Systolic inequality on Riemannian manifolds with bounded Ricci curvature

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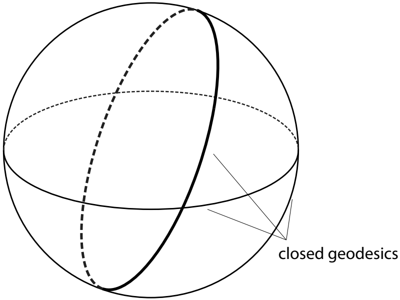
Definition

The systole sys(M) is the least length of a non-trivial closed geodesic.



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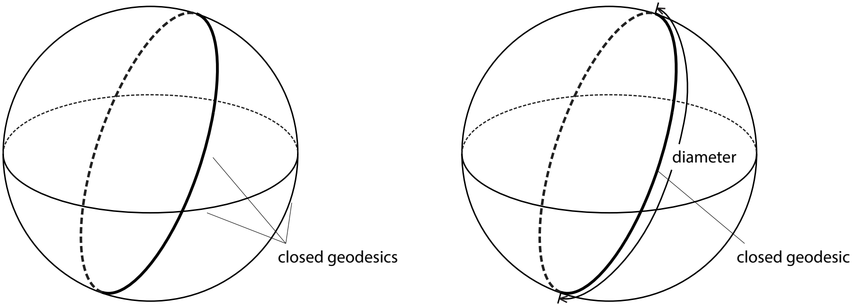

Loewner inequality

Suppose that M is homeomorphic to T^2,

$$sys^2 \leq \frac{2}{\sqrt{3}} Area(M)$$

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Pu's inequality (1949)

Suppose that M is homeomorphic to $\mathbb{R}P^2$,

$$sys^2 \leq \frac{\pi}{2} Area(M)$$

C. Croke (88), improved by A. Nabutovsky-R. Rotman(02), S. Sabourau(04), and Rotman(06)

Suppose that M is homeomorphic to S^2,

$$sys^2 \leq 32 Area(M)$$

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Conjecture. (E. Calabi, J. Cao (92); Croke (88))

The best constant is $2\sqrt{3}$.

Question. (M. Gromov (83))

Is it true that the systole of an n-dimensional Riemannian manifold can be bounded by constant(n)vol$(M)^{1/n}$?

Remark.

Similar questions can be asked about diameter and other geometric quantities. Note that if M is not simply-connected, then an upper-bound of systole in terms of diameter is trivial.

Example. (F. Balacheff, C. Croke and M. Katz (09))

There exists (Zoll) Riemannian metric on S^2 such that $sys > 2D$, where D is the diameter.

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Croke (88) (9D), improved by M. Maeda (94) (5D), Sabourau(04) (4D), and Nabutovsky-Rotman(09) (4D)

Suppose that M is homeomorphic to S^2,

$$sys \leq 4D$$

Higher dimensional manifolds, Nabtovsky-Rotman (03)

Let M be a closed Riemannian manifold with sectional curvature ≤ 1 and volume $\leq V$. Then

$$sys \leq 2\pi(V+1)^{c(n)V^n}.$$

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Nabtovsky-Rotman (03)

Let M be a closed Riemannian manifold with sectional curvature ≥ -1, diam $\leq D$ and volume $\geq V > 0$. Then

$$sys \leq exp(\frac{exp(c_1(n)D)}{min\{1, V\}^{c_2(n)}}).$$

N. Wu and Z. (19)

Let M be a closed simply-connected 4-dimensional Riemannian manifold with Ricci curvature $|Ric| < 3$, diam $\leq D$ and volume $\geq V > 0$. Then

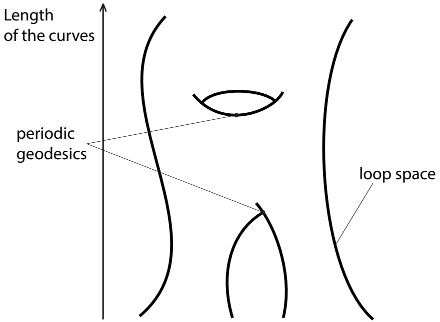
$$sys \leq F(V, D).$$

Moreover, F can be explicitly computed if M is Einstein.

Intuition

- Morse theory
- Width of a homotopy
- Cheeger-Naber Structural theorem

Morse theory (Lusternik-Fet)



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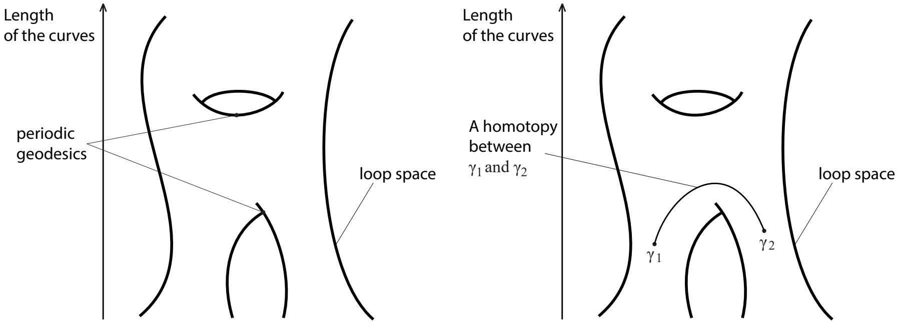
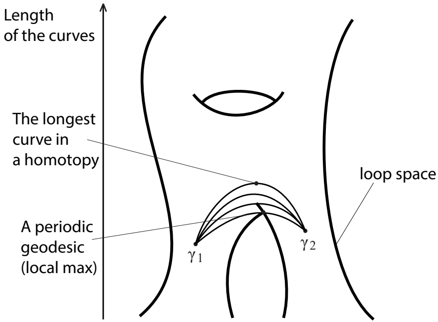
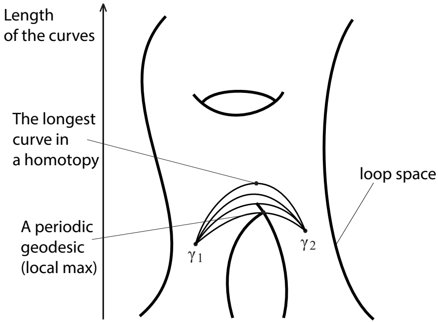
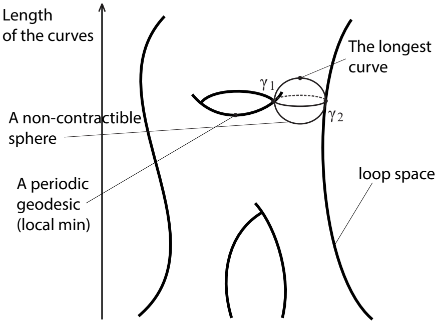


Morse theory




Length
of the curves
The longest
curve in
a homotopy
A periodic
geodesic
(local max)
γ1
γ2
loop space


Length
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γ1
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sphere
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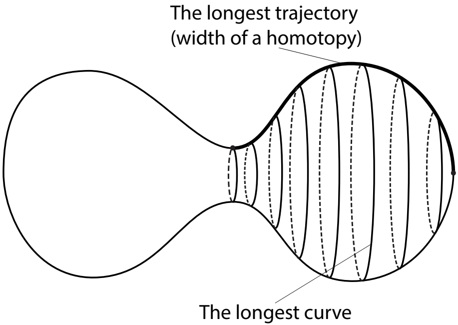
geodesic
(local min)
loop space

Width of a homotopy



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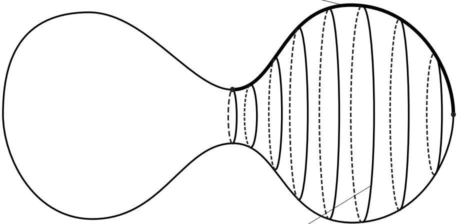
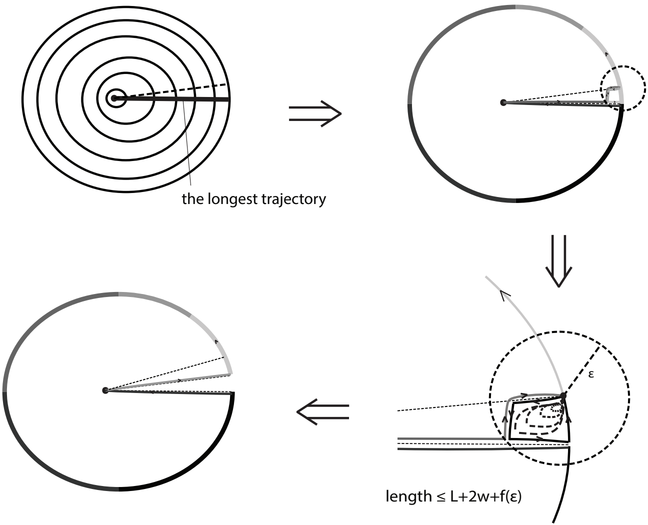


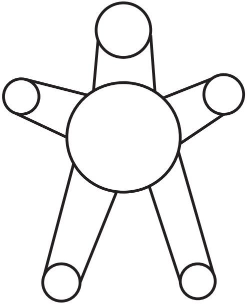
Lemma. (Alex Nabutovsky and Regina Rotman)

Control of the width. $\Rightarrow$ Control of the longest curve during a homotopy.

Width of a homotopy



Structural theorem (Jeff Cheeger and Aaron Naber, 2015)



A manifold
(with body and
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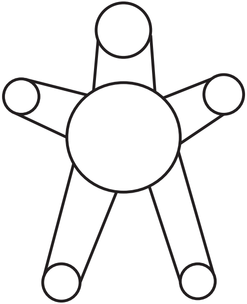

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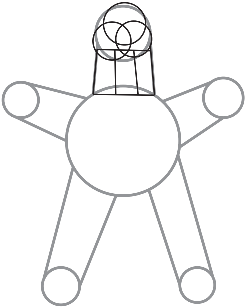
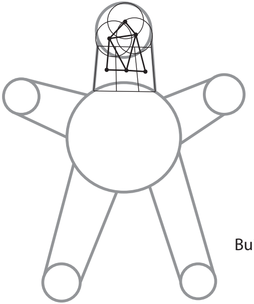


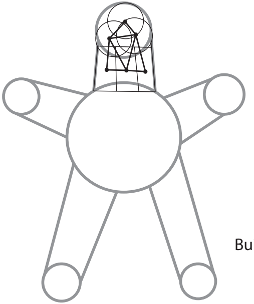
A manifold
covered by
“good sets”

Structural theorem



Build a skeleton

Structural theorem



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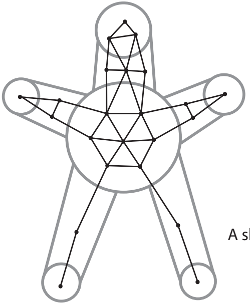

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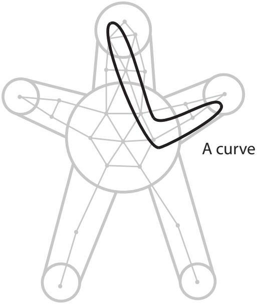

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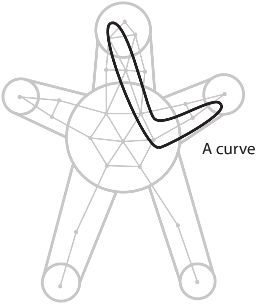

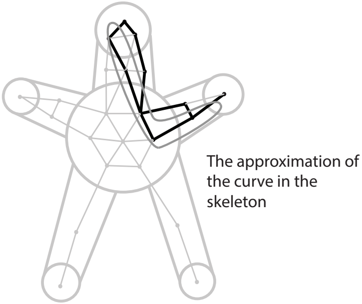

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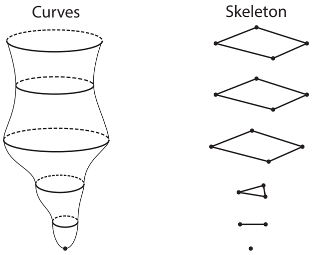

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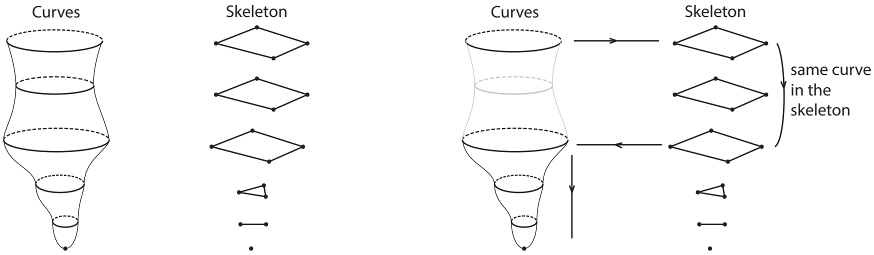

Reducing the width

Difficulty

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Observation 1

Every closed curve is homotopic to a wedge of "almost" geodesic digons α_i through a homotopy of width bounded by $2D + \varepsilon$.

Structural theorem

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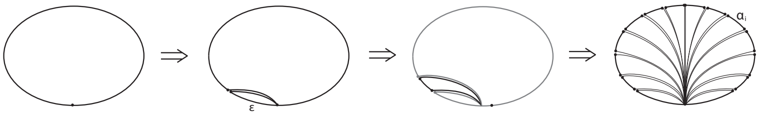

γ is homotopic to ∨ᵢαᵢ

Observation 2

If each α_i can be contracted to a point with width $< W_i$, then $\vee \alpha_i$ can be contracted with width $2 \cdot \max_i W_i$.

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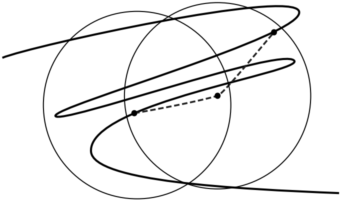
Observation 3

The number of the edges in the approximation of a minimizing geodesic must be small (≤ 5).



Summary

- Morse theory on the loop space vs Sweep-out of the manifold.
- Width of a homotopy: geometrically approachable.
- Cheeger-Naber Structural theorem: compute width via combinatorics.