E_{\tau,t}$ and $E_{t,\tau}$ are mirror duals.

Lastly, at some point we will talk about the SYZ conjecture due to Strominger, Yau and Zaslow. In the heads of people who understand this better, it *very very very very roughly* looks like two bundles over the same base.

$$\begin{tikzcd}
T^{(n)} \arrow[r] & M^{(2n)} \arrow[dr] & & \tilde{M}^{(2n)} \arrow[dl] & \tilde{T}^n = \mathrm{Hom}(T^{(n)}, \mathbb{C}^*) \arrow[l] \\
& & B^{(n)} & &
\end{tikzcd}$$

Maybe this will make more sense to us later on.

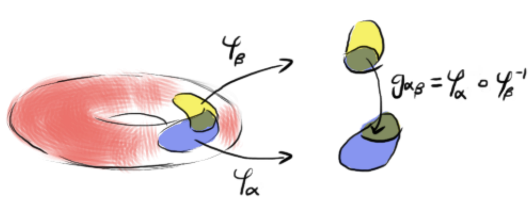
2 Differential Topology and Geometry (21/3/13)

Speaker: Andrew Elvey-Price
Note-taker: Dougal Davis

2.1 Manifolds

A manifold is something which can be broken into pieces which look like pieces of $\mathbb{R}^n$. More formally, a manifold is a topological space M together with a collection or *atlas* of *coordinate charts* $\{(U_\alpha, \varphi_\alpha)\}$, where $\{U_\alpha\}$ is an open cover of M and each $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ is a homeomorphism onto an open subset of $\mathbb{R}^n$. Given a manifold M with atlas $\{(U_\alpha, \varphi_\alpha)\}$, we define the *transition functions*

$$g_{\alpha\beta} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$
$$g_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$$



If all the transition functions are a thing, the manifold is called that thing. For example, if all transition functions are smooth, we have a smooth manifold. If the transition functions are holomorphic when we identify $\mathbb{R}^n = \mathbb{R}^{2m}$ with $\mathbb{C}^m$, we have a *complex manifold*.

It turns out that we can recover the whole manifold just from the sets $\varphi_\alpha(U_\alpha)$ and the transition functions $g_{\alpha\beta}$. Notice that the transition functions satisfy

$$g_{\alpha\beta} \circ g_{\beta\gamma} \circ g_{\gamma\alpha} = \mathrm{id}$$
$$g_{\alpha\beta} = g_{\beta\alpha}^{-1}$$

Example 2.1 (2-sphere)**.** *The* 2*-sphere is*

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

Let

$$U_1 = S^2 \setminus \{(0, 0, 1)\} \qquad U_2 = S^2 \setminus \{(0, 0, -1)\}$$

and define

$$\varphi_1(x, y, z) = \left(\frac{-x}{z-1}, \frac{y}{z-1}\right)$$
$$\varphi_2(x, y, z) = \left(\frac{x}{z+1}, \frac{y}{z+1}\right)$$

The transition function is

$$\begin{aligned} g_{12} : \mathbb{R}^2 &\to \mathbb{R}^2 \\ (u, v) &\mapsto \left(\frac{u}{u^2+v^2}, \frac{-v}{u^2+v^2}\right) \end{aligned}$$

This function and its inverse g_{21} are smooth on $\varphi_2(U_1 \cap U_2) = \mathbb{R}^2 \setminus \{(0, 0)\}$, so this makes S^2 a smooth manifold. If we identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way, we can write

$$g_{12}(w) = \frac{1}{w}$$

which is holomorphic on $\varphi_2(U_1 \cap U_2) = \mathbb{C} \setminus \{0\}$, so we can also think of S^2 as a complex manifold. S^2 is often identified with $\mathbb{C} \cup \{\infty\}$ as a complex manifold.

2.2 Vector bundles

A *vector bundle* is a manifold M together with a vector space at every point. Locally a vector bundle should look like $M \times \mathbb{R}^r$.
More formally, a vector bundle over a manifold M is a manifold E with a function $\pi : E \to M$ such that for every point $x \in M$, there exists a neighbourhood U of x such that

$$\pi^{-1}(U) \simeq U \times \mathbb{R}^r$$

The number r appearing here is called the *rank* of the vector bundle. We also keep track of the vector space structure on the fibre $\pi^{-1}(x) \simeq \mathbb{R}^r$.
A vector bundle is *trivial* if it is globally isomorphic to $M \times \mathbb{R}^r$.

Example 2.2 (Vector bundles on the circle)**.** *Take*

$$M = S^1 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$$

There are two non-isomorphic rank 1 vector bundles on the circle: the cylinder and the Möbius strip.
We can write these explicitly as follows. For the cylinder, write

$$\begin{tikzcd} E = \{re^{i\theta} \,|r \neq 0\} = \mathbb{C}^\times = \mathbb{R}^2 \setminus \{(0,0)\} \arrow[d, "\pi"'] \\ S^1 = \{e^{i\theta}\} \end{tikzcd}$$

Take $U = S^1$ so that

$$\begin{aligned} \pi^{-1}(U) = E &\simeq S^1 \times \mathbb{R} \\ re^{i\theta} &\mapsto \left(e^{i\theta}, \log r\right) \end{aligned}$$

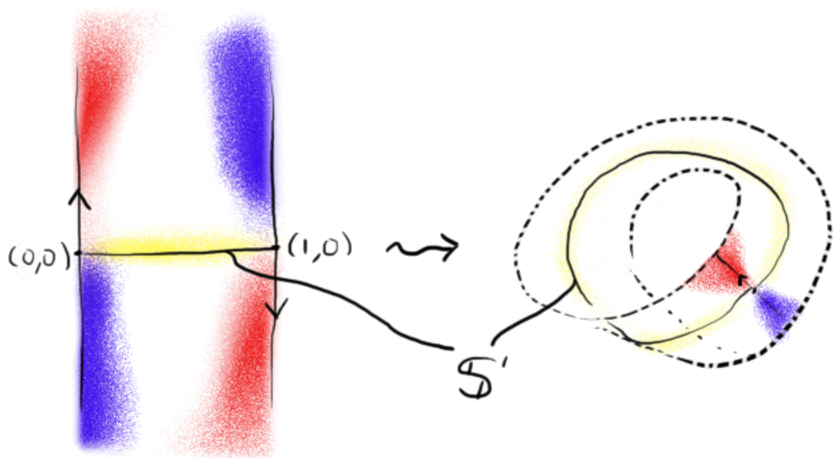
The vector space structure on the fibres comes from this isomorphism to $S^1 \times \mathbb{R}$. Explicitly, addition and scalar multiplication are given by

$$\left(r_1 e^{i\theta} + r_2 e^{i\theta}\right) = r_1 r_2 e^{i\theta}$$
$$\lambda \left(re^{i\theta}\right) = r^\lambda e^{i\theta}$$

For the Möbius strip, we can write

$$\begin{tikzcd} E' = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 1\}/\sim \arrow[d, "\pi'"'] \\ S^1 = \{(x, 0) \in \mathbb{R} \times \{0\} \,|\, 0 \leq x \leq 1\}/\sim \end{tikzcd}$$

where $(0, y) \sim (1, -y)$.



2.3 Sections, tangent bundles and vector fields

Consider the circle S^1 as in Example 2.2. The *tangent bundle* TS^1 of S^1 is the set of pairs (p, v) such that $p \in S^1$ and $v \in \mathbb{R}^2$ is a vector tangent to S^1 at p. More explicitly,

$$\begin{array}{c} TS^1 = \{(\cos t, \sin t, -\alpha \sin t, \alpha \cos t\} \subseteq \mathbb{R}^2 \times \mathbb{R}^2 \\ \pi \big\downarrow \\ S^1 = \{(\cos t, \sin t)\} \subseteq \mathbb{R}^2 \end{array}$$

Then TS^1 is isomorphic to the trivial bundle on S^1. For if we set

$$\rho : TS^1 \to S^1 \times \mathbb{R}$$
$$(\cos t, \sin t, -\alpha \sin t, \alpha \cos t) \mapsto (\cos t, \sin t, \alpha)$$

then ρ is an isomorphism.
A section of this vector bundle is a function

$$f : S^1 \to TS^1$$

such that

$$(\cos t, \sin t) \mapsto (\cos t, \sin t, ?)$$

For example,

$$(\cos t, \sin t) \mapsto ((\cos t, \sin t), -3(\sin t, \cos t))$$
$$(\cos t, \sin t) \mapsto ((\cos t, \sin t), \cos t(\sin t, \cos t))$$

are sections of TS^1.

In general, a section of a vector bundle $\pi : E \to M$ is a function

$$\nu : M \to E$$

such that $\pi \circ \nu = \mathrm{id}_M$. Notice that the Möbius strip in Example 2.2 has no nonzero continuous sections.

It turns out that a nice way to look at the tangent bundle is through *differential operators*. For M an arbitrary manifold, we define the tangent bundle TM of M by

$$TM = \{(p, X_p) \mid p \in M, X_p \text{ is a differential operator at } p\}$$

A differential operator at p is a map

$$\{\text{smooth functions defined in a neighbourhood of } p\} \to \mathbb{R}$$

which in local coordinates looks like

$$X_p f = \left(a_1 \frac{\partial f}{\partial x_1} + a_2 \frac{\partial f}{\partial x_2} + \cdots a_n \frac{\partial f}{\partial x_n} \right)\Bigg|_p, \quad a_i \in \mathbb{R}$$

The local coordinates $(x_1, \ldots, x_n)$ on M give local coordinates $(x_1, \ldots, x_n, a_1, \ldots, a_n)$ on TM. The transition functions are given by the chain rule.

We generally think of $X_p f$ as the directional derivative of f in the direction X_p. The sections of TM are sometimes called *differential operators*. We can also think of a differential operator X as a map

$$\{\text{smooth functions on} M\} \to \{\text{smooth functions on} M\}$$

which in local coordinates look like

$$Xf = a_1(p) \frac{\partial f}{\partial x_1} + a_2(p) \frac{\partial f}{\partial x_2} + \cdots a_n(p) \frac{\partial f}{\partial x_n}$$

where the a_i are now functions on M.

Example 2.3 (2-sphere)**.** *Recall the manifold $S^2 = \mathbb{C} \cup \{\infty\}$ from Example 2.1. We have the coordinate charts*

$$\begin{aligned} \varphi_1 : U_1 = \mathbb{C} &\to \mathbb{R}^2 = \mathbb{C} \\ (x, y) &\mapsto (x, y) \\ z &\mapsto z \end{aligned}$$

$$\begin{aligned} \varphi_2 : U_2 = \mathbb{C} \cup \{\infty\} \setminus \{0\} &\to \mathbb{R}^2 = \mathbb{C} \\ (x, y) &\mapsto \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right) \\ z &\mapsto \frac{1}{z} \end{aligned}$$

The corresponding coordinate charts for TS^2 are

$$\begin{aligned}\tilde{\varphi}_1 : U_1 \times \mathbb{R}^2 \subseteq TS^2 &\to \mathbb{R}^2 \times \mathbb{R}^2 \\ (z; u, v) &\mapsto (x, y; u, v)\end{aligned}$$

$$\begin{aligned}\tilde{\varphi}_2 : U_2 \times \mathbb{R}^2 \subseteq TS^2 \to \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{C} \times \mathbb{R}^2& \\ (z; u, v) \mapsto \left(z; u\frac{y^2 - x^2}{(x^2+y^2)^2} + v\frac{-2xy}{(x^2+y^2)^2}, u\frac{2xy}{(x^2+y^2)^2} + v\frac{y^2-x^2}{(x^2+y^2)^2} \right)&\end{aligned}$$

In terms of differential operators, we have

$$X_p f = (u, v)_z f = u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y}$$

for any smooth function $f : U_1 \to \mathbb{R}$.

2.4 Constructing new vector bundles from old ones

In this section we will describe the following constructions on vector spaces.

1. Direct sum ($V \oplus U$);
2. Dual (V^*);
3. Tensor product ($V \otimes U$); and
4. Hom construction ($Hom(V, U) = V^* \otimes U$)[6].

In our heads, vector bundles over M are just vector spaces over the points of M with some nice gluing properties that pieces all the vector spaces together. From linear algebra, we are familiar with the above constructions for vector spaces. The corresponding definitions of these constructions for vector bundles is then to do each thing fibrewise. Essentially this is what this little section is about and if that makes sense to you you can probably skip this bit for now (like I would if I wasn't typing these notes). However, drawing matrices is always fun!

2.4.1 Transition Maps

Let $E \to M$ be a vector bundle. Let $U_\alpha \times \mathbb{R}^n$ and $U_\beta \times \mathbb{R}^n$ be local trivialisations of the bundle E over coordinate patches of M. For simplicity, we'll also call the coordinate patches U_α and U_β.
We have two pieces of date. We have a transition map

$$g_{\beta\alpha} : U_\alpha \to U_\beta,$$

[6] where the equality holds as long as the bundles have finite rank

which comes from the underlying manifold structure of M. We also have a transition map

$$s_{\beta\alpha} : U_\alpha \times \mathbb{R}^n \to U_\beta \times \mathbb{R}^n$$

which tells us how to transition from fibre to fibre. In particular, at each $x \in U_\alpha$, the map $s_{\beta\alpha}(x) : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation (a matrix!). These are linear maps of the fibres. These satisfy the usual compatibiliity conditions that one would expect.

2.5 Direct Sum

Let E and F be vector bundles over M. Recall that if E and F were vector spaces (ie. vector bundles over a point), then the underlying set of the direct sum is

$$E \oplus F := \{(u, v) \mid u \in U, v \in V\}.$$

For vector bundles, we just have to keep track of the base space as well.

$$E \oplus F := \{(x, (v, u)) \mid x \in M, v \in E_x, u \in F_x\}.$$

Here E_x is notation for the fibre above x. Similarly for F_x. Once we have the underlying set, the transition functions should come naturally. Do it bit by bit!

2.6 Dual bundles

Recall that the dual of a vector space V is $V^* := Hom_{Vect}(V, R)$. If we write vectors in V as column vectors, then we can write elements of the dual space as row vectors. Then a dual vector $[y_1, \ldots, y_n]$ applied to a vector in V is just multiplication

$$[y_1, \ldots, y_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 y_1 + \ldots + x_n y_n.$$

For vector bundles, the fibres of the dual bundle are the dual of the original fibres.
We can also define the transition maps. Let $f V_x \to \mathbb{R}$ be a dual fibre. Then

$$s^*_{\alpha\beta}(x)(f) = f \circ s_{\beta\alpha}.$$

It's also useful to think of what things look like has matrices. We have

$$[s^*_{\alpha\beta}(x)] = [s_{\beta\alpha}(x)]^T = ([s_{\alpha\beta}(x)]^T)^{-1}.$$

In a bit more detail,

$$[s^*_{\alpha\beta}(x)] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = ([y_1, \ldots, y_n] s_{\beta\alpha}(x))^T = s^T_{\beta\alpha} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = [s^{-1}_{\alpha\beta}]^T \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

2.6.1 Cotangent bundle

The *cotangent bundle* is the dual to the tangent bundle. We denote it by T^*M. As sets we can write the tangent bundle and cotangent bundle as

$$TM = \{(m, (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})_{\text{at } m}) \mid m \in M\}; and$$
$$T^*M = \{(m, (dx_1, \ldots, dx_n)) \mid m \in M\}.$$

The elements of the fibre should really be the span of what is written down. In particular a basis for the cotangent bundle is $dx_1, \ldots, dx_n$.
The point is just like we can think of elements of TM as differential operators, we can think of elements of the cotangent bundle as differential forms. We have the relation $dx_j \frac{\partial}{\partial x_i} = \delta_{ij}$.
Here are some matrices to convince you we have the right definitions.

$$\begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_1} \\ \vdots & & \vdots \\ \frac{\partial x_1}{\partial y_n} & \cdots & \frac{\partial x_n}{\partial y_1} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial y_1} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{bmatrix}$$
$$\begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{bmatrix} \begin{bmatrix} dy_1 \\ \vdots \\ dy_n \end{bmatrix} = \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix}$$

2.7 Tensor Product

Recall the definition of tensor products for vector spaces. Let $u_1, \ldots, u_n$ be a basis for the vector space U and let $v_1, \ldots, v_n$ be a basis for V. Then a basis for the tensor product $U \otimes V$ is

$$\{v_i \otimes v_j \mid 1 \leq i \leq n, 1 \leq j \leq n\}.$$

In particular, $\dim(U \otimes V) = \dim(U) \times \dim(V)$. It is often useful to think of tensor products in terms of matrices.
In terms of elements, if we let $u = (u_1, \ldots, u_n)$ and $v = (u_1, \ldots, u_m)$, then we can think of $u \otimes v$ as the matrix

$$[u_1, \ldots, u_n] \otimes \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} u_1 v_1 & \ldots & u_n v_1 \\ \vdots & & \vdots \\ u_1 v_m & \ldots & u_n v_m \end{bmatrix}$$

If instead we wrote $v = \sum x_i v_i$ and $u = \sum y_j u_j$ then $u \otimes v$ is $\sum x_i y_j (v_i \otimes u_j)$ which is what we would get from "expanding out" the brackets.
If we write our vectors in this form, then if we think of these vectors as sitting on fibres above a point $x \in M$ of a vector bundle $E \to M$, then we can define

our transition functions as

$$s^{E\otimes F}_{\alpha\beta}(x)(\sum_{i,j} x_{i,j} v_i \otimes u_j) = \sum_{i,j} x_{i,j} s^E_{\alpha,\beta}(x)(v_i) \otimes s^F_{\alpha,\beta}(x)(u_j).$$

2.7.1 Hom construction

We didn't talk too much about this except to note that in finite dimensional cases, $\mathrm{Hom}_{\mathbb{R}}(U, V) \cong U^* \otimes V$.

2.7.2 Symmetric and antisymmetric (wedge) products

Lastly we noted that we can write tensor products of bundles of themselves as a direct sum of symmetric parts and anti-symmetric parts. In terms of vector spaces, we have

$$V \otimes V = (V \otimes_S V) \oplus (V \otimes_a V).$$

In the symmetric tensor product, we have $x \otimes_s y = y \otimes_s x$, so a basis for $V \otimes_s V$ is

$$\{v_i \otimes_s v_j \mid i \leq j\}.$$

In the antisymmetric product, we have the relation $x \otimes_a y = -y \otimes_a x$, so a basis for $V \otimes_a V$ is

$$\{v_i \otimes_a v_j \mid i < j\}.$$

From this we see that $V \otimes_s V$ is $n(n+1)/2$ dimensional and $V \otimes_a V$ is $n(n-1)/2$ dimensional so that their direct sum is n^2 dimensional. So at least we have the dimensions right.
For an explicit isomorphism, we can define $\phi : (V \otimes_s V) \oplus (V \otimes_a V) \to V \otimes V$ by

$$(av_i \otimes_s v_j, bv_i \otimes_a v_j) \mapsto \frac{a+b}{2} v_i \otimes v_j + \frac{a-b}{2} v_j \otimes v_i.$$

The inverse send $x \otimes y \mapsto (x \otimes_s y, x \otimes_a y)$.
Just like for tensor products, we can define the symmetric and anti-symmetric products for vector bundles. The anti-symmetric product will be the more interesting one to us. In the literature it is often called the wedge product and is denoted

$$E \wedge E.$$

We can iterate this process to get the p^{th} wedge product, which we denote by

$$\wedge^p = E \wedge \ldots E.$$

A basis for the fibres over a point has the form

$$\{v_{i_1} \wedge \ldots \wedge v_{i_p} \mid i_1 < \ldots < \imath_p\}.$$

Thus $\wedge^p E$ is a rank $\binom{n}{p}$ vector bundle. An interesting thing happens when $p = n$, the dimension of the base manifold M. For example $\wedge^n T^* M$ has basis

$$dx_1 \wedge \ldots \wedge dx_n.$$

This gives a notion of volume on our manifold and should remind you of the determinant! Buzz word: volume form.

2.7.3 Pullback bundle

Given a vector bundle E over N and a smooth map $f : M \to N$ we can form the pullback bundle $f^*E \to M$ which is a bundle over M. Here is a picture:

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

The fibre over a point $p \in M$ of f^*E is the fibre over $f(p)$ in N.

2.7.4 Kernels and cokernels

Give a morphism of vector bundles $f : V \to U$ We can form the kernel bundle $ker(f)$ which is a bundle over the base of V and the cokernel bundle which is a bundle over the base of U.
In terms of vector spaces, the kernel is the kernel of f (null space if we think f is a matrix) and the cokernel is $U/im(f)$.

2.8 Metrics

In this section, we will define a notion of metric that will allow us to talk about lengths of curves on M.
A *metric* g associates to each $p \in M$ an inner product

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

varying smoothly with p, i.e. $g : TM \times TM \to \mathbb{R}$ is smooth.
If $c : [0, 1] \to M$ is a curve, then we define the length of c to be

$$\text{length}(c) = \int_0^1 \|c'(t)\| \text{d}t = \int_0^1 \sqrt{g_{c(t)}(c'(t), c'(t))} \text{d}t.$$

In local coordinates $x^1, \ldots, x^n$, we may consider g as a matrix $[g_{ij}]$, where

$$g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}).$$

Note that in these coordinates, $g(X, Y)$ for two vector fields X, Y may be calculated by writing X and Y in terms of local coordinates as $\sum \alpha_i \frac{\partial}{\partial x^i}$ and $\sum \beta_j \frac{\partial}{\partial y^j}$ and then computing the matrix multiplication:

$$[\alpha_1, \ldots, \alpha_n] \begin{bmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & & \vdots \\ g_{n1} & \cdots & g_{nn} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}.$$

Example 2.4 (Spherical coordinates)**.** *Let $S^2 \subset \mathbb{R}^3$ be the unit sphere. We have a parameterization $r : (0, 2\pi) \times (0, \pi) \to S^2 \subset \mathbb{R}^3$ defined by the equation*

$$r(\theta, \varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi).$$

In θ, φ coordinates, consider the metric defined by:

$$[g_{ij}] = \begin{bmatrix} r_\theta . r_\theta & r_\theta . r_\varphi \\ r_\varphi . r_\theta & r_\varphi . r_\varphi \end{bmatrix} = \begin{bmatrix} \sin^2\varphi & 0 \\ 0 & 1 \end{bmatrix}$$

Then $r_\varphi . r_\theta = g(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\theta})$ and so forth. Note that this metric enables us to calculate the lengths of paths or the areas of regions on the sphere considered as the unit sphere in $\mathbb{R}^3$.

Sometimes it is convenient to use Einstein notation.

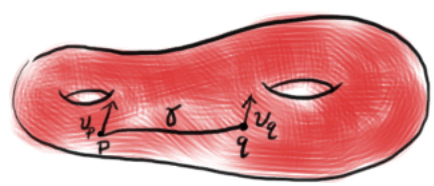
$$v^i \partial_i := \sum_{i=1}^{m} v^i \partial_i.$$

If we fix a vector in our tangent space, then we can define an element of the cotangent space using g.

$$g_p(v, -) : T_pM \to \mathbb{R}$$

The map $v \mapsto g_p$ is an isomorphism from $T_pM \to (T_pM)^*$. This is easily seen from the local coordinates matrix multiplication.

2.9 Connections



Let $\gamma : [0, 1] \to M$ be a path with $\gamma(0) = \gamma(1) = q$. The idea of a *connection* is the it "connects" the tangent spaces

$$T_pM \xrightarrow{\cong} T_qM$$

by associating to each vector v_p with a vector v_q.
In Euclidean space, we have the notion of parallel transport. For example, take the vector $(a, b) \in T_p\mathbb{R}^2 = \mathbb{R}^2$ to $(a, b) \in T_q\mathbb{R}^2 = \mathbb{R}^2$.
On manifolds, this is a bit trickier. Can we construct a connection so that on in local coordinates on every patch, parallel transport is given by this intuitive idea of $(a, b) \mapsto (a, b)$? The answer is no, and so we need to say what it actually means to have parallel transportation. A vector field V is parallel transported along a path γ if:

$$\frac{\mathrm{d}}{\mathrm{d}t} V_{\gamma(t)} = 0.$$

And we say that V is parallel transported along a vector field W if V is parallel transported along all the integral curves of W.

2.9.1 Affine Connections

Recall that $\Gamma(TM)$ is our notation for sections of the tangent bundle. In other words, they are vector fields.
An *affine connection* is a map

$$\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM).$$

We denote $\nabla(X, Y)$ by $\nabla_X Y$ and this is the generalisation of the derivative of Y in the direction of X.

This motivates us to impose the following conditions upon ∇: if $f, g \in C^\infty(M)$, $X, Y, Z \in \Gamma(TM)$, then

1. $\nabla_{fX+gY}(Y + Z) = f\nabla_X Z + g\nabla_Y Z$;
2. $\nabla_X(Y + z) = \nabla_X Y + \nabla_X Z$; and
3. $\nabla_X(fY) = X(f)Y + f\nabla_X Y$.

If $X = x^i\partial_i$, $Y = y^j\partial_j$, then

$$\nabla_X Y = \sum_{j=1}^{n} X(y^j)\partial_j + \sum_{i=1}^{n} x^i y^j \nabla_{\partial_i}\partial_j$$

As an example, if $M = \mathbb{R}^2$ consider the connection

$$\nabla_V W = [dW][V],$$

where $[dW]$ denotes the Jacobian of the vector field W considered as a map from $M = \mathbb{R}^2$ to its "globally parallelised" vector space $\mathbb{R}^2$. This is precisely the connection that takes (a, b) to (a, b) that first motivated all this parallel transportation business. In particular, the vector field V is parallel transported along W if and only if W is a constant vector field.

2.9.2 Metric connection

The metric (or Levi-Civita) connection for (M, g) is the unique affine connection compatible with the metric, that is:

$$g(v, w) = g(v', w').$$

In particular, the Levi-Civita connection is *the* torsion-free metric connection. Meaning that the parallel transportation of vectors does not "twist" unnecessarily.

2.10 Curvature

We drew a picture of a sphere. If you pick a vector at the north pole, walked to the equator, walked a bit along the equator and then walked back to the north pole, all while trying to hold your vector pointing in the same direction, you will find that your vector is now pointing in a different place.
What we have done is try to parallel transport our vector along a closed loop in S^2. How much the vector changes will then be a measure of the curvature.
The *curvature* of a manifold M is a map $R(.,.).:\Gamma(TM)^3 \to \Gamma(M)$. Intuitively, this is what you do.

1. Start with vectors X, Y, Z.
2. Extend X and Y to vector fields so that the flow along X for t, Y for t, X for $-t$, Y for $-t$ lets you get back to the same point.
3. Go along this parallelogram and see how Z parallel transports. Then take the limit as $t \to 0$.

The formula is

$$R(X,Y)Z = \nabla_x \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[x,Y]} Z.$$

2.11 Differential Forms

A k-form ω on M assigns to each $p \in M$ a multilinear, antisymmetric function

$$\omega_p : T_pM \times \ldots \times T_pM \to \mathbb{R}$$

where there are k copies of T_pM in the domain.
We will not go into much detail. Instead, we will say what differential forms look like on $\mathbb{R}^3$.

0-form $f \in C^\infty(M)$.

1-form $f_1 \mathrm{d}x + f_2 \mathrm{d}y + f_3 \mathrm{d}z$.

2-form $f_3 \mathrm{d}x^1 \wedge \mathrm{d}x^2 + f_1 \mathrm{d}x^2 \wedge \mathrm{d}x^3 + f_2 \mathrm{d}x^3 \wedge \mathrm{d}x^1$.

3-form $f \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$.

There are no k-forms for $k \geq 4$. We denote the set of k-forms (which is easily made into a vector space) by $\Omega^k(M)$.
Differential forms are closely related to determinates. For example a formula for a 2-form looks like

$$f_2 \mathrm{d}x^2 \wedge \mathrm{d}x^3(v,w) = f_2 . \begin{vmatrix} \mathrm{d}x^2(v) & \mathrm{d}x^2(w) \\ \mathrm{d}x^3(v) & \mathrm{d}x^3(w) \end{vmatrix}.$$

The *Hodge star* is $*dz = dx \wedge dy$.

Differential forms are the right thin to integrate over. We can integrate n form over M^n. We can integrate a $(n-1)$-form over $(n-1)$ dimensional submanifolds of M^n and so on. For example

$$\int_{\mathbb{R}^3} f dx^1 \wedge dx^2 \wedge dx^3 := \int_{\mathbb{R}^3} f \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3.$$

On (M, g) there is a volume form, which is a top dimensional form locally given by

$$dV = \sqrt{\det(g_{ij})}\, dx^1 \wedge \dots dx^n.$$

For example, if M is the 2-sphere and $g_{ij} = \begin{bmatrix} \sin^2\varphi & 0 \\ 0 & 1 \end{bmatrix}$, then

$$\int_{S^2} dV = \int_0^\pi \int_0^{2\pi} \sin\varphi \; d\theta d\varphi = 4\pi.$$

2.11.1 de Rham Cohomology

We can define the exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ for $k \in \mathbb{Z}_{\geq 0}$. Instead we just give an example of how to compute it in an example.
Let $f(x, y) = x^2 y$ be a 0-form. Then

$$df = f_x dx + f_y dy = sxydx + x^2 dy.$$

Also

$$d(df) = (\frac{\partial}{\partial x}(2xy)dx + \frac{\partial}{\partial y}dy) \wedge dx + (\frac{\partial}{\partial x}(x^2)dx + \frac{\partial}{\partial y}(x^2)dy) \wedge dy = 0$$

The fact that $d^2 = 0$ is not unique to this example. This gives us a chain complex

$$0 \to \Omega^0 \xrightarrow{d^0} \Omega^1 \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} \omega^n \to 0.$$

We define the k^{th} de Rham cohomology of M to be the vector space

$$H^j_{dR}(M; \mathbb{R}) := \frac{ker d^k}{im d^{k-1}}$$

As an example, $H^0(T^2) = \mathbb{R}$, $H^1(T^2) = \mathbb{R}^2$ and $H^2(T^2) = \mathbb{R}$.

3 Projective Varieties

$\mathbb{CP}^n$ is the set of lines through the origin in $\mathbb{C}^{n+1}$. We can think of it is

$$\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus 0)/\mathbb{C}^\times.$$

Here $\mathbb{C}^\times$ is the group of units of $\mathbb{C}$ and acts by multiplication. This way of representing $\mathbb{CP}^n$ also tells us one way that we could topologise it.

In fact, $\mathbb{CP}^n$ is a manifold. It's coordinate charts can be defined as follows. Define $u_i = \{(x_0, \dots, x_n) \mid x_i \neq 0\}$. Then

$$\varphi((x_0, \dots, x_n)) := (\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \hat{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}).$$

For $\mathbb{CP}^1$ the charts look like

$$U_0 = \{(x_0, x_1) \mid x_0 \neq 0\}$$

$$U_1 = \{(x_0, x_1) \mid x_1 \neq 0\}$$

which should remind you of charts for S^1.
Remark: The linear transformations on $\mathbb{C}^{n+1}$ are $GL(n+1, \mathbb{C})$. If we allowing for scaling, then we get $PGL(n+1, \mathbb{C})$ as all the holomorphic automorphisms on projective space.

3.1 Varieties in $\mathbb{CP}^n$

The zero sets of a homogenous polynomial in $n+1$ variables is a projective subvariety of $\mathbb{CP}^n$. For example

$$\{[X, Y] \mid X^2 + XY + Y^2 = 0\}.$$

Example 3.1. *A general degree 3 homogeneous polynomial looks like*

$$f = a_1 X^3 + a_2 Y^3 + \ldots + a_{10} Y Z^2$$

We can reduce this to one parameter. This should be intuitively clear. At the moment, we have 10 parameters. Since $PGL(3, \mathbb{C})$ is 8 dimensional (elements are 3 by 3 matrices up to a constant) so we can reduce down to 2 parameters. The final parameter that we can remove comes from the fact that our variety is projective.
This is an elliptic curve with structure parameter an algebraic function of parameters [UM WHAT DID I JUST TYPE? - I think that you wanted to be talking about this 1-parameter space of elliptic curves, and that this space is parametrised by the j-invariant. But I dun have your notes. - Yi.]

Sometimes, you can get more parameters. For example, a similar dimension count tells us that a degree 5 polynomial in $\mathbb{CP}^4$ has 101 parameters. To get this number, you can do the same sort of dimensional counting as before. Try it!

3.2 Weighted projective varieties

Let $\mathbb{C}^\times$ be the multiplicative group of units (non-zero elements) of $\mathbb{C}$. Given $(w_0, \dots, w_n) \in \mathbb{Z}^n$, we can define an action of $\mathbb{C}^\times$ on $\mathbb{C}^{n+1}$ by

$$\lambda.(x_0, \dots, x_n) = (\lambda^{w_0} x_0, \dots, \lambda^{w_n} x_n).$$

We can define the weighted projective spaces to be the quotient

$$\mathbb{CP}^n_{(w_0,\ldots,w_n)} := \frac{\mathbb{C}^{n+1}\setminus\{0\}}{\mathbb{C}^\times}.$$

Unfortunately, the $\mathbb{C}^\times$ action is not free. For example, let's consider $\mathbb{CP}^3_{(w_0,w_1,w_2,w_3)}$. If $w_0 \neq 1$ we can choose $\lambda \neq 1$ such that $\lambda^{w_0} = 1$ so that

$$\lambda(x_0, 0, 0, 0) = (\lambda^{w_0} x_0, 0, 0, 0) = (x_0, 0, 0, 0).$$

Thus in our quotient, there is a $\mathbb{Z}/w_0\mathbb{Z}$ singularity at $(1, 0, 0, 0)$. This means the tangent space looks like $\mathbb{C}^4/\frac{\mathbb{Z}}{w_0\mathbb{Z}}$.
However, because this singularity has codimension 3, this is okay. In general, these types of singularities have codimension n and so subvarieties in general position almost always miss these singularity points (by some version of Sard's theorem).
There is a worse possibility than this. Let's look at the same example, but suppose that $gcd(w_1, w_2) \neq 1$. We can choose $k \neq 1$ so that $k|w_1$ and $k|w_2$. Now choose λ so that $\lambda^k = 1$. Then

$$\lambda.(0, x_1, x_2, 0) = (0, \lambda^{w_1} x_1, \lambda^{w_2} x_2, 0) = (0, x_1, x_2, 0).$$

This is now a codimension 2 singularity and cannot generally be avoided. There is however a way to treat this problem by a process known as "smoothing" which we might look at in future.
Away from these types of singularities, we expect that in general, subvarieties of $\mathbb{CP}^n_{(w_0,\ldots,w_n)}$ to be smooth if $gcd(w_i, w_j) = 1$ for $i \neq j$.

3.3 Toric Varieties

Take $\mathbb{C}^N$ and an action by $(\mathbb{C}^\times)^m$ where $m < N$. Take away a subset U fixed by a continuous subgroup of $(\mathbb{C}^\times)^m$. We then get a toric variety given by

$$\frac{\mathbb{C}^N \setminus U}{(\mathbb{C}^\times)^m}.$$

The circle action is by the group $(\mathbb{C}^\times)^{N-m}$.

Example 3.2. *Consider $\mathbb{C}^3$ with $U = \{(x, 0, 0) \mid x \in \mathbb{C}\}$. $(\mathbb{C}^\times)^1$ acts on $\mathbb{C}^3$ by $\lambda(x, y, z) = (x, \lambda y, \lambda z)$.*

SOME PICTURES FOR YI TO DRAW. THIS IS THE BIT WHERE WE REPRESENT THE TORIC VARIETIES BY DIAGRAMS. SINCE I DON'T REALLY GET IT, IT'D BE GOOD IF SOMEONE EXPLAINED IT CLEARLY TOO.

4 Sheaves

Let X be a topological space and $Top(X)$ be the category of open sets of X with morphisms being inclusions. Let $\mathbb{C}$ be another category. Usually for us it will be an abelian category like groups or rings. A *sheaf* is a contravariant functor from $Top(X)$ to $\mathbb{C}$ satisfying

1. (Locality). Let $\{U_i\}$ is an open cover of an open set U. If $s, t \in F(U)$, then $s|_{U_i} = t|_{U_i} \implies s = t$.

2. (Gluing). Let $\{U_i\}$ be an open cover of U. Suppose there exists $s_i \in F(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. Then there exists $s \in F(U)$ so that $s|_{U_i} = s_i$ for all i.

Here is a brief explanation. The contravariant condition means that for each inclusion of open sets $U \subset V$, there is a map $F(V) \to F(U)$ in $\mathbb{C}$ making the following diagram commute.

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ F(U) & F(V) \arrow[l] \end{tikzcd}$$

We call elements of $F(U)$ sections. We think of the map $F(V) \to F(U)$ by $s \mapsto s|_U$ as restrictions.
For the locality condition, this says that if two coming from a bigger open set agree on smaller open sets of a cover, then they must be equal. The second condition says that if there are sections defined on open sets that agree on their overlap, then they must have come from a section of a bigger open set. We can glue the sections together.
The following examples should help make the notion of a sheaf clearer.

Example 4.1. *Let M be a complex manifold. We will define $\mathcal{O}$, the sheaf of holomorphic functions. For an open set U of M, we define $\mathcal{O}(U)$ to be*

$$\{\textit{Holomorphic functions } U \to \mathbb{C}\}.$$

$\mathcal{O}$ defines a sheaf of rings, where multiplication and addition of sections is defined point-wise. The restriction maps are actual restrictions. The locality and gluing conditions are satisfied because holomorphicity is a local condition.

Let $\mathbb{Z}$ be the sheaf of locally constant integer valued functions. Let $\mathcal{O}^*$ be the sheaf of nowhere zero holomorphic functions. Each is a sheaf of abelian groups (respectively additive and multiplicative). These sheaves fit naturally into a short exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0.$$

where the map $\mathbb{Z} \to \mathcal{O}$ is the obvious inclusion and $\mathcal{O} \to \mathcal{O}^*$ is given by $f \mapsto e^{2\pi i f}$. This example is important mainly because it illustrates what “exactness” means in the context of sheaves. This is subtler than one might naïvely imagine: it does *not* mean that we get an exact sequence

$$0 \to \mathbb{Z}(U) \to \mathcal{O}(U) \to \mathcal{O}^*(U) \to 0$$

for each open set U: there exist non-vanishing holomorphic functions f on subsets U of the complex plane which cannot be written globally (on U) as e^g for some holomorphic function g (alas, the logarithm is not so well behaved). However, we *can* write f in this form in some small open neighbourhood of any point in U and then invoke the gluing axiom to stitch these locally defined functions together. We will have constructed a function on U which is locally equal to f on each member of an open cover of U and so is equal to f by the identity axiom for sheaves. Thus $\mathrm{im}(\mathcal{O}) = \mathcal{O}^*$.
[7]

Exactness (just like any other property of sheaves) should thus be understood as a local condition. It is worth noting that checking the exactness of a sequence of abelian sheaves is equivalent to checking the exactness (in the usual sense for $\mathbb{Z}$-modules) of the induced sequence of stalks [8] at each point: this is not very surprising in light of the example, since we verified "sheaf-surjectivity" by checking that the map was eventually "set-surjective" on suitably small open neighbourhoods.
Here is a very important example of a sheaf.

Example 4.2. *Let $E \xrightarrow{\pi} X$ be a vector bundle. If U is an open set of X, then a section is a map $s : U \to E$ such that $\pi \circ s(p) = p$. Sometimes these are called local sections if $U \neq X$. When $U = X$, we call these global sections. Define $\Gamma(U, E)$ to be the set of sections on U. Γ is the* sheaf of sections *of π. It is a vector space valued sheaf. Addition and scalar multiplication are defined point-wise, which is okay since the fibres are vector spaces.*

4.1 Stalks

Sheaves have values on open sets. The point of a stalk is to be able to talk about what our sheaf looks like at points of our space.

Definition 4.3. *Let $x \in X$ and $\mathcal{F}$ be a sheaf. The* stalk of $\mathcal{F}$ at x *is*

$$\mathcal{F}_x := \lim_{\to} \mathcal{F}(U)$$

where the limit is taken over all U that contain x.

The limit means that we identify sections which agree on small open sets containing x. Another way to define the stalk is to let it be

$$\{\langle U, s \in \mathcal{F}(U)\rangle\}/\sim$$

[7] For experts/pedants: you might worry that image of $\mathcal{O}$ could somehow fail to be closed under such gluings (which are, after all, really being performed inside the sheaf $\mathcal{O}^*$). There is a sneaky application of the so-called "sheaffification" functor involved in defining the image of a sheaf map which rules out this sort of badness.

[8] See the next section.

where we mod out by the equivalence relation $\langle U_1, f_1\rangle \sim \langle U_2, f_2\rangle$ if and only if there exists $\langle U, f\rangle$ with $U \subset U_1, U \subset U_2$ such that $f|_{U_1} = f|_{U_2}$. Basically, sections are equal if they agree on a smaller open set around x.
As an example, if $\mathcal{F} = \mathcal{O}$ the sheaf of holomorphic functions, then $\mathcal{O}_x$ would be the vector space of taylor expansions of holomorphic functions at x.

THIS IS THE BIT WHERE JAMES PUT UP AN EXERCISE. THERE WAS SOME DISCUSSION SO MAYBE SOMEONE CAN SAY WHAT IT WAS ALL ABOUT. AT THE MOMENT, I HAVE SOME SCATTERED COMMENTS WHICH DON'T SEEM THAT IMPORTANT.

4.2 Čech Cohomology

In this section, we will define the Čech Cohomology of a space. Choose a covering $\{U_i\}$ of X. A q-simplex σ is an ordered set of $q+1$ open sets

$$\sigma_{(U_0,\ldots,U_q)}$$

such that $|U| := \cap U_i \neq \emptyset$. A j-boundary of σ is $\partial_j \sigma = \sigma_{(U_0,\ldots,U_{j-1},U_{j+1},U_q)}$. The boundary map is

$$\partial\sigma = \sum_{j=0}^{q} (-1)^{j+1}(\partial_j \sigma).$$

A cochain is then a map from q-simplices to elements of $F(|\sigma|)$. Let

$$C^q(U, F)$$

be the set of all q-cochains. It is an abelian group.
There are some more notes that should probably go here, but due to some interruptions, we never quite wrote things out properly. Basically you should try to be comfortable with Cěch cohomology for the next section.

5 Line bundles

Recall that a line bundle over a complex manifold X is a rank 1 vector bundle over $\pi : E \to X$. This means that for each point $x \in X$, there is an open set $U_\alpha \subset X$ containing x and an isomorphism

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\simeq} U_\alpha \times \mathbb{C}.$$

We have transition functions

$$s_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta).$$

We can write $s_{\alpha\beta}(x, y) = (x, g_{\alpha\beta}(x)(y))$, where $g_{\alpha\beta} U_\alpha \cap U_\beta \to \mathbb{C}^{times}$ is in $\mathcal{O}^*(U_\alpha \cap U_\beta)$. We observed previously that they satisfied the following conditions.

1. $s_{\alpha\beta}s_{\beta\alpha} = 1$; and
2. $s_{\alpha\beta}s_{\beta\gamma}s_{\gamma\alpha} = 1$.

This says that these transition functions are 1 co-cycles.
Now suppose we had two isormophic line bundles. Given U_α, there are two trivialisations φ_α and φ'_α. We have the following commutative diagram.

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "\sim", "\varphi_\alpha"'] \arrow[dr, "\sim", "\varphi'_\alpha"'] & U_\alpha \times \mathbb{C} \arrow[d, "f_\alpha"', "\sim"] \\ & U_\alpha \times \mathbb{C} \end{tikzcd}$$

where $f_\alpha(x, v) = (x, g_\alpha(x)v)$, $g_\alpha : U_\alpha \to \mathbb{C}^\times \in \mathcal{O}^*(U_\alpha)$. We have have a similar diagram

$$\begin{tikzcd} \pi^{-1}(U_\beta) \arrow[r, "\sim", "\varphi_\beta"'] \arrow[dr, "\sim", "\varphi'_\beta"'] & U_\beta \times \mathbb{C} \arrow[d, "f_\beta"', "\sim"] \\ & U_\beta \times \mathbb{C} \end{tikzcd}$$

Here's a computation: $s_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} = f_\alpha\varphi'_\beta \circ \frac{1}{f_\beta}(\varphi'_\beta)^{-1} = \frac{f_\alpha}{f_\beta}s'_{\alpha\beta}$. The notation $\frac{1}{f_\alpha}\varphi(x, v)$ means $\varphi(x, \frac{1}{g_\alpha(x)}v)$.
The upshot is that $s'_{\alpha\beta} = \frac{f_\alpha}{f_\beta}s_{\alpha\beta}$. In particular $\frac{f_\alpha}{f_\beta} \in Im\delta_0$ where δ_0 is the δ_0 in the cochain complex one gets when computing Cech Cohomology. Thus isomorphic line bundles differ by a constant, which is in $Im\delta_0$ and so define the same element of $\check{H}^1(X, \mathcal{O}^*)$.

5.0.1 The Picard Group

The *Picard group on* X is the abelian group of all line bundles on X modulo isomorphisms. It is denoted by $Pic(X)$. There is an isomorphism

$$Pic(X) \cong \check{H}^1(X, \mathcal{O}^*).$$

Our previous discussion show that we at least have a map from $Pic(X)$ to $\check{H}^1(X, \mathcal{O}^*)$. The fact that this is an isomorphism shouldn't be too hard to work out after that.

6 Divisors

Let X be a variety. A *Hypersurface* is a codimension 1 subvariety of X defined locally by the zero locus of holomorphic functions. In other words, we have a decomposition of our hypersurface

$$H = \bigcup_\alpha H \cap U_\alpha$$

where $H \cap U_\alpha = \{z \in U_\alpha \mid f_\alpha = 0\}$, where $f : U_\alpha \to \mathbb{C}$ with the property that in $U_\alpha \cap U_\beta$, the quotient $\frac{f_\alpha}{f_\beta}$ will be nonzero in $\mathcal{O}^*(U_\alpha \cap U_\beta)$. The point is that the zeroes of functions on different patches agree up to multiplicity on overlaps. Here's the cool bit. Since $\frac{f_\alpha}{f_\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^\times$ we can use these to define a line bundle. In particular, these will form the transition functions of the line bundle.

Definition 6.1. *A* divisor *is a formal sum, $D = \sum u_i V_i$ of hypersurfaces with integer coefficients.*

What we said earlier about hypersurfaces defining line bundles carries over to divisors. That is, given a divisor, we can get a line bundle.

Example 6.2. *Consider $\mathbb{P}^1$. A divisor is given by $D = N + S$ where N and S are the north and south poles respectively. Let $U_\alpha = \mathbb{P}^1 - S$ and $U_\beta = \mathbb{P}^1 - N$ (both are isomorphic to $\mathbb{C}$). We have*

$$f_\alpha = u, \quad f_\beta = 1/u \quad \text{and} \quad \frac{f_\alpha}{f_\beta} = u^2.$$

The corresponding line bundle is the tangent bundle of $\mathbb{P}^1$. This is an exercise!

Here's the **point:** You can get line bundles from divisors.

7 Algebraic and Differential Topology

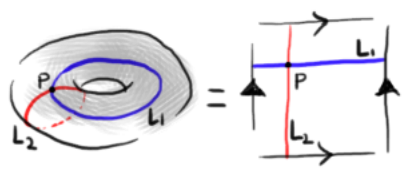
For us, algebraic topology will mean homology and cohomology. We will be doing a lot of cohomology calculations. We will move on to also include a discussion on characteristic classes, Morse theory and some mention of the moduli space of curves.

7.1 Some cohomology calculations

Let $T^2 = \mathbb{C}/\langle 1, i\rangle$ be the torus.
A few weeks ago[9], we worked out that that the homology of T^2 was

$$H_0(T^2, \mathbb{Z}) = \mathbb{Z}, \quad H_1(T^2, \mathbb{Z}) = \mathbb{Z}^2 \quad \text{and} \quad H_2(T^2, \mathbb{Z}) = \mathbb{Z}.$$

We can be a little more specific on the generators of the homology classes. Let p be a point, L_1 a circle in the 1 direction and L_2 a circle in the i direction.



[9]We worked this out on the boards but did not write notes for it.

The $[p]$ is the generator of H_0; $[L_1]$ and $[L_2]$ are the generators of H_1; and $[T_2]$ is the generator of H_2. We also worked out the de Rham cohomology of T^2. It was

$$H^0(T^2) = \mathbb{R}, \;\; H^1(T^2) = \mathbb{R}^2 \;\text{ and }\; H^2(T^2) = \mathbb{R}.$$

Let's go into a little more detail. Recall that de Rham cohomology is obtained from the cochain complex

$$0 \to \Omega^0(T^2) \xrightarrow{d^0} \Omega^1(T^2) \overset{d}{\to}^{1} \Omega^2(T^2) \xrightarrow{d^2} 0,$$

Where $\Omega^i(X)$ are the differential i-forms on x and the $d's$ are given by the total derivative map. The de Rham cohomology is then

$$H^i(T^2) = \frac{ker d^i}{im d^{i-1}}.$$

In the case of T^2,

$$H^0(T^2) = \mathbb{R}\{[1]\}$$

$$H^1(T^2) = \mathbb{R}\{[dx], [dy]\}$$

Note that x and y are not functions on T^2, so dx and dy are non zero in H^1. Lastly,

$$H^2(T^2) = \mathbb{R}\{[dx \wedge dy]\}.$$

The notation $\mathbb{R}\{S\}$ means we are taking the $\mathbb{R}$ span of S. Notice that there is a relationship between homology and cohomology. We'd like the relationship to take the form a a pairing. For example,

$$\int_{L_1} dx = 1, \;\; \int_{L_2} dx = 0, \;\; \int_{L_1} dy = 0, \;\; \int_{L_2} dy = 1.$$

We would like to define a map $\int : H_i(X) \times H^i(X) \to \mathbb{R}$ given by $([c], [\omega]) \mapsto \int_c \omega$. This is in fact what we get, but firstly we need to check that this map is independent of choice of homology class of c and cohomology class of ω. In both cases, this more or less follows from Stokes' theorem.
Changing homology class: Suppose $[L_1] = [L_1']$. Then $\int_{L_1} dx = \int_{L_1'} dx \iff \int_{\partial R} dx = 0 \iff \int_R d(dx) = 0$. We used Stokes' theorem and the fact that $d^2 = 0$.
Changing cohomology class: Suppose $[\omega] = [\omega']$. This means $\omega - \omega' = d\theta$ for some θ in $\Omega^{i-1}(X)$. Then $\int_c(\omega - \omega') = \int_c d\theta - \int_{\partial c} \theta = 0$, since $\partial c = \emptyset$.
Thus we do indeed get a map

$$\int L H_i(X) \times H^i(X) \to \mathbb{R}.$$

Let's again look at the case of the torus. Let's use real coefficients for homology as well just so that everything lines up nicely. We have $H_0(T^2) = \mathbb{R}\{[p]\}$ and $H^0(T^2) = \mathbb{R}\{[1]\}$. Since $\int_p 1 = 1, H^0(T^2) = H_0(T^2)^*$. Similarly, $H^1(T^2) = H_1(T^2)^*$ and $H^2(T^2) = H_2(T^2)^*$. Moreover, the generators that we described earlier are dual.
This always happens.

Theorem 7.1 (de Rham's Theorem)**.** *$H^i(X) = H_i(X, \mathbb{R})^*$.*

7.2 Cup products and Poincaré duality

There is a wedge product on differential forms. The map is $\Omega^i(X) \otimes \Omega^j(X) \to \Omega^{i+j}(X)$ given by $(\omega_1, \omega_2) \mapsto \omega_1 \wedge \omega_2$. This induces a well product

$$\cup : H^i(X) \otimes H^j(X) \to H^{i+j}(X)$$

given by $([\omega_1], [\omega_2]) \mapsto [\omega_1 \wedge \omega_2]$. This is called the *cup product.* We leave it to the reader to check that this map is well defined. Cup products are one of the reasons why cohomology is often better than homology. The cup products puts a graded ring structure on $H^*(X) = \oplus_i H^i(X)$.
Let's again look at the cohomology of the torus. It was $\mathbb{R}, \mathbb{R}^2$ and $\mathbb{R}$ in dimensions $0, 1$ and 2 respectively. Notice that the $H^i = H^{n-i}$, where $n = 2$ is the dimension of T^2. This turns out to be not a coincidence and is an example of Poincaré duality in action. We will describe it in what follows.
The cup product gives a map

$$\cup : H^i(X) \otimes H^{n-i}(X) \to H^n(X)$$

where $dimX = n$. We thus get a map

$$H^i(X) \otimes H^{n-i}(X) \to \mathbb{R}$$

by $[\omega_1] \otimes [\omega_2] \mapsto \int_X \omega_1 \wedge \omega_2$.
For the torus, this induces an isomorphism $H^i(T^2) \to H^{2-i}(T^@)^* = H_{2-i}(T^2)$. This happens in general (at least for compact spaces) and is known and the phenomenon is known as Poincaré duality..

Theorem 7.2 (Poincaré Duality)**.** *Let X be compact of dimension n. The map $H^i(X) \to H^{n-i}(X)^* = H_{n-i}(X)$ is an isomorphism.*

This gives us a way of interpreting $H^i(X)$ as "{ codimension i submanifolds of X }".

Example 7.3. *In this example, we compute the Poincar'e dual of $\omega = 2[dx] + 3[dy] \in H^1(T^2)$. We need to work out the corresponding map in $H^1(T^2)*$. We compute what omega does to basis vectors $[dx]$ and $[dy]$.*

$$\int_{T^2} [\omega] \cup [dx] = \int_{T^2} 3dy \wedge dx = -3 = \int_{-3[L_1]+2[L_2]} [dx]$$

$$\int_{T^2} [\omega] \cup [dy] = \int_{T^2} 2dx \wedge dy = 2 = \int_{2[L_2]-3[L_1]} [dx]$$

So $[\omega]$ corresponds to the same element of $H^1(T^2)*$ as $-3[L_1]+2[L_2] \in H_1(T^2, \mathbb{R})$. Thus the dual of $2[dx] + 3[dy]$ is $-3[L_1] + 2[L_2]$.

7.3 Chern classes

If X is a complex manifold and $E \to X$ is a holomorphic line bundle then we get a divisor D which corresponds to a homology class $[D] \in H_{2n-2}(X)$, where $n = dim X$. If X is compact, we can use Poincaré duality to get a cohomology class $c_1(E) \in H^2(X)$. This element of $H^2(X)$ is called the *first chern class of* E. If $E = TX$ the holomorphic tangent bundle, then we sometimes write $c_1(X)$ for $c_1(TX)$.
Let's look at an example. Let $X = T^2$ and consider $E = TX$, the holomorphic tangent bundle. Pick a section, say $s = \frac{d}{dz}$. To get a divisor, we find the zeroes of this section. There are none! Thus $c_1(E) = c_1(T^2) = 0 \in H^2(T^2)$. We will see later that this means that T^2 is Calabi-Yau.
Let's look at another example. Let $X = \mathbb{P}^2 = \frac{\mathbb{C}-\{0\}}{\mathbb{C}^\times} = \{[x_1 : x_2 : x_3]\}$. Again let $E = TX$. The cohomology of $\mathbb{P}^2$ is as follows.

$$\begin{aligned}
&H^0 = \mathbb{R}\{[1]\} \\
&H^1 = 0 \\
&H^2 = \mathbb{R}\{[\omega]\}, \ \ \omega \in \Omega^2(\mathbb{P}^2) \\
&H^3 = 0 \\
&H^4 = \mathbb{R}\{[\omega \wedge \omega]\} = \mathbb{R}\{[\omega]^2\}
\end{aligned}$$

To work out the chern classes, we need to get a divisor. To do this, take the following two "generic" sections of $E = T\mathbb{P}^2$:
Use coordinate charts on $\mathbb{P}^2$ given by

1. $U = \{x_1 \neq 0\}$ with coordinates $u_1 = \frac{x_2}{x_1}$ and $u_2 = \frac{x_3}{x_1}$.
2. $V = \{x_2 \neq 0\}$ with coordinates $v_1 = \frac{x_1}{x_2}$ and $v_2 = \frac{x_3}{x_2}$.
3. $W = \{x_3 \neq 0\}$ with coordinates $w_1 = \frac{x_1}{x_3}$ and $w_2 = \frac{x_2}{x_3}$.

The two generic sections are defined on U (and hence can be extended onto the rest of $\mathbb{P}^2$) as

$$\begin{aligned}
s_1 &= u_1 \frac{\partial}{\partial u_1} - u_2 \frac{\partial}{\partial u_2} \\
s_2 &= u_1 \frac{\partial}{\partial u_1} - u_1 \frac{\partial}{\partial u_2}.
\end{aligned}$$

In V the sections are

$$\begin{aligned}
s_1 &= -v_1 \frac{\partial}{\partial v_1} - 2v_2 \frac{\partial}{\partial v_2} \\
s_1 &= -v_1 v_2 \frac{\partial}{\partial v_1} + (1 - v_2^2) \frac{\partial}{\partial v_2}.
\end{aligned}$$

In W the sections are

$$s_1 = w_1 \frac{\partial}{\partial w_1} 2w_2 \frac{\partial}{\partial w_2}$$

$$s_1 = -w_1 w_2 \frac{\partial}{\partial w_1} + (1 - w_2^2) \frac{\partial}{\partial w_2}.$$

The zeroes of s_1 are $[1:0:0], [0:1:0], [0:0:1]$. Thus $c_2(E) = 3[\omega]^2 \in H^4(\mathbb{P}^2)$. The first chern class is part of a family of characteristic classes (called chern classes). The i^{th} chern class lives in $H^{2i}(X)$. For the first chern class, we looked at where two sections s_1 and s_2 are linearly independent i.e., when $s_1 \wedge s_2 = 0$. The other chern classes come from generalising this notion. We will describe this in later sections. We can package together all the chern classes into one big *total chern class*, $c(E)$ which is

$$c(E) = \oplus_i c_i(E) \in H^*(X).$$

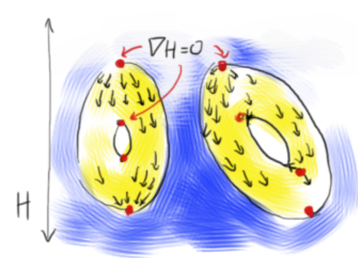
7.4 Morse Theory

Note: the approach to Morse theory in this section is by no means the usual approach. It's even possible that some of our claims in terms of separating the dynamics of gradient flows into classes of separatrices may be false in dimensionals greater than 2. Moreover, it lacks the rigour of Milnor's classical approach of growing your given manifold using fixed-point information. However, these are very simple and geometric notions that give an intuitive feel for why Morse theory should be the same as cellular (co)homology theory. Plus, it totes gives a justification for modern combinatorial versions of Morse theory such as that developed by Forman.

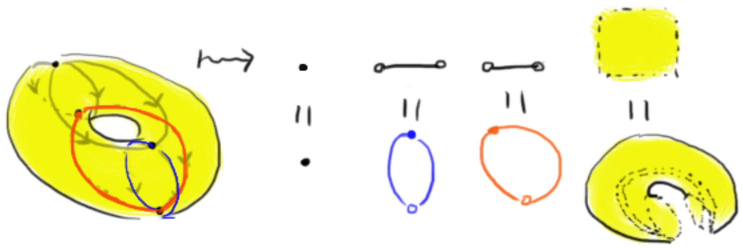
Morse theory is another way to understand (co)homology. Let $M^n \subset \mathbb{R}^N$ be a smooth compact embedded manifold in some high dimensional euclidean space. A *height function* is a smooth function

$$H : M \to \mathbb{R}.$$

One should think of a surface (say a torus) turned vertically, and the height function that tells us the height at each point of the torus.



Consider the gradient ∇H of H. It is a vector field on M that points in the (unscaled) direction of steepest descent (or ascent depending on signs). Let's study the situation by drawing flow lines and consider the dynamics of these lines. We get distinct families for different types of flow, partitioned based on the *source* of the flow line - i.e.: the limit of a flow line as $t \to \infty$. So doing gives us a cell decomposition of our surface, using which we can do (co)homology. Let's partition the cells in accordance to their "origing/source" stationary point (when $\nabla H = 0$).



Here's the main point:

To each maximum, we get a top dimensional cell. For each stationary point of the form $-(x_1^2 + \ldots + x_\mu^2) + x_{\mu+1}^2 + \ldots + x_n^2$, we get a μ-dimensional cell. Having a cell decomposition lets us do (co)homology with stationary points of ∇H. In particular, generic (i.e.: Morse-Smale systems) cell decompositions will satisfy that the boundaries of a cell corresponding to a stationary point of the form $-(x_1^2 + \ldots + x_\mu^2) + x_{\mu+1}^2 + \ldots + x_n^2$ will have boundary consisting only of cells corresponding to stationary point with (strictly) fewer negative terms in its local expansion, thereby giving us a natural boundary map. To be fair, this boundary probably still makes perfect sense for non-generic systems, it just takes a bit more work (e.g.: you can probably show that any boundary cells corresponding to stationary points which don't have strictly fewer negative terms will cancel out).

7.5 More on Chern classes

The last time we talked about chern classes, we saw that the zeros of a generic section of a vector bundle $E \to X$ of rank r gives $c_r(E)$. In this section we will look at two other approaches to studying chern classes.

7.5.1 Chern classes from Grassmannians

Recall that projective space is $\mathbb{P}^n := \{[x_0 : \ldots : x_n]\}$. We have inclusions

$$\mathbb{P}^n \subset \mathbb{P}^{n+1} \subset \ldots$$

by $[x_0 : \ldots : x_n] \mapsto [x_0 : \ldots : x_n : 0]$. We can take the direct limit (colimit) of these maps to get

$$\mathbb{P}^\infty := \{[x_0 : x_1 : \ldots] \mid \text{ at most finitely many } x_i \text{ are nonzero } \}.$$

$\mathbb{P}^\infty$ parametrizes complex lines in $\mathbb{C}^\infty$. It has a tautological line bundle $\mathcal{O}(-1) \to \mathbb{P}^\infty$ whose total space is

$$\mathcal{O}(-1) := \{([x_0 : x_1 : \ldots], \alpha(x_0, x_1, \ldots)) \mid \alpha \in \mathbb{C}\} \subset \mathbb{P}^\infty \times \mathbb{C}.$$

Similarly, it is possible to define a space Gr_k which parametrizes k-dimensional vector subspaces of $\mathbb{C}^\infty$. It is called the *Grassmannian*. There is also a tautological plane bundle $E_k \to Gr_k$, where

$$E_k := \{(V, v) \subset Gr_k \times (\mathbb{C})^k \mid v \in V\}.$$

The point is that $E_k \to Gr_k$ is the universal rank k vector bundle. Given a rank k vector bundle $F \to X$, there is a map $\varphi_k \to Gr_k$ such that $F \cong \varphi^*(E_k)$. That is, we have the following pull back diagram.

$$\begin{array}{ccc} F & \longrightarrow & E_k \\ \downarrow & & \downarrow \\ X & \xrightarrow[\varphi]{} & Gr_k \end{array}$$

Now we can define the total chern class of E_k, $c(E_k) = 1 + c_1(E_k) + \ldots c_k(E_k)$. Then we can get every other chern class by

$$c(f) = \varphi^* c(E_k),$$

where $\varphi^* : H^*(Gr_k) \to H^*(X)$.
Chern classes have the following nice properties.

- $c_i(F) \in H^{2i}(X, \mathbb{Z})$, $c_0(F) = 1$, $c_{i>k}(F) = 0$.
- $c(f^*F) = f^*c(F)$.
- $c(F \oplus G) = c(F) \cup c(G)$.
- $-c_1(\mathcal{O}(-1)) = e(\mathcal{O}(1))$ is the generator of $H^2(G_k)$, where e is the euler class.

It turns out that these properties completely characterize chern classes. Thus, we can forget all we know about them and just take these properties as axioms if we like.

7.5.2 Chern classes from differential geometry

Let E be a smooth rank k complex vector bundle over M. Let Ω be something called the *curvature form* of E. Then chern classes come out as the determinant

$$\det(I + \frac{it\Omega}{2\pi}) = \sum_m c_m(E) t^m.$$

To get Ω, pick *any* connection for the bundle E. A *connection* is a map

$$\nabla : \Gamma(E) \to \Gamma(\Omega^1 M \otimes E).$$

We think of ∇ as a $\mathbb{C}$-linear map between vector fields on E and vector fields of 1-form valued vectors in E. ∇ must also satisfy the Leibnitz rule

$$\nabla(fs) = df \otimes s + f\nabla(s).$$

Here is a simple example.

Example 7.4. *Let E be the trivial complex 3-bundle over $M = \mathbb{C}^2$. We expect $c(E) = 1$. Let's work this out. To specify the connection, we need coordinates to specify a generic section. Let ∇, let α, β and γ be the sections of $E \to M$ of the form*

$$\alpha : (x, y) \mapsto ((x, y), (1, 0, 0))$$

$$\beta : (x, y) \mapsto ((x, y), (0, 1, 0))$$

$$\gamma : (x, y) \mapsto ((x, y), (0, 0, 1)).$$

A generic section is then of the form $f_\alpha \alpha + f_\beta \beta + f_\gamma \gamma$. Then $\nabla : \Gamma(E) \to \Gamma(\Omega^1 M \otimes E)$ is given by

$$((x, y) \mapsto ((x, y), (f_\alpha, f_\beta, f_\gamma)) \mapsto ((x, y) \mapsto ((x, y), df_\alpha \otimes \alpha + df_\beta \otimes \beta + df_\gamma \otimes \gamma).$$

Note that $\nabla(\alpha) = \nabla(\beta) = \nabla(\gamma) = 0$. Let's calculate $c(E)$ using this connections stuff.
The first step is to pick sections for E on a patch U of $\mathbb{C}^2$ so that theses sections are a $C^\infty(M, \mathbb{C})$ basis for E on this patch $U \subset \mathbb{C}^2$. For example we can use $\{\alpha, \beta, \gamma\}, \{e^x\alpha, e^y\beta, e^{x+2y}\gamma\}$ or $\{e^x\alpha, e^x\alpha + e^y\beta, e^x\alpha + e^y\beta + e^{x+2y}\gamma\}$.
Let's use the middle one. We can specify all the data of ∇ using such a basis and the Leibnitz rule. We have

$$\nabla(e^x\alpha) = e^x dx \otimes \alpha$$

$$\nabla e^{2y}\beta) = 2e^{2y} dy \otimes \beta$$

$$\nabla(e^{x+2y}\gamma) = x^{x+2y}(dx + 2dy) \otimes \gamma.$$

We can write this in a matrix:

$$\omega = [\omega^i_j]_{ij} = \begin{bmatrix} e^x dx & 0 & 0 \\ 0 & 2e^{2y} dy & 0 \\ 0 & 0 & e^{x+2y} dx + 2dy \end{bmatrix}$$

Then Ω is defined to be

$$\Omega := d\omega + \omega \wedge \omega.$$

$d\omega$ is defined to be d of the entries. To calculate the wedge of two matrices we simply multiply out the two matrices, but replacing multiplication of entries

from the left and the right matrices with wedge products. In our case, one can check that $d\omega = 0$ and $\omega \wedge \omega = 0$. Thus, $\Omega = 0$.

$$\det(I + \frac{i\Omega t}{2\pi}) = \det(I) = 1 = c(E)$$

as we expected.

Here is another example.

Example 7.5. *For $T\mathbb{P}^1$ take the metric $h = \frac{dzd\overline{z}}{(1+|z|^2)^2} = \frac{dx^2+dy^2}{(1+x^2+y^2)^2}$. If we can show that*

$$\Omega = \left[\frac{2dz \wedge d\overline{z}}{(1+|z|^2)^2}\right] = \left[\frac{-4idx \wedge dy}{(1+x^2+y^2)^2}\right].$$

Then $\det(I + \frac{4t}{2\pi}\left[\frac{dx\wedge dy}{(1+(x^2+y^2))^2}\right]) = 1 + \frac{2}{\pi}\frac{dx\wedge dy}{(1+(x^2+y^2))^2}$.

The term $\frac{4}{2\pi}\frac{dx\wedge dy}{(1+(x^2+y^2))^2}$ is a 2-form representative of the cohomology class $c_1(T\mathbb{P}^2)$. We need to determine if it's nullcohomologous. Let's compute

$$\int_{\mathbb{R}^{\not 2}} \frac{2dz \wedge d\overline{z}}{\pi(1+|z|^2)^2} = \frac{2}{\pi}\int_{\mathbb{R}^2} \frac{rdr \wedge d\theta}{(1+r^2)^2} = 2.$$

So it isn't! In addition, we should note that this is the Euler characteristic of $\mathbb{P}^1$ as the n-th Chern class for the tangent bundle of a complex n-dimensional manifold is also its Euler class. Thus the Euler characteristic of the manifold may be calculated by the pairing of $c_n(TM)$ and the fundamental class of the manifold (basically, you think of the manifold AS a homology class) - which is precisely integrating over the manifold.

Let's now compute the the curvature form $\Omega = \mathrm{d}\omega + \omega \wedge \omega$ for the Levi-Civita connection $\nabla : \Gamma(T\mathbb{P}^1) \to \Gamma(\Omega^1\mathbb{P}^1 \otimes T\mathbb{P}^1)$ over the usual coordinate patch for $\mathbb{C} \subset \mathbb{C} \cup \{\infty\} = \mathbb{P}^1$. In terms of real coordinates, the Levi-Civita connection has Christoffel symbols:

$$\Gamma^x_{xx} = -\Gamma^x_{yy} = \Gamma^y_{xy} = \Gamma^y_{yx} = \frac{-2x}{1+x^2+y^2},$$
$$\Gamma^y_{yy} = -\Gamma^y_{xx} = \Gamma^x_{yx} = \Gamma^x_{xy} = \frac{-2y}{1+x^2+y^2}.$$

So, given that the Levi-Civita connection satisfies that $\nabla_{\vec{e}_i}\vec{e}_j = \Gamma^k_{ij}\vec{e}_k$, we have that:

$$\nabla(\partial_x) = (\Gamma^x_{xx}\mathrm{d}x + \Gamma^x_{yx}\mathrm{d}y) \otimes \partial_x + (\Gamma^y_{xx}\mathrm{d}x + \Gamma^y_{yx}\mathrm{d}y) \otimes \partial_y,$$
$$\nabla(\partial_y) = (\Gamma^x_{xy}\mathrm{d}x + \Gamma^x_{yy}\mathrm{d}y) \otimes \partial_x + (\Gamma^y_{xy}\mathrm{d}x + \Gamma^y_{yy}\mathrm{d}y) \otimes \partial_y.$$

Therefore, using the addtional facts that $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and that $\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$, we see that:

$$\nabla(\partial_z) = \frac{1}{2}\nabla(\partial_x) - \frac{i}{2}\nabla(\partial_y) = \frac{1}{1+z\overline{z}}[(-\overline{z}\mathrm{d}x - i\overline{z}\mathrm{d}y) \otimes \partial_x + (i\overline{z}\mathrm{d}x - \overline{z}\mathrm{d}y) \otimes \partial_y]$$
$$= \frac{1}{1+z\overline{z}}(-\overline{z}\mathrm{d}z \otimes \partial_x + i\overline{z}\mathrm{d}z \otimes \partial_y) = \frac{-2\overline{z}}{1+z\overline{z}}\mathrm{d}z \otimes \partial_z.$$

In retrospect, there was probably an easier way to do that computation, but it'd probably involve us knowing something about Hermitian metrics? Anyhoo, this tells us that our connection form *is:*

$$\omega = \left[\frac{-2\bar{z}}{1 + z\bar{z}} \mathrm{d}z\right].$$

Since this connection form is a 1×1*-matrix, we see that* $\omega \wedge \omega = [0]$*. Hence the curvature form is given by:*

$$\Omega = \mathrm{d}\omega = \left[\mathrm{d}\frac{-2\bar{z}}{1 + z\bar{z}} \wedge \mathrm{d}z\right] = \left[\frac{2\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1 + z\bar{z})^2}\right],$$

as desired. Also note that we may obtain the Gaussian curvature of our manifold with respect to this metric by feeding into Ω *a pair of orthonormal vector fields such as* $\frac{1}{\sqrt{2}}(1 + |z|^2)\partial_z$ *and* $\frac{1}{\sqrt{2}}(1 + |z|^2)\partial_{\bar{z}}$*.*

Before we leave this example, consider the following question: all of these computations that we've just done are over the patch $\mathbb{C}$ *in* $\mathbb{P}^1$*, and should all still apply to the restriction of the tangent bundle of* $\mathbb{P}^1$ *to* $\mathbb{C}$*. But we know that any vector bundle over* $\mathbb{C}$ *should be trivial and cannot have nontrivial Chern classes. So, what's going on?*

7.5.3 The Chern character

Recall the following: Let X be a complex n-manifold, E a rank r vector bundle over X. Let $s_1, \ldots, s_r$ be a generic set of r sections of E. For $1 \leq k \leq r$, the k^{th} Cher class of E

$$c_k(E) \in H^{2k}(X)$$

is given by the Poincar'e dual to the set of points $x \in X$ where

$$\{s_1(x), \ldots, s_{r-k+1}(x)\}$$

are linearly dependent.
Here are some properties of Chern classes.

1. $c_k(E)$ depends only on the topology of E.
2. If $0 \to E \to F \to G \to 0$ is an exact sequence of vector bundles over X, then $c(F) = c(E)c(G)$ where $c(E) = 1 + c_1(E) + \ldots + c_r(E) \in \oplus_i H^i(X)$. In particular, if $G = E \oplus F$, then $c(G) = c(E)c(F)$.
3. (Chern character). Suppose we can factorise $c(E) = (1+x_1)(1+x_2)\cdots(1+x_r)$ where $x_i \in H^2(X)$. The x_i are called Chern roots. The *Chern character* is
$$ch(E) = \sum_{i=1} e^{x_i}$$

where $x^{x_i} = 1 + x_i + \frac{x_i^2}{2!} + \ldots + \frac{x^n}{n!}$. Then

$$ch(E \otimes F) = ch(E)ch(F)$$

$$ch(E \oplus F) = ch(E) + ch(F).$$

Example 7.6. *If $X = \mathbb{P}^2$ and $E = T\mathbb{P}^2$, we calculated that $c(E) = 1+3\omega+3\omega^2$, $\omega \in H^2(\mathbb{P}^2)$ is the Poincaré dual to a hyperplane. We can factorise $c(E) = (1 + \frac{3+i\sqrt{3}}{2}\omega)(1 + \frac{3-i\sqrt{3}}{2}\omega)$. If we let $x_\pm = \frac{3\pm i\sqrt{3}}{2}\omega$, then the Chern character is*

$$ch(E) = e^{x_+} + e^{x_-} = 2 + 3\omega + \frac{3}{2}\omega^2.$$

Here is a special chase of the equality $ch(E \otimes F) = ch(E)ch(F)$. If E and F are line bundles, we have $c(E) = 1 + c_1(E)$. Thus

$$ch(E) = 1 + c_1(E) + \frac{c_2(E)^2}{2!} + \ldots = 1 + c_1(E) + \ldots$$

Then $ch(E \otimes F) = ch(E)ch(F)$ gives

$$1 + c_1(E \otimes F) + \ldots = 1 + c_1(E) + c_1(F) + \ldots$$

so we have the equality

$$c_1(E \otimes F) = c_1(E) + c_1(F).$$

7.5.4 Some vector bundles on $\mathbb{P}^n$

The *tautological line bundle on* $\mathbb{P}^n$ is

$$\mathcal{O}(-1) = \{(p, v) \mid p \in \mathbb{P}^n, v \in p\}.$$

The *hyperplane bundle on* $\mathbb{P}^n$ is $H = \mathcal{O}(1) = \mathcal{O}(-1)^*$. It has first Chern class $c_1(H) = \omega$ where ω is dual to a hyperplane. Since there is an isomorphism $\mathcal{O}(1) \otimes)(-1) \xrightarrow{\sim} \mathbb{C} = \prime(0)$, we have

$$c_1(\mathbb{C}) = c_1(\mathcal{O}(1)) + c_1(\mathcal{O}(-1)).$$

Since $c_1(\mathbb{C}) = 0$, we get $c_1(\mathcal{O}(-1)) = -\omega$.
Let's look at the tangent bundle for $\mathbb{P}^n$. We have an exact sequence

$$0 \to \mathbb{C} \to H^{\oplus n+1} \to T\mathbb{P}^n \to 0$$

The map $H^{\oplus n+1} \to T\mathbb{P}^n$ sends $(s_0, \ldots, s_n) \mapsto s_0\frac{\partial}{\partial x_0} + \ldots + s_n\frac{\partial}{\partial x_n}$, where $s_i \in \Gamma(H)$.
We have $c(H^{\oplus n+1}) = c(\mathbb{C})c(T\mathbb{P}^n)$. From this, we get $c(T\mathbb{P}^n) = (1 + \omega)^{n+1}$. In particular, $c_k(T\mathbb{P}^n) = \binom{n+1}{k}\omega^k$. The top Chern class is $c_n(T\mathbb{P}^n) = (n + 1)\omega^n$. After integration, we get that the Euler characteristic is

$$\chi(T\mathbb{P}^n) = \int_{\mathbb{P}^n} c_n = n + 1$$

7.5.5 Adjunction Formulas

We don't really know why this section was called "adjunction formulas".
Let X be a smooth hypersurface in $\mathbb{P}^n$ defined by a homogeneous degree d polynomial

$$P(x_0, \ldots, x_n) = 0.$$

Here's a **fun fact:** The *normal bundle* to X in $\mathbb{P}^n$ is $N_X := T\mathbb{P}^n/TX \cong \mathcal{O}(d)|_X$. Here $\mathcal{O}(d) := \mathcal{O}(1)^{\otimes d}$ and $|_X$ means we are restricting to the hypersurface $X \subset \mathbb{P}^n$. There is an exact sequence

$$0 \to TX \to T\mathbb{P}^n|_X \to \mathcal{O}(d)|_X \to 0.$$

Again, using the properties of chern classes, we have $c(T\mathbb{P}^n|_X) = c(TX)c(\mathcal{O}(d)|_X)$, so $(1+\omega)^{n+1} = c(TX)(1 + d.\omega)$. Here the d is **not** the total derivative, but is instead just the degree d of the polynomial. So we have

$$c(TX) = \frac{(1+\omega)^{n+1}}{1+d.w} = (1+\omega)^{n+1}(1 - d.\omega + d^2\omega^2 - \ldots)$$

.

Example 7.7. *If X is a degree d curve in $\mathbb{P}^2$ then*

$$c(TX) = \frac{(1+\omega)^3}{1+d.\omega} = (1 + 3\omega + 3\omega^2)(1 - d.\omega + d^2\omega^2) = 1 + (3-d)\omega.$$

So $c_1(TX) = (3-d)$, which gives us that the Euler characteristic is

$$\chi(X) = \int_X c_1(X) = \int (3-d)\omega = (3-d)\int_{\mathbb{P}^2} d.\omega^2$$

Since $\chi(X) = 2 - 2g$, where g is the genus of the curve X, we get the formula (after rearranging)

$$g = \frac{(d-1)(d-2)}{2}.$$

This is quite a nice relationship between the genus of a curve and the degree of the polynomial defining it.

Example 7.8. *This will be an important example for us. It is the quintic hypersurface in $\mathbb{P}^4$. Let X be a degree 5 hypersurface in $\mathbb{P}^4$. Then*

$$c(X) = \frac{(1+\omega)^5}{1+5\omega} = 1 + 10\omega^2 - 40\omega^3.$$

NB: $c_1(X) = 0$ so X is Calabi-Yau. The Euler characteristic is $\chi(X) = \int_X -40\omega^3 = \int_{\mathbb{P}^4}(5\omega)(-40\omega^3) = -200$.

8 Classical Mechanics

(P) We now switch gears into the world of Physics. A dynamical system consists of 2 parts.

1. A phase space manifold M. We think of this as the space of all states that the system can be in.

2. A Lagrangian $\mathcal{L} : M \to \mathbb{R}$.

   Example 8.1 (A free particle)**.** *This system consists of the following.*

- *Phase space:* $\mathbb{R}^6 = \{(x, y, z, \dot{x}, \dot{y}, \dot{z})\}$.

- $\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{x}^2 + \dot{z}^2)$.

The time evolution is determined as follows.

Definition 8.2 (Action)**.** *Given a trajectory* $q_i : [t_0, t_2] \to M$*, we define the* action *of* q *to be*

$$s[q] = \int_{t_0}^{t_1} dt\mathcal{L}(q_i).$$

The *true path* of the system is the trajectory for which

$$\frac{\delta s}{\delta q}|_{q=q_true} = 0.$$

This gives the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_i} + \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}) = 0.$$

8.1 Classical Field Theory

So we (some) begin with a "space-time" manifold. The simplest realistic example in *Minkowski space*: M^4. The coordinates are (t, x, y, z). It is pseudo-Riemannian. The space is $\mathbb{R}^4$ with metric $g = diag(-1, 1 \ldots, 1)$ where $diag$ is the diagonal matrix with entries -1 in the top left, and 1 in the other diagonal positions.
We turn M^4 into a fibre bundle (add fibres to it, so we are talking about fibre bundles over M^4). *Fields* are sections of the fibre bundle. The phase space is the set of all sections.
Some examples include:

- line bundles: temperateure.

- vector bundles: electric field.

- tensor bundles: gravitational field.

For fields, we define a *Lagrangian density*

$$\mathcal{L} : \{\Gamma(E \to M^4)\} \to C^\infty(M^4, \mathbb{R})$$

The action of a field F is

$$S[F] = \int_{\Omega \subset M^4} \mathcal{L}[F] dx \wedge dy \wedge dz \wedge dt$$

We minimise the action over the subspace over all sections F in $\Gamma(E)$ with some specified boundary conditions denoted by $F(\partial\Omega)$.

Example 8.3 (Electromagnetic field in a vacuum)**.** *Consider M^4 which is our spacetime manifold.*

$$\mathcal{L}[a] = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} F_{\mu\nu} F^{\mu\nu},$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $F^{\mu\nu} = q^{\mu\alpha} q^{\beta\nu} F_{\alpha\beta}$ and $A = A^0, A^1, A^2, A^3$. The metric is the one we saw earlier, $g = diag(-1, 1, 1, 1)$.

So the Lagrangian density is a map $\mathcal{L} : \Gamma(E) \to \Omega^n(M)$ where E is a fibre bundle over M and $\dim(M) = n$. The action of the field configuration is

$$S[F] = \int \mathcal{L}(F).$$

Here the field F is a smooth section of E.

Example 8.4 (Einstein-Hilbert action)**.** *The field is the metric tensor over $\mathbb{R}^4$, known as the Loretzian signature. The action for g is*

$$S[g] = \int d^4 x R(\sqrt{-g})$$

where R is the Ricci scalar and $g = \det(metric)$.

8.2 Non-relativistic quantum mechanics

The basic structures we will look at are as follows. There is a Hilbert space H over $\mathbb{C}$, the elements of which are called *kets*. We write

$$H = \{|\alpha\rangle\}.$$

We think of these as preparation states of our system.
The initial state of our system is an equivalence class of kets, under the equivalence relation

$$|\alpha\rangle \sim |\beta\rangle \iff |\alpha\rangle = \lambda|\beta\rangle$$

for $\lambda \in \mathbb{C} \setminus \{0\}$.

The dual space H^* has elements written as $\langle\alpha|$, and they are called *bras*. We think of these as measurement outcomes. We subject bras to the same equivalence relation of multiplication by a nonzero scalar.

does H^* have this relation or is this for something else later?

The next element of quantum mechanics gives us time evolution. It is a unitary operator $\hat{U}(t;t_0)$ and is generated by a Hermitian operator called the Hamiltonian as

$$\hat{U}(t;t_0) = \exp(-i\hat{H}(t-t_0))$$

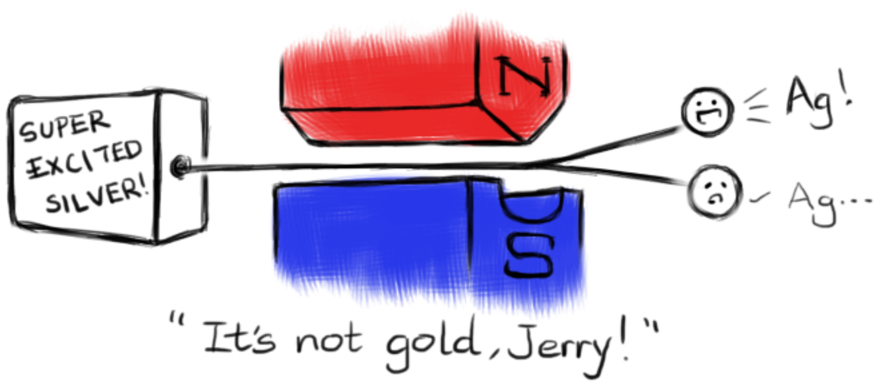
where $\exp(\hat{A}) = \mathbb{1} + \hat{A} + \frac{1}{2!}\hat{A}^2 + \cdots$.
The "probability amplitude" associated to a process is

$$\langle\alpha \mid U(t;t_0) \mid \beta\rangle = \langle\alpha \mid e^{-i(t-t_0)\hat{H}} \mid \beta\rangle.$$

The probability associated to this input/outcome combination is

$$Pr(\beta \to \alpha) = \frac{|\langle\alpha \mid e^{-i(t-t_0)\hat{H}} \mid \beta\rangle|^2}{|\langle\alpha \mid \alpha\rangle||\langle\beta \mid \beta\rangle|}$$

Example 8.5 (Sterm-Gerlach experiment)**.**



Imagine an oven if silver atoms inside it bouncing around. Imagine shooting these atoms through a hole in the oven through a non uniform magnetic field and seeing which ones are spin up $\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *or spin down* $\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ *(having specified a z-axis).*
Now send the spin up beam through a second Sterm Gerlach apparatus oriented in the x-direction. This again splits into two beams $\langle +|$ *and* $\langle -|$*. We represent these as*

$$\langle = (1/\sqrt{2}, 1/\sqrt{2}), \quad \langle -| = (1/\sqrt{2}, -1/\sqrt{2})$$

Then

$$Pr(\uparrow \to +) = |\langle + \mid \uparrow\rangle|^2 = |(1/\sqrt{2}, 1/\sqrt{2}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}|^2 = 1/2$$

8.3 Path Integrals

The raison d'être of quantum physicists is to find the value of $\langle \alpha \mid e^{-i\Delta t\hat{H}} \mid \beta\rangle$. For a point particle, we have a Hilbert space of kets $H = span(|q\rangle)$ where $q \in \mathbb{R}$ (i.e., q indexes space). We want to evaluate

$$\langle q_f \mid e^{iHT} \mid q_I\rangle$$

where $\langle q_f|$ is some fixed position and $|q_I\rangle$ is some fixed initial position. Here, we've replaced Δt with T.
To go about computing this, we divide T into N pieces $\delta_t = T/N$ and write

$$\langle q_F \mid e^{-i\delta_t H} e^{-i\delta_t H} \dots e^{-i\delta_t H} \mid q_I\rangle.$$

The basis $\{|q\rangle\}$ is othonormal in the following sense:

$$\langle q \mid q'\rangle = \delta(q - q')$$

Where δ is the Dirac delta function. We can represent momentum states as $\{|p\rangle\}$. These also form a basis for H and satisfy:

- $\langle p \mid p'\rangle = \delta(p - p')$.
- $\langle p \mid q\rangle = e^{-ipq}$.

Switching from p to q is equivalent to a Fourier transform.
The Hamiltonian for a free particle is:

$$\hat{H} = \frac{1}{2}\hat{p}^2,$$

where $\hat{p}$ is an operator which satisfies $\hat{p}|p\rangle = p|p\rangle$.
Since $\{|q\rangle\}$ forms a basis for H, we can write

$$\int dq|q\rangle\langle q| = \mathbb{1}.$$

Thus we can write our probability amplitude as

$$\langle q_f \mid e^{-i\delta_t H}(\int dq_{N-1}|q_{N-1}\rangle\langle q_{N-1}|) \cdots e^{-i\delta_t H}(\int dq_{N-1}|q_{N-1}\rangle\langle q_{N-1}|) \mid q_I\rangle$$

$$= (\prod_{j=1}^{N-1} \int dq_j)\langle q_F \mid e^{-i\delta_t H} \mid q_{N-1}\rangle \cdots \langle q_1 \mid e^{-i\delta_t H} \mid q_I\rangle$$

Now we pull the same trick that $\int \frac{dp}{2\pi}|p\rangle\langle p| = \mathbb{1}$ to get

$$\begin{aligned}
\langle q_{j+1} \mid e^{-i\delta_t H} \mid q_j\rangle &= \int \frac{dp}{2\pi}\langle q_{j+1} \mid e^{-i\delta_t H} \mid p\rangle\langle p \mid q_j\rangle \\
&= \int \frac{dp}{2\pi} e^{-i\delta_t \frac{1}{2}p^2} e^{ip(q_{j+1}-q_j)} \\
&= \left(\frac{-i}{\pi\delta_t}\right)^{\frac{1}{2}} \exp(\frac{i}{2}\frac{(q_{j+1}-q_j)^2}{\delta_t})
\end{aligned}$$

So all together we get

$$\langle q_f \mid e^{iHT} \mid q_I \rangle = \left(\frac{-i}{2\pi\delta_t} \right)^{\frac{N}{2}} ((\prod_{k=1}^{N-1} \int dq_k) \exp\left(i\frac{\delta_t}{2} \sum_{j=0}^{N-1} \left(\frac{q_{j+1} - q_j}{\delta_t} \right)^2 \right)$$

Letting $\delta_t \to 0$, we have $\left(\frac{q_{j+1}-q_j}{\delta_t}\right)^2 \to \dot{q}^2$ and $\delta_t \sum_{j=0}^{N-1} \to \int_o^T dt$. So we get

$$\langle q_f \mid e^{iHT} \mid q_I \rangle = \int D[q] \exp(i \int_0^T dt \frac{1}{2}\dot{q}^2)$$

where $\int D[q] = \lim_{N\to\infty} \left(\frac{-i}{\pi\delta_t}\right)^2 (\prod_{k=1}^{N-1} \int dq_k)$. But

$$\int_0^T dt \frac{1}{2}\dot{q}^2$$

is nothing other than the classical action for a point particle. In general, $\int D[q] \exp(i\delta[q])$ defines the "quantized" version of the classical system defined by the action S.

8.3.1 Wick Rotation

Take once more the action for a point particle

$$S[q] = \int_0^T dt[\frac{1}{2}(\dot{q})^2 - V(q)].$$

Suppose the integrand admits an analytic continuation to $\mathbb{C}$, i.e., we let t in $q(t)$ take complex values. The integral is now over a contour in $\mathbb{C}$. Via a change of variables we get

$$\begin{aligned} i \in_0^T dt[\frac{1}{2}(\dot{q})^2 - V(q)] &= -\int_0^T d(-it)[\frac{1}{2}\left(\frac{d(-it)}{dt}\frac{dq}{d(-it)}\right)^2 - V(q)] \\ &= -\int_0^{-iT} d\tau[\frac{1}{2}\left(\frac{dq}{d\tau}\right)^2 + V(q)] \end{aligned}$$

The result, $\int d\tau[\frac{1}{2}\left(\frac{dq}{dt}\right)^2 + V(q)]$ is called the Euclidean Action. It is called that because when $t \mapsto -it$ then $-dt^2 + dx^2 \mapsto dt^2 + dx^2$.
Anyway, this means we are now studying

$$Z = \int D[X] e^{-S_E[X]}$$

which is called the partition function. From now on, we will be given the Euclidean action and this will be our starting point.

Recall that a classical field consists of a fibre bundle and an action $S : \Gamma(E) \to \mathbb{R}$. A quantum field theory consists of evaluating

$$\int D[X]e^{-S_E[X]}$$

over the space $\Gamma(E)$ subject to boundary conditions.

9 Classifying spaces and equivariant cohomology

The goal of this section is to learn to compute cohomology - especially when our spaces of a group acting on them.

Here's the idea. Let X be a space with an action of a group G. We will be defining the G-equivariant cohomology of X, which we will denote by appending a subscript G to our normal notation for cohomology: $H^*_G(X)$. In our heads, we should think think

$$H^*_G(X) \text{``} = \text{''} H^*(X/G),$$

the cohomology of the quotient space. This is NOT the definition because if G doesn't act freely on X, then the quotient won't be very nice. For example, if X were a manifold, then we could get orbifold points in X/G, so it will be hard to get statements like Poincaé duality. Basically, this definition would only work if G acted freely. So what should we do?

Here is the solution: we will replace X with a (weakly) homotopy equivalent space Y on which G does act freely. We can then define $H^*_G(X) := H^*(Y/G)$. The question now becomes: how can we find such a Y? The answer comes from the theory of classifying spaces.

Definition 9.1. *Let G be a group. The* classifying space *of G, denoted BG is the base space of the universal principal G-bundle*

In our heads, we should think of a G-bundle like a vector bundle, where the fibres are G. The "universal" part of the definition means that BG satisfies the following universal property. Denote the universal bundle by $\pi : EG \to BG$. If $p : E \to B$ is a principal G-bundle, then there exists a map $f : B \to BG$ such that

$$(p, E) \simeq (f^*\pi, EG).$$

i.e., we have the following diagram.

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "p"'] & EG \arrow[d, "\pi"] \\
B \arrow[r, "f"'] & BG
\end{tikzcd}$$

From this, we get that there is a correspondence between principal G-bundles over X and homotopy classes of maps from $X \to BG$.

9.1 Real life

In real life, if we can find a weakly contractible [10] space EG with a free G action, then we can set $BG = EG/G$. This will give us a universal principal G bundle $EG \to BG$ where the map is the quotient map. This is one way to find classifying spaces.

Example 9.2. *$B\mathbb{Z}$. $\mathbb{Z}$ acts freely on $\mathbb{R}$ via translations and $\mathbb{R}$ is contractible. Thus $B\mathbb{Z} = \mathbb{R}/\mathbb{Z} = S^1$.*

Example 9.3. *What about $B(\mathbb{Z}/2\mathbb{Z})$? Well $\mathbb{Z}/2\mathbb{Z}$ acts freely on the n-sphere S^n by $x \mapsto -x$. Unfortunately $\pi_n S^n = \mathbb{Z} \neq 0$, so S^n is not weakly contractible. However, that is fine because we have S^∞! S^∞ is weakly contractible[11] and $\mathbb{Z}/2\mathbb{Z}$ acts freely upon it (again via $x \mapsto -x$). The upshot is: we can take $E\mathbb{Z}/2\mathbb{Z} = S^\infty$ so that $B\mathbb{Z}/2\mathbb{Z} = \mathbb{RP}^\infty$.*

A similar method shows that $ES^1 = S^\infty$ and $BS^1 = \mathbb{CP}^\infty$. Work it out! (Hint: S^1 is homotopic to $\mathbb{C}^\times$)

9.2 Equivariant cohomology

Example 9.4. *For X a single point, and G any group. From our initial philosophy, we should have*

$$H^*_G(X) = H*(EG/G) = H^(BG).$$

This will indeed be true! Actually, for those that know about this stuff, it is also the case that $H^*_G(X) = H^*(G)$, the group cohomology of X (we haven't really defined what this is yet, but the take home message is that in this case equivariant cohomology should really only know about the group itself). For others, the group cohomology $H^*(G)$ is defined *to be $H^*(BG)$ (at least with $\mathbb{Z}$ coefficients) so this is either mildly interesting or tautological depending on your background.*

So for example, based on our previous work,

$$H^*_{S^1}(pt) = H^*\mathbb{CP}^\infty = \mathbb{Q}[t]$$

is a polynomial algebra. More generally, for $G = \mathbb{T}$ a torus (here we mean $\mathbb{C}^\times)^n$, then we can use the Kunneth theorem to show that

$$H^*_{\mathbb{T}}(pt) = \mathbb{Q}[t]^{\otimes n} = \mathbb{Q}[t_1, \ldots, t_n].$$

Anyway, here's the definition.

[10] A space is X "weakly contractible" if $\pi_n(X) = 0$ for all n

[11] One way to see that S^∞ is contractible is to embed it in $L^2[0,1]$ (a countable dimensional Hilbert space) as the elements of norm 1 ie. as functions $f : [0,1] \to \mathbb{R}$ with $\int_[0,1]f^2 = 1$. Given $f \in S^\infty \subset L^2$, we can define a family of functions $f_t, 0 \leq t \leq$ by setting $f_t(x) = f(x/t)$ (ie. f but *faster* for $x \leq t$ and $f_t(x) = 1$ thereafter. Letting t tend to zero we get an explicit deformation retract of S^∞ to the constant function with value 1.

Definition 9.5. *$H_G^*(M) := H^*(M \times_G EG)$ where $M \times_G EG$ is the quotient of the cartesian product $M \times EG$ by the equivalence relation $(m, ge) \sim (mg, e)$ for all $g \in G$ together with the G-action.* [12]

I guess at this point you should check that this definition works and the right hand side has a free G-action (before quotienting) and stuff like that. Check it!
Remark: The map $M \to pt$ is G-equivariant and induces a map $H_G^*(pt) \to H_G^*(M)$. This means that $H_G^*(M)$ is a $H_G^*(pt)$ module.
Facts:

1. If G acts trivially on M, then $H_G^*(M) = H_G^*(pt) \otimes H^*(M, \mathbb{Q})$.
2. If G acts freely, then M/G is already nice and $H_G^*(M) = H^*(M/G)$.

Okay, here's a theorem to help us eventually compute some cohomology:

Theorem 9.6. *Let G be a torus, $((\mathbb{C}^\times)^m)$. Let $F = M^G$ be the G fixed points. Then, at least up to torsion, we have*

$$H_G^*(M) = H^*(F)[t_1, \ldots, t_m].$$

By "up to torsion", we mean that this becomes true after tensoring with $\mathbb{Q}(t_1, \ldots, t_n)$ ie we allow rational functions and not simply polynomials.

9.2.1 De Rham model

We are going to build a De Rham-ish way to think of equivariant cohomology for circle actions.
Let $G = S^1$ act on M. Let X be a vector field generating the action. This is okay since S^1 acts by a 1 parameter family of diffeomorphisms, which gives us flows. Let $i(x)$ be the "interior derivative" with respect to X. What is the interior derivative? Well for $\omega \in \Omega^{k+1}$

$$i(X)(\omega(Y_1, \ldots, Y_k) = \omega(X, Y_1, \ldots Y_k).$$

So $i(X) : \Omega^* \to \Omega^{*-1}$.
Define

$$d_X = d + ui(X)$$

where d is the exterior derivative and u is a parameter with formal degree 2 i.e., u is an indeterminate element of Ω^2. We'd like this new d_X to be a differential, but $d_X^2 \neq 0$ in general.

[12] Dougal: I *think* this is what the "G-fibred product" notation $\times_G$ means. In TriThang's original version of the notes, he thought it meant the subset of $M \times G$ consisting of pairs (m, e) satisfying $(m, ge) = (mg, e)$ for all $g \in G$. But that seems weird, since it would just give $M^G \times EG^G$. We also quotiented by a G-action $g(m, e) = (mg, e) = (m, ge)$ before taking cohomology. But this isn't actually a group action for non-abelian groups, and that's not what's in the book anyway. We wanted to quotient because we should have $H_G^*(M) = H^*(M/G)$ for G acting trivially on M, but I think this is achieved anyway by the quotient in the definition of $M \times_G EG$.

However, if we restrict to the subcomplex $\Omega_X^* \subset \Omega_X$ where $di_X + i_x d = 0$ (the operator on the left is sometimes called the Lie derivative), then d_X is a differential on this subcomplex.
Moral: (Ω_X^*, d_X) is a chain complex and $H_{S^1}^*(M) = \mathrm{ker} d_X / \mathrm{im} d_X$. Magic! Maybe (really maybe) we will say how this works next time.

9.3 Atiyah-Bott localisation

We will state the theorem, then explain what it means after which we will give some examples.

Theorem 9.7 (Atiyah-Bott localisation)**.** *Let M be a compact manifold with a $T = (\mathbb{C}^\times)^n$ action. If $\varphi \in H_T^*(M)$, then*

$$\varphi = \sum_F \frac{i_* i^* \varphi}{\underline{e}(N_{f/M})}$$

where the sum is over connected componentts F of the fixed locus M^T and $i : F \hookrightarrow M$ is its inclusion into M.

9.3.1 Explaining the theorem

What are the i things?

$$i^* : H_T^*(M) \to H_T^*(F)$$

is the usual pullback of equivariant cohomology that one expects. To define the pushforward

$$i_* : H_T^*(F) \to H_T^*(M)$$

we need to use Poincaré duality. It is defined as the sequence of maps

$$H_T^p(F) \xrightarrow{\sim} H_T^{\dim F - p}(F)^\wedge \xrightarrow{(i^*)^\wedge} H_M^{\dim F - P}(M)^\wedge \xrightarrow{H}{}_T^{\dim M - \dim F + p}(M).$$

One might ask if this is right since we haven't said that Poincaré duality holds for equivariant cohomology. This is something to check, but is part of the reason why we introduced the funny definition of equivariant cohomology i.e., to make nice properties like Poincaré duality work.
What is $\underline{e}(N_{F/M})$?
In the Mirror Symmetry book we have been following [1], the notation does not have the underline as we do. But we think that this is just notational convention to not have the underline and that to understand it, it really should. Without the underline, $e(N_{F/M})$ is the equivariant Euler class of the normal bundle $N_{F/M}$ to F in M. With the underline, $\underline{e}(N_{F/M})$ is the class in $H_T^*(M)$ with $i^*\underline{e}(N_{F/M}) = e(N_{F/M})$. It turns out that $\underline{e}(N_{F/M}) = i_*1$.
In particular $i^*i_*1 = e(N_{F/M}$. Reason: 1 is Poincaré dual to the homology class of F in F. i_*1 is dual to the homology class of F in M. i^*i_* is the intersection of F with some generic representative of its homology class. $N_{F/M}$ looks like a neighbourhood of F.

YI TO DRAW A SECTIONY PICTURE AND ADD SOME WORDS IF NECESSARY

Why are we allowed to divide?
$H_T^*(M)$ is an algebra over $H_T^*(pt) = \mathbb{Q}[t_1, \dots, t_n]$, $t_i \in H_T^2(pt)$. The hard part of the theorem is that once we localise the ring $H_T^*(M)$, i.e., replace it with

$$H_T^*(M) \otimes_{H_T^*(pt)} \mathbb{Q}(t_1, \dots, t_n)$$

we *can* divide by $\underline{e}(N_{F/m})$.
Interpretation There is a projection map.

$$\pi^M : M \to \{pt\}$$

From this, we get an induced map

$$\pi_*^M : H_T^*(M) \to H_T^{*-\dim M}(pt) \cong \mathbb{Q}(t_1, \dots t_n)$$

For top dimensional classes, this is integration of differential forms. Applying the localisation theorem to π_*^M we get

$$\int_M \varphi = \sum_F \int_F \frac{i^*\varphi}{e(N_{F/M})}$$

9.3.2 Examples

Corollary 9.8. *Suppose T acts on M with exactly m fixed points. Then $\chi(M) = m$.*

Proof.

$$\begin{aligned}
\chi(M) &= \int_M e(TM) \\
&= \sum_F \int_F \frac{i^*e(TM)}{e(N_{F/m})} \\
&= \sum_F \int_F \frac{i^*(e(TM))}{e(i^*TM)} \\
&= \sum_F 1 \\
&= m
\end{aligned}$$

□

Example 9.9. $\mathbb{P}^{m-1} = \{[X_0 : X_1 : \cdots : X_{m-1}]\}$. $(\mathbb{C}^\times)^{m-1}$ *acts on* $\mathbb{P}^{m-1}$ *by*

$$\lambda_1, \dots, \lambda_{m-1}.[X_0 : X_1 : \cdots : X_{m-1}] = [X_0, \lambda_1 X_1 : \cdots : \lambda_{m-1} X_{m-1}]$$

The fixed points are $\{[1 : 0 : \cdots 0], [0 : 1 : \cdots : 0], \dots, [0 : 0 : \cdots : 1]\}$. *There are* M *of them so* $\chi(\mathbb{P}^{m-1}) = m$.

9.3.3 Application to calculating Gromov-Witten invariants of Calabi-Yau 3-folds

Quantum cohomology has ordinary cohomology plus other “numbers” which are called Gromov-Witten invariants. We can think of these numbers as Taylor coefficients to a Taylor expansion in some parameter q.
There is something called the Sigma model of string theory. The idea is that we want to count holomorphic maps from Riemann surface (or algebraic curves / complex 1-folds) to a fixed target manifold X. Often X is a Calabi-Yau 3-fold.

[YI DRAW]

Gromov Witten invariants are some of these counts.
To do this rigorously, we come up with a moduli space of maps from Riemann surfaces to X denoted by

$$\mathcal{M}_{g,n}(X,\beta).$$

The g stands for the genus of the domain curve, n the number of marked points and $\beta \in H_2(X)$ is the homology class of the image. We can break the moduli space into these bits because they are disconnected components of $\mathcal{M}(X)$. Unfortunately, this moduli space is not compact. Happily, there is a way to compactify it, and we denote its compactification by $\overline{\mathcal{M}}_{g,n}(X,\beta)$. Basically, this allows our domain to curves to have nodal singularities which are limits of non-nodal curves.
If X is a quintic hypersurface in $\mathbb{P}^4$, then X is a Calabi-Yau 3-fold. This means that $c_1(TX) = 0$. We also have an inclusion

$$\overline{\mathcal{M}}_{g,n}(X,\beta) \subset \overline{\mathcal{M}}_{g,n}(\mathbb{P}^4,\beta').$$

We want to count maps to $\mathbb{P}^4$ that end up in X. X is the zero set of a degree 5 polynomial. We saw earlier (in the section about sections) that degree 5 polynomials correspond to sections of $\mathcal{O}(5)$. So X is the zero set of a section of $\mathcal{O}(5)$. $\mathcal{O}(5)$ gives us a vector bundle on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^4,\beta')$ with the fibre over a “point” $f : C \to \mathbb{P}^4$ of $\overline{\mathcal{M}}_{g,n}$ being the space of global sections of $f^*(\mathcal{O}(5))$. s induces a global section $\tilde{s}$ of E. The zero set of $\tilde{s}$ is the set of maps with image in X. So to count maps to X, we can compute the Euler class of E. It turns out that for X and Calabi-Yau 3-fold, the dimension

$$\dim \overline{\mathcal{M}}_{g,n}(X,\beta) = 0.$$

So we can count it by integrating the Euler class of E :

$$\int_{\mathcal{M}_{g,0}(\mathbb{P}^4,\beta')} e(E).$$

In particular, localisation is useful for computing stuff like this!

10 Complex manifolds

Here are some examples of complex manifolds to keep in mind: $\mathbb{C}, \mathbb{C}^\times, \mathbb{C}^n, \mathbb{C}^n/\wedge^n = T^n, \mathbb{P}^n, Gr_n(\mathbb{C}^{m+n})$.
Complex manifolds are spaces that look locally like $\mathbb{C}^n$ with holomorphic transistion functions. Holomorphic can mean one of three things.

1. Exapandable in power series.
2. Satisfy Cauchy-Riemann equations + square integrable.
3. Complex differentiable in each variable.

Here is the motivating question. Given a complex manifold of dimension n, we get a real manifold of dimension $2n$. When can we go back the other way?
Given a complex manifold M, it has a tangent bundle TM where everything can be written in terms of complex coordinates. We can multiply by i on each tangent space. In a sense, i determines a notion of angle on M since it tells us how to rotate by $\pi/2$. This is the key idea that we will be focusing on.

Definition 10.1. *Let M be a real manifold M (we probably want of even dimension). M is* almost complex *if there is a smoothly varying map*

$$J : TM \to TM$$

such that $J^2 = -1$ on tangent vectors. i.e., for $p \in M$, the map $J_p : T_pM \to T_pM$ is represented by a matrix whose square is $-I$.

Coming back to the motivating question. Let M be a real manifold of even dimension equipped with an almost complex structure

$$J : TM \to TM.$$

Theorem 10.2 (Newlander-Ninenberg)**.** *If*

$$[JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] = 0$$

for all vector fields X and Y then M can be made uniquely into a complex manifold where $J : TM \to TM$ corresponds to multiplication by i.

11 Sheaf cohomology

11.1 Čech cohomology

Let M be a manifold. Recall that a presheaf $\mathcal{F}$ of abelian groups on M assigns to every open set $U \subseteq M$ an abelian group $\mathcal{F}(U)$ and to every inclusion $U \subseteq V$ an abelian group homomorphism $\mathcal{F}(V) \to \mathcal{F}(U)$. We call the elements of $\mathcal{F}(U)$ sections of $\mathcal{F}$ over U. The maps $\mathcal{F}(V) \to \mathcal{F}(U)$ restriction maps, and the notation $s|_U$ is often used for the image of $s \in \mathcal{F}(V)$ in $\mathcal{F}(U)$. We say that $\mathcal{F}$ is a sheaf if $\mathcal{F}$ also satisfies the following gluing condition.

Definition 11.1 (Gluing condition)**.** *Let $U \subseteq M$ be open, let $\{U_i\}_{i \in I}$ be a cover of U by open sets $U_i \subseteq U$, and let $(s_i)_{i \in I}$ be a family of sections $s_i \in \mathcal{F}(U_i)$ such that for all $i, j \in I$,*

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}.$$

Then there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

Example 11.2 (Constant sheaves)**.** *Let A be any abelian group. The constant A-sheaf on M is the sheaf $\underline{A}$ defined by*

$$\underline{A}(U) = \{s : U \to A \mid s \textit{ is locally constant.}\}$$

Example 11.3 (Sheaf of sections)**.** *Let $\pi : F \to M$ be a vector bundle on M. Define*

$$\mathcal{F}(U) = \{s : U \to F \mid \pi \circ s = \mathrm{id}_U\}$$

Then $\mathcal{F}$ is a sheaf of abelian groups on M, called the sheaf of sections of F.

It turns out that a very useful way to study sheaves is to study something called sheaf cohomology. For manifolds, sheaf cohomology is the same as Čech cohomology, which is defined as follows. Let M be a manifold, $\mathcal{F}$ a sheaf of abelian groups on M. Let $\{U_i\}_{i \in I}$ be an open cover of M such that the intersection of finitely many of the U_i is either empty or contractible to a point. Such a cover is called a "good cover". For all $p \in \mathbb{Z}_{\geq 0}$, define

$$C^p = \left\{ \sigma = (\sigma_{i_0, i_1, \cdots, i_p})_{i_0, i_1, \cdots, i_p \in I} \;\middle|\; \begin{array}{c} \sigma_{i_0, \cdots, i_p} \in \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_p}) \\ \sigma_{i_0, \cdots, i_p} \text{ is antisymmetric in } i_0, i_1, \ldots, i_p \end{array} \right\}.$$

Define $d^p : C^p \to C^{p+1}$ by

$$(d^p \sigma)_{i_0, \cdots, i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{i_0, \cdots, \hat{i_j}, \cdots, i_{p+1}}$$

for all $\sigma \in C^p$. On the right hand side, all terms are restricted to the set $\mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_{p+1}})$ so that the sum makes sense. The Čech complex is

$$0 \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \cdots$$

where d^{-1} is defined to be the zero map. An easy check shows that $d^p \circ d^{p-1} = 0$ for all $p \in \mathbb{Z}_{\geq 0}$, so we can define

$$H^p(M, \mathcal{F}) = \frac{\ker d^p}{\operatorname{im} d^{p-1}}$$

called the Čech (or sheaf) cohomology of $\mathcal{F}$. It turns out that this is independent of the choice of good cover $\{U_i\}$.

Example 11.4. *Let M be any manifold, $\mathcal{F}$ any sheaf of abelian groups on M. Then it follows from the gluing condition for $\mathcal{F}$ that*

$$\begin{aligned}
H^0(M, \mathcal{F}) &= \frac{\ker d^0}{\operatorname{im} d^{-1}} \\
&= \ker d^0 \\
&= \{\sigma = (\sigma_i)_{i \in I} \mid d\sigma = 0\} \\
&= \{\sigma = (\sigma_i)_{i \in I} \mid \sigma_j|_{U_i \cap U_j} - \sigma_i|_{U_i \cap U_j} = 0 \text{ for all } i, j \in I\} \\
&= \{\sigma = (\sigma_i)_{i \in I} \mid \text{There exists } s \in \mathcal{F}(M) \text{ such that } \sigma_i = s|_{U_i} \text{ for all } i \in I\} \\
&= \mathcal{F}(M)
\end{aligned}$$

Example 11.5. *The Čech cohomology of the constant $\mathbb{Z}$-sheaf on M is the same as the singular cohomology of M, i.e.*

$$H^p(M, \underline{\mathbb{Z}}) = H^p(M, \mathbb{Z})$$

In particular, for $M = \mathbb{CP}^1 = S^2$, we have[13]

$$H^0(M, \underline{\mathbb{Z}}) = \mathbb{Z}; \quad H^1(M, \underline{\mathbb{Z}}) = 0; \quad H^2(M, \underline{\mathbb{Z}}) = \mathbb{Z}$$

11.2 Čech-de Rham isomorphism

Theorem 11.6 (Čech-de Rham isomorphism)**.** *Let M be a smooth manifold. Then for all $p \in \mathbb{Z}_{\geq 0}$,*

$$H^p(M, \underline{\mathbb{R}}) = H^p_{\mathrm{dR}}(M)$$

where $H^p_{\mathrm{dR}}(M)$ is the de Rham cohomology of M.

The aim of this section is to understand enough of the proof of the Čech-de Rham isomorphism so that we can generalise it to sheaves other than $\underline{\mathbb{R}}$. The first ingredient is the idea of an acyclic resolution.

Definition 11.7 (Acyclic resolution)**.** *A sheaf $\mathcal{A}$ on a manifold M is called* acyclic *if for every $p > 0$, we have $H^p(M, \mathcal{A}) = 0$. If $\mathcal{F}$ is any sheaf of abelian groups on M, an* acyclic resolution *of $\mathcal{F}$ is a complex of sheaves of abelian groups on M*

$$0 \xrightarrow{d^{-1}} \mathcal{A}^0 \xrightarrow{d^0} \mathcal{A}^1 \xrightarrow{d^1} \mathcal{A}^2 \xrightarrow{d^2} \cdots$$

such that there is an exact sequence of sheaves of abelian groups

$$0 \to \mathcal{F} \to \mathcal{A}^0 \xrightarrow{d^0} \mathcal{A}^1 \xrightarrow{d^1} \mathcal{A}^2 \xrightarrow{d^2} \cdots$$

Note that for sheaves, "exact sequence" means that everything in the kernel of an arrow is *locally* in the image of the preceding arrow. Acyclic resolutions are nice because we can use them to compute Čech cohomology.

[13] Dougal: I did some of these calculations on the board. There is a very small chance that I will fill in the details later.

Theorem 11.8. *Let $\mathcal{F}$ be a sheaf of abelian groups on M and let*

$$0 \xrightarrow{d^{-1}} \mathcal{A}^0 \xrightarrow{d^0} \mathcal{A}^1 \xrightarrow{d^1} \mathcal{A}^2 \xrightarrow{d^2} \cdots$$

be an acyclic resolution for $\mathcal{F}$. Then for all $p \geq 0$,

$$H^p(M, \mathcal{F}) = \frac{\ker d^p(M)}{\operatorname{im} d^{p-1}(M)}$$

where $d^p(M) : \mathcal{A}^p(M) \to \mathcal{A}^{p+1}(M)$ is the map on global sections induced by d^p.

Proof. Play around with long exact sequences of cohomology and use the fact that $H^p(M, \mathcal{A}^q) = 0$ for $p > 0$, and that $H^0(M, \mathcal{G}) = \mathcal{G}(M)$ for all sheaves $\mathcal{G}$. I may fill this in later. $\square$

To use this theorem, we need to know some acyclic sheaves. Fortunately, there are lots of them.

Proposition 11.9. *Let $\mathcal{A}$ be a fine sheaf (i.e. a sheaf with "partitions of unity"). Then $\mathcal{A}$ is acyclic. In particular, the sheaf of smooth sections of any vector bundle on M is acyclic.*

Let Ω^p denote the sheaf of smooth sections of the (real) vector bundle $\bigwedge^p T^*M$. Recall that $\Omega^0 = \mathcal{O}_M^{sm}$ is the sheaf of smooth functions on M. By Proposition 11.9, Ω^p is an acyclic sheaf for each $p \geq 0$. Consider the complex of sheaves on M

$$0 \to \underline{\mathbb{R}} \to \mathcal{O}_M^{sm} = \Omega^0 \to \Omega^1 \to \Omega^2 \to \cdots \tag{3}$$

Lemma 11.10 (Poincaré lemma)**.** *The complex (3) is an exact sequence of sheaves.*

Thus,

$$0 \xrightarrow{d^{-1}} \Omega^0 \xrightarrow{d^0} \Omega^1 \xrightarrow{d^1} \Omega^2 \xrightarrow{d^2} \cdots$$

is an acyclic resolution of $\underline{\mathbb{R}}$, so by Theorem 11.8,

$$H^p(M, \underline{\mathbb{R}}) = \frac{\ker d^p(M)}{\operatorname{im} d^{p-1}(M)} =: H^p_{\mathrm{dR}}(M).$$

11.3 Cohomology of holomorphic vector bundles

Let M be a complex manifold. Our aim is to come up with de Rham models for the cohomology of arbitrary holomorphic vector bundles on M. We'll start with the simplest example: the trivial line bundle on M. Its sheaf of sections is just the sheaf $\mathcal{O}_M^{hol}$ of holomorphic functions on M. Recall that a smooth, complex-valued function f is holomorphic if and only if $\bar{\partial} f = 0$.[14] So we have an exact sequence of sheaves

$$0 \to \mathcal{O}_M^{hol} \to \mathcal{O}_M^{sm} = \Omega^{0,0} \xrightarrow{\bar{\partial}} \Omega^{0,1}$$

[14]This is just the Cauchy-Riemann equations written in a fancy way.

where $\mathcal{O}_M^{sm}$ is the sheaf of smooth, complex-valued functions on M and $\Omega^{p,q} = \bigwedge^p T^*M \otimes \bigwedge^q \bar{T}^*M$ is the sheaf of smooth (p,q)-forms. To extend this, we use the following lemma.

Lemma 11.11 ($\bar{\partial}$-Poinaré lemma)**.** *The complex of sheaves on M*

$$0 \to \mathcal{O}_M^{hol} \to \Omega^{0,0} \xrightarrow{\bar{\partial}^{0,0}} \Omega^{0,1} \xrightarrow{\bar{\partial}^{0,1}} \Omega^{0,2} \xrightarrow{\bar{\partial}^{0,2}} \cdots$$

is exact.

Therefore,

$$0 \xrightarrow{\bar{\partial}^{0,-1}} \Omega^{0,0} \xrightarrow{\bar{\partial}^{0,0}} \Omega^{0,1} \xrightarrow{\bar{\partial}^{0,1}} \Omega^{0,2} \xrightarrow{\bar{\partial}^{0,2}} \cdots$$

is an acyclic resolution of $\mathcal{O}_M^{hol}$. So we can apply Theorem 11.8 again to get

$$H^q(M, \mathcal{O}_M^{hol}) = \frac{\ker \bar{\partial}^{0,q}(M)}{\operatorname{im} \bar{\partial}^{0,q-1}(M)}$$

More generally, let F be any holomorphic vector bundle on M, and let $\mathcal{F}$ be its sheaf of holomorphic sections. Then we get an exact sequence of sheaves on M

$$0 \to \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_M^{hol}} \mathcal{O}_M^{sm} = \mathcal{F} \otimes_{\mathcal{O}_M^{hol}} \Omega^{0,0} \xrightarrow{\bar{\partial}^{0,0}} \mathcal{F} \otimes_{\mathcal{O}_M^{hol}} \Omega^{0,1} \xrightarrow{\bar{\partial}^{0,1}} \mathcal{F} \otimes_{\mathcal{O}_M^{hol}} \Omega^{0,2} \xrightarrow{\bar{\partial}^{0,2}} \cdots$$

Note that the maps here make sense since $\bar{\partial}$ is $\mathcal{O}_M^{hol}$-linear. Exactness follows from the $\bar{\partial}$-Poincaré lemma since $\mathcal{F}$ is "locally free". The sheaf $\mathcal{F} \otimes_{\mathcal{O}_M^{hol}} \Omega^{0,q}$ is just the sheaf of smooth sections of the vector bundle $F \otimes \bigwedge^q \bar{T}^*M$, and is therefore acyclic. So

$$0 \to \mathcal{F} \otimes_{\mathcal{O}_M^{hol}} \Omega^{0,0} \xrightarrow{\bar{\partial}^{0,0}} \mathcal{F} \otimes_{\mathcal{O}_M^{hol}} \Omega^{0,1} \xrightarrow{\bar{\partial}^{0,1}} \mathcal{F} \otimes_{\mathcal{O}_M^{hol}} \Omega^{0,2} \xrightarrow{\bar{\partial}^{0,2}} \cdots$$

is an acyclic resolution for $\mathcal{F}$, so we can use it to compute $H^q(M, \mathcal{F})$. In particular, if we set $F = \bigwedge^p T^*M$, we get the following result.

Theorem 11.12 (Čech-Dolbeault isomorphism)**.** *Let $\bigwedge^p T^*M$ denote the sheaf of holomorphic $(p,0)$-forms. Then*

$$H^q\left(M, \bigwedge^p T^*M\right) = \frac{\ker \bar{\partial}^{p,q}}{\operatorname{im} \bar{\partial}^{p,q-1}} =: H^{p,q}(M)$$

12 Hodge diamonds

We want to compute the "Hodge numbers" $h^{p,q} = \dim H^{p,q}(M)$ for Calabi-Yau 3-folds M. We'll start by studying some properties.

12.1 Serre duality

One very deep relationship between sheaf cohomology groups is the following. Let M be a compact complex manifold of complex dimension n, let F be a vector bundle on M and let $\mathcal{F}$ be its sheaf of holomorphic sections. Denote by K_M the sheaf of holomorphic sections of the canonical bundle of M. Then

$$H^p(M, \mathcal{F}) = (H^{n-p}(M, \mathcal{F}^* \otimes K_M))^*$$

In particular, if M is Calabi-Yau, then $K_M = \mathcal{O}_M$, so for $\mathcal{F} = \mathcal{O}_M$, this gives

$$H^{0,q}(M) = H^q(M, \mathcal{O}_M) = (H^{n-q}(M, \mathcal{O}_M))^* = (H^{0,n-q}(M))^*$$

and hence

$$h^{0,q} = h^{0,n-q} \tag{4}$$

12.2 Hodge theory

In Hodge theory, we try to understand the Dolbeault cohomology groups $H^{p,q}(M)$ of a compact complex manifold M by choosing a canonical representative for each cohomology class. We do this as follows.
Fix a Hermitian metric h on M. There is a natural way, depending only on the choice of h, to construct Laplacians

$$\Delta_d : \Omega^r(M) \to \Omega^r(M); \quad \Delta_\partial : \Omega^{p,q}(M) \to \Omega^{p,q}(M); \quad \Delta_{\bar\partial} : \Omega^{p,q}(M) \to \Omega^{p,q}(M).$$

A differential form ω is called $\bar\partial$-harmonic (respectively d-harmonic) if $\Delta_{\bar\partial}(\omega) = 0$ (respectively $\Delta_d(\omega) = 0$. Let $\mathcal{H}^{p,q}_{\bar\partial}(M)$ denote the set of $\bar\partial$-harmonic (p,q)-forms on M, and let $\mathcal{H}^r_d(M)$ denote the set of d-harmonic r-forms on M. Then

Theorem 12.1 (Hodge Theorem). *The maps*

$$\begin{aligned} \mathcal{H}^{p,q}_{\bar\partial}(M) &\to H^{p,q}(M) \\ \omega &\mapsto [\omega]_{\bar\partial} \end{aligned}$$

and

$$\begin{aligned} \mathcal{H}^r_d(M) &\to H^r(M) \\ \omega &\mapsto [\omega]_d \end{aligned}$$

are isomorphism of vector spaces. Here we can take $H^r(M)$ and $\mathcal{H}^r_d(M)$ to be given in terms of either real or complex-valued differential forms.

If our metric g is Kähler, we in fact have $\Delta_d = 2\Delta_{\bar\partial} = 2\Delta_\partial$, so in particular, ω is $\bar\partial$-harmonic if and only if ω is d-harmonic. So we have

$$\mathcal{H}^r_d(M) = \bigoplus_{p+q=r} \mathcal{H}^{p,q}_{\bar\partial}(M)$$

which gives via the Hodge Theorem a decomposition

$$H^r(M) = \bigoplus_{p+q=r} H^{p,q}(M).$$

Note that this Hodge decomposition depends on the choice of Kähler metric h! The Hodge decomposition gives us the relation

$$b_r = \sum_{p+q=r} h^{p,q} \tag{5}$$

where $b_r = \dim H^r(M)$ is the rth Betti number of M.
We can also use Hodge theory to construct a complex conjugation map $H^{p,q}(M) \to H^{q,p}(M)$. Notice that we always have a complex conjugation map $\Omega^{p,q}(M) \to \Omega^{q,p}(M)$. Since $\Delta_d : \Omega^r(M) \to \Omega^r(M)$ is a real operator, i.e. it comes from an $\mathbb{R}$-linear map $\Omega^r_{\mathbb{R}}(M) \to \Omega^r_{\mathbb{R}}(M)$ on real r-forms, we have $\Delta_d(\bar{\omega}) = \overline{\Delta_d(\omega)}$. In particular, $\bar{\omega}$ is d-harmonic if and only if ω is, so for h Kähler, we get a conjugate linear isomorphism

$$\mathcal{H}^{p,q}_{\bar{\partial}}(M) \to \mathcal{H}^{q,p}_{\bar{\partial}}(M)$$

and hence a conjugate linear isomorphism

$$H^{p,q}(M) \to H^{q,p}(M).$$

In particular,

$$h^{p,q} = h^{q,p}. \tag{6}$$

12.3 Hodge star

A Hermitian metric h on M induces a Hermitian inner product on the fibres of $\wedge^r T^*M$ as follows. If $\{v_1, v_2, \ldots, v_{2n}\}$ is an orthonormal basis for T_xM, and $\{\theta_1, \theta_2, \ldots, \theta_n\}$ is the dual basis for T^*_xM, then $\{\theta_{i_1} \wedge \theta_{i_2} \wedge \cdots \wedge \theta_{i_r} \,|\, i_1 < i_2 < \cdots < i_r\}$ is an orthonormal basis for $\wedge^r T^*_xM$. Let ω be $(1,1)$-form associated to h. The Hodge star operator is defined to be the conjugate-linear map

$$* : \Omega^{p,q}(M) \to \Omega^{n-p,n-q}(M)$$

satisfying

$$\langle \theta, \psi \rangle \frac{\omega^n}{n!} = \theta \wedge *\psi$$

for all $\theta, \psi \in \Omega^{p,q}(M)$. Here $n = \dim M$. For Kähler metrics, $*\psi$ is harmonic if and only if ψ is. So we have a conjugate linear isomorphism

$$* : \mathcal{H}^{p,q}_{\bar{\partial}}(M) \to \mathcal{H}^{n-p,n-q}_{\bar{\partial}}(M)$$

and hence

$$h^{p,q} = h^{n-p,n-q}. \tag{7}$$

12.4 Calculations of some Hodge diamonds for Calabi-Yaus

We want to work out the general form of the Hodge diamond for a simply connected Calabi-Yau 3-fold. We'll warm up by computing the Hodge diamond of a Calabi-Yau curve M. The Hodge diamond is

$$\begin{array}{ccc} & h^{1,1} & \\ h^{0,1} & & h^{1,0} \\ & h^{0,0} & \end{array}$$

Since M is a connected orientable real surface, we must have $b_0 = b_2 = 1$, so by (5), we have

$$1 = b_0 = h^{0,0}$$

and

$$1 = b_1 = h^{1,1}.$$

Since M is Calabi-Yau, we have that $K_M = T^*M$ is the trivial bundle, so

$$h^{1,0} = \dim H^{1,0}(M) = \dim H^0(M, T^*M) = 1$$

and by (6),

$$h^{0,1} = h^{1,0} = 1.$$

So the Hodge diamond is just

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$$

In particular, M is an elliptic curve.

Now consider a simply connected Calabi-Yau 3-fold M. The Hodge diamond is

$$\begin{array}{ccccccc}
 & & & h^{3,3} & & & \\
 & & h^{2,3} & & h^{3,2} & & \\
 & h^{1,3} & & h^{2,2} & & h^{3,1} & \\
h^{0,3} & & h^{1,2} & & h^{2,1} & & h^{3,0} \\
 & h^{0,2} & & h^{1,1} & & h^{2,0} & \\
 & & h^{0,1} & & h^{1,0} & & \\
 & & & h^{0,0} & & &
\end{array}$$

Since M is an orientable, connected real 6-manifold, we have $b_6 = b_0 = 1$, so by the Hodge decomposition (5),

$$h^{0,0} = h^{3,3} = 1.$$

Since M is simply connected, $b_1 = 0$, so

$$h^{0,1} = h^{1,0} = 0$$

by (5). This also gives, by the Hodge star (7), that

$$h^{3,2} = h^{2,3} = 0.$$

Applying Serre duality (4), we also get

$$h^{0,3} = 1; \quad h^{0,2} = 0.$$

By Hodge star (7), we also have

$$h^{3,1} = 0.$$

Applying complex conjugation (6), we get

$$h^{3,0} = 1; \quad h^{2,0} = h^{1,3} = 0.$$

Finally, the Hodge star (7) also gives us that

$$h^{2,2} = h^{1,1}; \quad h^{1,2} = h^{2,1}.$$

So the Hodge diamond for any simply connected Calabi-Yau 3-fold looks like

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

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