Loop decomposition of manifolds

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A Natural Question

- M = closed smooth manifold of finite dimension
- Homotopy group $\pi_n(M) = [S^n, M]$

A Natural Question

How to compute $\pi_*(M)$?

- It is an "impossible task" to get a complete answer!
- e.g. $\pi_n(S^m)$?
- We know a lot about spheres, etc

Homotopy groups of spheres

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^4	0	0	0	Z	Z_2	Z_2	$Z\times Z_{12}$	Z_2^2	Z_2^2	$Z_{24}\times Z_3$	Z_{15}	Z_2	Z_2^3	$Z_{120}\times Z_{12}\times Z_2$	$Z_{84}\times Z_2^5$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2	Z_{30}	Z_2	Z_2^3	$Z_{72}\times Z_2$
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	Z	Z_2	Z_{60}	$Z_{24}\times Z_2$	Z_2^3
S^7	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{120}	Z_2^3
S^8	0	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	$Z\times Z_{120}$

Alternative Questions

We can ask some more “accessible” questions

Reasonable Questions

- To describe $\pi_*(M)$ in terms of the homotopy groups of “smaller/simpler” spaces
- To study theoretical information of $\pi_*(M)$

- In several cases, the above questions can be approached
- One method: loop homotopy decomposition

A Motivational Question

One of the two big, driving conjectures in unstable homotopy theory:

Moore conjecture; '70s

For a simply connected finite complex Z, the following are equivalent

- at any prime p
$$p^N \cdot (p \text{ torsions of } \pi_*(Z)) = 0$$
for sufficiently large N;
- $\pi_*(Z) \otimes \mathbb{Q}$ is finite dimensional.

The conjecture is open

A Looped Question

- loop space $\Omega M := \mathrm{Map}_*(S^1, M)$
- loop homotopy decomposition $\Omega M \simeq Y \times Z$

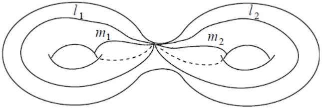


Figure: Loops on a surface
T. Lawson, Topology: a geometric approach

A Looped Question

To prove loop homotopy decomposition of M into more "familiar" pieces

- "smaller/simpler/familiar" spaces: CW complexes with a few number of cells, etc
- e.g. S^m, $P^m(p^r)$ Moore space, $S^n\{p^r\}$ the homotopy fiber of $p^r : S^n \to S^n$, etc

Classical examples: Hopf fibration

In 1931, Hopf discovered Hopf fibration $S^1 \longrightarrow S^3 \longrightarrow S^2$, which can be viewed as the beginning of homotopy theory.

- $S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbb{C}P^n$
- $S^3 \longrightarrow S^{4n+3} \longrightarrow \mathbb{H}P^n$
- $S^7 \longrightarrow S^{15} \longrightarrow S^8$

which imply

- $\Omega\mathbb{C}P^n \simeq S^1 \times \Omega S^{2n+1}$
- $\Omega\mathbb{H}P^n \simeq S^3 \times \Omega S^{4n+3}$
- $\Omega S^8 \simeq S^7 \times \Omega S^{15}$

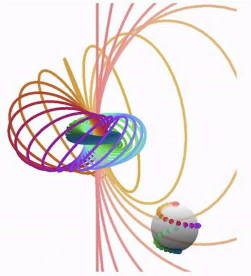


Figure: A family of fibres
https://www.youtube.com/watch?v=Rj6p3NDDtmE

Classical examples: Lie groups

In 1953, Serre showed homotopy decompositions of Lie groups rationally and localized at large primes.

- $p > 2n-1$, $SO(2n+1) \simeq_p S^3 \times \cdots \times S^{4n-1}$,
- $p > 2n-3$, $SO(2n) \simeq_p S^3 \times \cdots \times S^{4n-5} \times S^{2n-1}$
- $p > n-1$, $SU(n) \simeq_p S^3 \times \cdots \times S^{2n-1}$
- $p > 2n-1$, $Sp(n) \simeq_p S^3 \times \cdots \times S^{4n-1}$
- $p > 5$, $G_2 \simeq_p S^3 \times S^{11}$
- $p > 11$, $F_4 \simeq_p S^3 \times S^{11} \times S^{15} \times S^{23}$
- $p > 11$, $E_6 \simeq_p S^3 \times S^9 \times S^{11} \times S^{15} \times S^{17} \times S^{23}$
- $p > 17$, $E_7 \simeq_p S^3 \times S^{11} \times S^{15} \times S^{19} \times S^{23} \times S^{27} \times S^{35}$
- $p > 29$, $E_8 \simeq_p S^3 \times S^{15} \times S^{23} \times S^{27} \times S^{35} \times S^{39} \times S^{47} \times S^{59}$

Recall $\Omega BG \simeq G$.

The first modern study

- $M = (n-1)$-connected $2n$-manifold, $n \geq 2$
- $H^n(M) \cong \mathbb{Z}^{\oplus k}$, $k \geq 2$

Theorem (Beben-Theriault; '14)

When $n \neq 2, 4, 8$,

$$\Omega M \simeq \Omega(S^n \times S^n) \times \Omega\big(J_n \vee (J_n \wedge \Omega(S^n \times S^n))\big)$$

When $n = 2$,

$$\Omega M \simeq S^1 \times \Omega(S^2 \times S^3) \times \Omega\big(J_2 \vee (J_2 \wedge \Omega(S^2 \times S^3))\big)$$

where $J_n = \bigvee_{k-2} S^n$, $J_2 = \bigvee_{k-2} (S^2 \vee S^3)$.

- ΩM is homotopy equivalent to a product of loops on simply-connected spheres (with S^1 when $n = 2$).

Concrete studies

Studies on concrete cases of k-connected m-manifolds:

- $k = n - 1$, $m = 2n$: Beben-Theriault ('14);
- $k = n - 1$, $m = 2n + 1$: Beben-Wu ('15), Huang-Theriault ('21);
- $k = n - 2$, $m = 2n$ $(n > 3)$: Chenery ('22);
- $k = 1$, $m = 5$: Beben-Theriault ('18), Theriault ('20);
- $k = 1$, $m = 6$: Huang ('21).

The loop decompositions of these concrete manifolds support Moore conjecture.

One more example

- N= 1-connected 4-manifold
- $H^2(N) \cong \mathbb{Z}^{\oplus d}$, $d \geq 2$
- $S^n \longrightarrow M \longrightarrow N$, $n \geq 2$

Theorem (Huang; 2022)

If the sphere bundle is induced from a vector bundle, then

$$\Omega M \simeq S^1 \times \Omega S^n \times \Omega(S^2 \times S^3) \times \Omega\big(J \vee (J \wedge \Omega(S^2 \times S^3))\big)$$

where $J = \bigvee_{d-2} (S^2 \vee S^3)$.

- ΩM is homotopy equivalent to a product of loops on simply-connected spheres with S^1

Theoretical studies

From '21 to present, Huang-Theriault have made new progresses on homotopy of manifolds from more theoretical point of view, including

(Unstable/Integral/Local homotopy)

- manifolds with a prescribed embedding
- blow ups
- stabilized manifolds

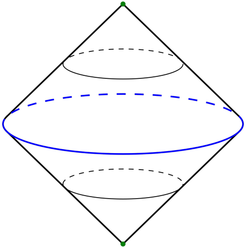
(Rational homotopy)

- open books
- free loop spaces

Some notations

Let $X = (X, *)$ be a based CW-complex.

- $\Omega X := \mathrm{Map}_*(S^1, X)$:
  (based) mapping space of base maps;
- $\Sigma X = S^1 \wedge X := \frac{[0,1]\times X}{(\{0,1\}\times X)\cup([0,1]\times\{*\})}$:
  suspension of X



If M is a closed m-manifolds,

- M_0: the manifold with a small disc D^m removed.
- Have the inclusion $S^{m-1} = \partial M_0 \overset{i}{\hookrightarrow} M_0$ of the boundary.

Stabilization of manifolds

- Kreck ('15): consider diffeomorphism classes of smooth manifolds modulo connected sum with a prescribed manifold T.
- Two typical choices of T: $S^n \times S^n$, or $\mathbb{C}P^n$.
- T-stabilization: $N \# T$.
- A problem in geometric topology: classify smooth manifolds up to T-stabilizations.

- $T = S^n \times S^n$: Kreck ('99);
- $T = \mathbb{C}P^2$: Kasprowski, Powell and Teichner ('21), Kreck ('99)

Homotopy of manifolds stabilized by projective spaces

- $N =$ connected closed $2n$-dimensional smooth manifold
- $F =$ homotopy fibre of $S^{2n-1} \xhookrightarrow{i} N_0$

Theorem (Huang-Theriault, '21)

- if $n \geq 2$ is even, $\Omega(N\#\mathbb{C}P^n) \simeq S^1 \times \Omega N_0 \times \Omega\Sigma^2 F$;
- if $n \geq 2$ is odd, $\Omega(N\#\mathbb{C}P^n) \simeq_{\{\frac{1}{2}\}} S^1 \times \Omega N_0 \times \Omega\Sigma^2 F$;
- if $n \geq 4$ is even, $\Omega(N\#\mathbb{H}P^{\frac{n}{2}}) \simeq_{\{\frac{1}{2},\frac{1}{3}\}} S^3 \times \Omega N_0 \times \Omega\Sigma^4 F$;
- if $n = 8$, $\Omega(N\#\mathbb{O}P^2) \simeq_{\{\frac{1}{2},\frac{1}{3},\frac{1}{5}\}} S^7 \times \Omega N_0 \times \Omega\Sigma^8 F$.

- A $\mathbb{C}P^n$-stabilization is a blow up at a point.
- (Duan) A canonical circle bundle over a $\mathbb{C}P^n$-stabilization is the effect of a 1-surgery.

Example

For $n \geq 2$, there are homotopy equivalences

$$\begin{aligned}
\Omega(\mathbb{C}P^{2n}\#\mathbb{C}P^{2n}) &\simeq S^1 \times S^1 \times \Omega S^3 \times \Omega S^{4n-1}, \\
\Omega(\mathbb{C}P^{2n}\#\mathbb{H}P^{n}) &\simeq S^1 \times S^3 \times \Omega S^5 \times \Omega S^{4n-1}, \\
\Omega(\mathbb{C}P^{8}\#\mathbb{O}P^{2}) &\simeq S^1 \times S^7 \times \Omega S^9 \times \Omega S^{15}, \\
\Omega(\mathbb{C}P^{2n+1}\#\mathbb{C}P^{2n+1}) &\simeq_{\{\frac{1}{2}\}} S^1 \times S^1 \times \Omega S^3 \times \Omega S^{4n+1}, \\
\Omega(\mathbb{H}P^{n}\#\mathbb{H}P^{n}) &\simeq_{\{\frac{1}{2},\frac{1}{3}\}} S^3 \times S^3 \times \Omega S^7 \times \Omega S^{4n-1}, \\
\Omega(\mathbb{H}P^{4}\#\mathbb{O}P^{2}) &\simeq_{\{\frac{1}{2},\frac{1}{3}\}} S^3 \times S^7 \times \Omega S^{11} \times \Omega S^{15}, \\
\Omega(\mathbb{O}P^{2}\#\mathbb{O}P^{2}) &\simeq_{\{\frac{1}{2},\frac{1}{3},\frac{1}{5}\}} S^7 \times S^7 \times \Omega S^{15} \times \Omega S^{15}.
\end{aligned}$$

- Duan ('22) has shown the results for $\mathbb{C}P^{2n}\#\mathbb{C}P^{2n}$ and $\mathbb{C}P^{2n}\#\mathbb{H}P^{n}$ by more geometrical arguments.
- He needs non-spin condition.

General blow ups

- N = connected closed oriented smooth manifold
- a codimension $2n$ embedding $B \hookrightarrow N$ with a complex normal bundle ν
- V = closed neighborhood of B in N, and $V \cong D\nu$
- N_c = the closure of the complement of V in N

$$
\begin{array}{ccc}
\partial V & \xrightarrow{\iota_\nu} & V \\
\downarrow{\scriptstyle \iota_c} & & \downarrow \\
N_c & \longrightarrow & N,
\end{array}
\qquad\qquad
\begin{array}{ccc}
\partial V & \xrightarrow{q} & P\nu \\
\downarrow{\scriptstyle \iota_c} & & \downarrow{\scriptstyle j} \\
N_c & \xrightarrow{r} & \widetilde{N},
\end{array}
$$

- P_ν = the projective bundle of ν

Definition (blow up)

$\widetilde{N}$ is called the blow up of N along B.

Homotopy of blow ups

- B is $(k-1)$-connected and of dimension m
- $F =$ homotopy fibre of $\partial V \overset{\iota_c}{\hookrightarrow} N_c$.

Theorem (Huang-Theriault; '22)

Suppose $m \leq 2n-4$. Let p be a prime such that

- $p > \frac{1}{2}(m-k)+1$, $(p-1) \nmid 2s$ for any $k+2 \leq 4s \leq m+2$, and
- $H_*(B;\mathbb{Z})$ is p-torsion free.

Then there is a p-local homotopy equivalence

$$\Omega\widetilde{N} \simeq_p S^1 \times \Omega N_c \times \Omega\Sigma^2 F.$$

Remark

- Let d_s is the denominator of $B_s/4s$, where B_s is the s-th Bernoulli number defined by

$$\frac{z}{e^z - 1} = 1 - \frac{1}{2}z - \sum_{s \geq 1} B_s \frac{z^{2s}}{(2s)!}.$$

- The arithmetical condition on p:

$$p \notin \{\text{prime } p \mid (p-1) \text{ divides } 2s,\ k+2 \leq 4s \leq m+2\}$$
$$= \{p \mid p \text{ divides } d_s,\ k+2 \leq 4s \leq m+2\}.$$

- We are localizing away from the image of stable J-homomorphism (Adams ('66) and Quillen ('71)).

- We introduce a type of "fibrewise surgery"
- A blow up is a "fibrewise $\mathbb{C}P^n$-stabilization"
- A canonical circle bundle over a fibrewise $\mathbb{C}P^n$-stabilization is the effect of a fibrewise 1-surgery.

Open books

Let M be a simply connected closed n-manifold.

- V simply connected compact
- $\dim(V) = n - 1$, $\partial V \neq \emptyset$
- $h : V \xrightarrow{\cong} V$
- $h_{|\partial V} = \mathrm{id}$
- $V_h =$ the mapping torus of h
- $M \cong_{\mathrm{diff}} (\partial V \times D^2) \cup_{\mathrm{id}} V_h$

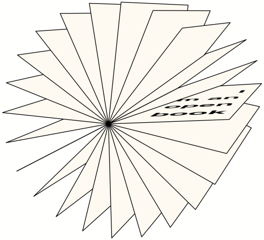


Figure: What is an ... open book Giroux, Notice AMS 52 (1), 2005

Definition (open book)

M is called an *open book*, with $V =$ page, $h =$ *monodromy*.

Classical rational dichotomy

(Félix, '89)

Any simply-connected finite CW-complex X is either:

- *rationally elliptic, that is, $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional, or else*
- *rationally hyperbolic, that is, $\pi_*(X) \otimes \mathbb{Q}$ grows exponentially.*

A special case

- $M =$ open book as before with the monodromy $h : V \to V$
- $F =$ homtopy fibre of $\partial V \overset{i}{\hookrightarrow} V$

Theorem (Huang-Theriault; '21)

Suppose $h \simeq \mathrm{id}$ relative to ∂V. Then there is a homotopy equivalence

$$\Omega M \simeq \Omega V \times \Omega \Sigma^2 F.$$

- One can obtain an extended rational dichotomy from the above theorem.
- Indeed, we can prove it in more general context.

Extended rational dichotomy of open books

- M = open book as before with the monodromy $h : V \to V$
- F = homtopy fibre of $\partial V \overset{i}{\hookrightarrow} V$

Theorem (Huang-Theriault; '21; weak version)

Suppose the monodromy is of finite order and acts nilpotently on $\pi_*(V)$. Then either:

(1) M is rationally elliptic, in which case
- V is also rationally elliptic
- $F \simeq_{\mathbb{Q}} S^l$ for some $l \in \mathbb{Z}^+$, and
- $\pi_*(M) \otimes \mathbb{Q} \cong (\pi_*(V) \otimes \mathbb{Q}) \oplus (\pi_*(S^{l+2}) \otimes \mathbb{Q})$;

(2) M is rationally hyperbolic, in which case
- either V is rationally hyperbolic, or
- $F \not\simeq_{\mathbb{Q}} S^l$ for any $l \in \mathbb{Z}^+$.

A few words on methodology: unstable homotopy theory

homotopy fibration : $F \xrightarrow{i} E \xrightarrow{p} B$, with $r: B \dashrightarrow E$, $\Omega p \circ r \simeq \mathrm{id}$

$$\begin{array}{ccc} F & \xrightarrow{i} & E \\ \downarrow f & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

: homotopy pushout

Theorem (Huang-Theriualt; '22)

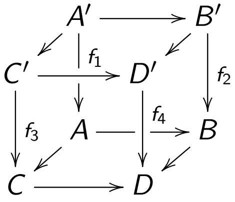
Under reasonable conditions, there is a homotopy equivalence

$$\Omega Y \simeq \Omega B \times \Omega X \times \Omega(H * \Omega B),$$

where H is the homotopy fibre of $F \xrightarrow{f} X$.

- It generalizes a classical theorem of (Ganea; '65)

A cubic diagram technique



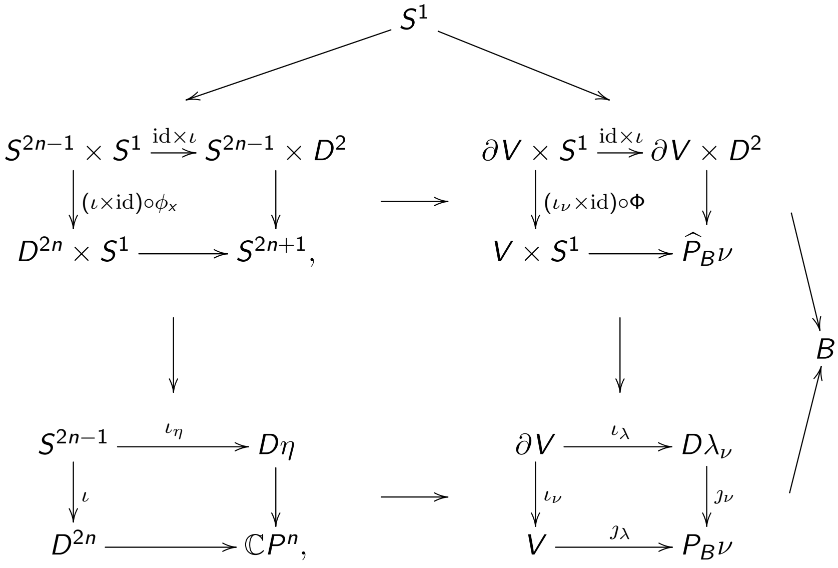
Mather's Cube Lemma; '76

Suppose in the above homotopy commutative diagram

- the vertical faces are homotopy pullbacks, and
- the bottom face is a homotopy pushout.

Then the top face is a homotopy pushout.

Thanks very much


S^1
$S^{2n-1} \times S^1$
$\mathrm{id}\times\iota$
$S^{2n-1} \times D^2$
$(\iota\times\mathrm{id})\circ\phi_x$
$D^{2n} \times S^1$
S^{2n+1},
$\partial V \times S^1$
$\mathrm{id}\times\iota$
$\partial V \times D^2$
$(\iota_\nu\times\mathrm{id})\circ\Phi$
$V \times S^1$
$\widehat{P}_B\nu$
B
S^{2n-1}
ι_η
$D\eta$
ι
D^{2n}
$\mathbb{C}P^n$,
∂V
ι_λ
$D\lambda_\nu$
ι_ν
$\jmath_\nu$
V
$\jmath_\lambda$
$P_B\nu$

The homotopy conditions are necessary

Consider the Milnor's open book decompositions of S^{2n+1} determined by the complex polynomial

$$f(z_1, \ldots, z_{n+1}) = (z_1)^{a_1} + \cdots + (z_n)^{a_n} + (z_{n+1})^{a_{n+1}}.$$

- If $3 \le a_1 \ge \cdots \ge a_{n+1} \ge 2$, then
  - S^{2n+1} is elliptic,
  - the page $V \simeq \bigvee_{\mu} S^n$ is hyperbolic, $\mu = \prod_{j=1}^{n+1} (a_i - 1)$.

  It follows that either the monodromy is of infinite order, or it acts non-nilpotently on the homology groups $H_*(V; \mathbb{Z})$.

The homotopy conditions are necessary

- If $a_1 = \cdots = a_{n+1} = 2$, we can construct an open book decomposition of a homotopy sphere Σ^{2n+1}

$$\Sigma^{2n+1} \cong ((\partial V \# \partial V) \times D^2) \cup_{\mathrm{id}} (V \# V)_{h\#h}.$$

satisfying that

  - its monodromy $h\#h$ is of order at most 8
  - its page $V\#V \simeq S^n \vee S^n$ is rationally hyperbolic, and
  - $\Sigma^{2n+1} \simeq S^{2n+1}$ is rational elliptic

It follows that the monodromy $h\#h$ acts non-nilpotently on the homology groups $H_*(V\#V;\mathbb{Z})$.