Equivalent Curves on Surfaces

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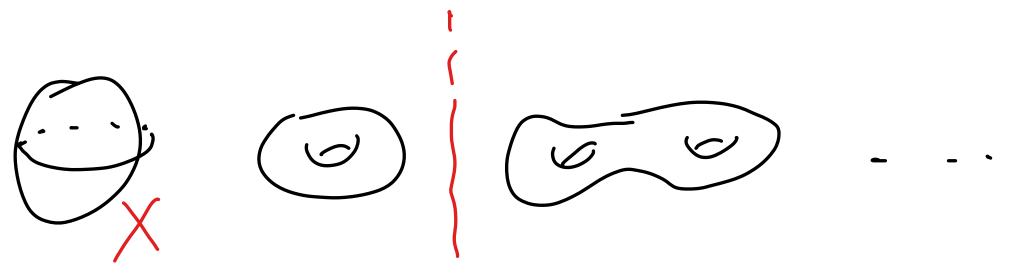
Surfaces: 2-dim topo mfld oriented, without ∂, finite type (finitely presented π_1)

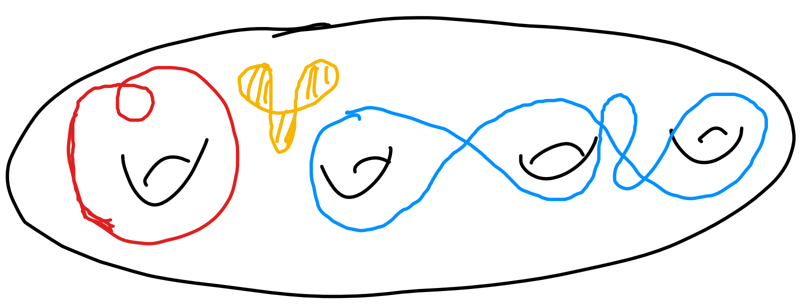
Σ

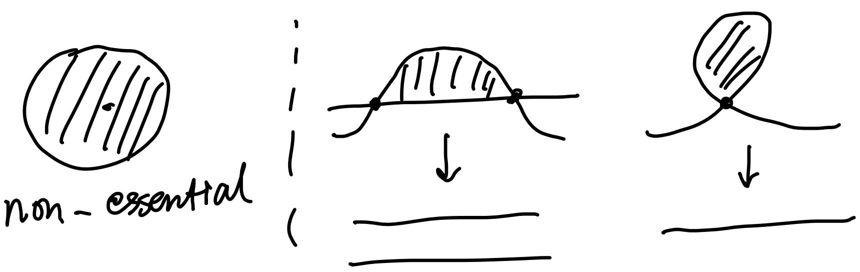


$g =$ 0 1 2 ...

Curves: image of $\gamma : S^1 \to \Sigma$ continuous map.

we consider γ up to homotopy





— we'll consider only essential curves up to homotopy.

Rmk: Curves are important in the study of sf's.

— topology partition
— geometry geodesic
— dynamic periodic orbit.

Question: How to tell a curve from another ?

Any "coord's system" for $\mathcal{C}(\Sigma) = \{\text{curves}\}/\sim$

Key Tool in studying of curves: intersection

. γ, η curves on Σ $\quad \gamma : S^1 \to \Sigma$
$\quad \eta : S^1 \to \Sigma$

$i(\gamma, \eta) := \# \{ (s,t) \mid \gamma(s) = \eta(t) \} \subset S^1 \times S^1$

$i([\gamma],[\eta]) := \min \{ i(\gamma', \eta') \mid \gamma' \sim \gamma, \ \eta' \sim \eta \}$.

$\gamma : S^1 \to \Sigma$

$t_1, t_2 \in S^1$

$\gamma(t_1) = \gamma(t_2)$

$p = \gamma(t_1) = \gamma(t_2)$

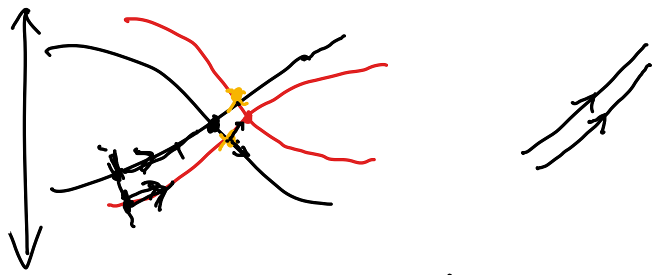
(t_1, t_2)

$\gamma(t_1) = \gamma(t_2)$

(t_2, t_1)

$\gamma(t_2) = \gamma(t_1)$

$i([\gamma],[\gamma]) = 2k \in 2\mathbb{N}$ $\qquad$ $k \in \mathbb{N}$



Notation. $[\gamma] \longrightarrow \gamma.$

γ has k selfintersection pts.

Def: $\gamma \in \mathcal{C}(\Sigma)$. γ is a k-curve if $i(\gamma,\gamma) = 2k$

$k = 0$ $\quad$ γ is simple

Q. $\exists ?\; \underset{A}{\{\gamma_i \mid i \in \Omega\}} \subset \mathcal{C}(\Sigma)$

$\forall \alpha, \beta \in \mathcal{C}(\Sigma)$

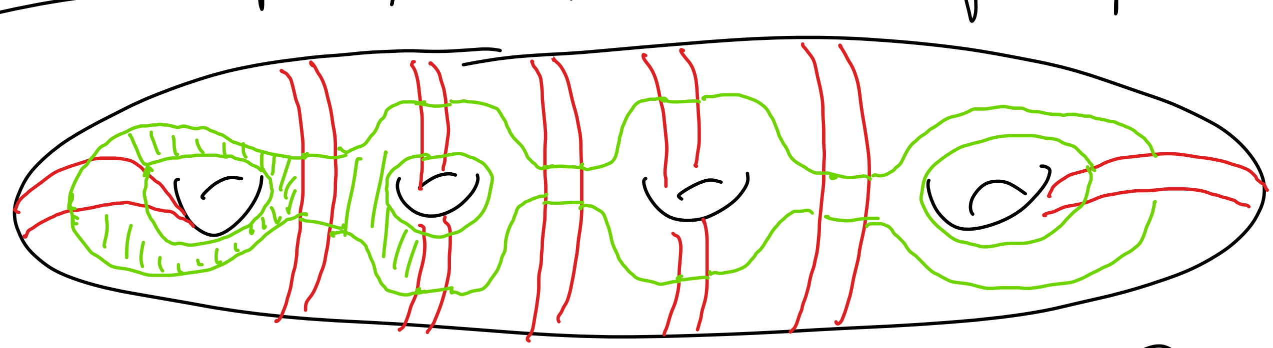
$\forall \gamma_i \in A,\; i(\alpha, \gamma_i) = i(\beta, \gamma_i) \Rightarrow \alpha \sim \beta.$

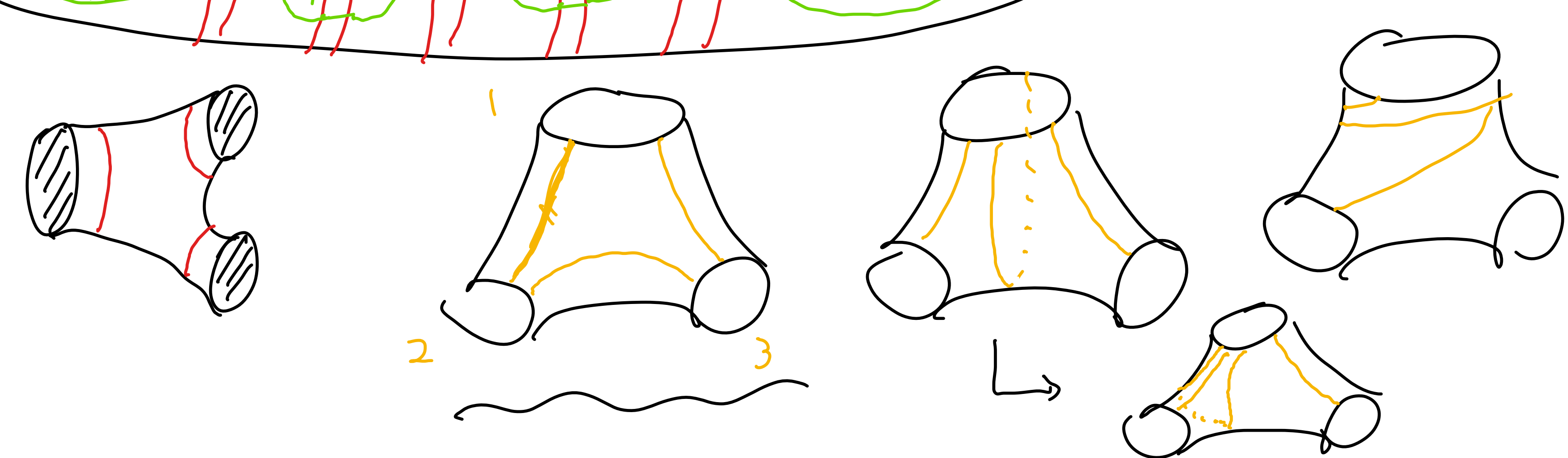
Simple case.

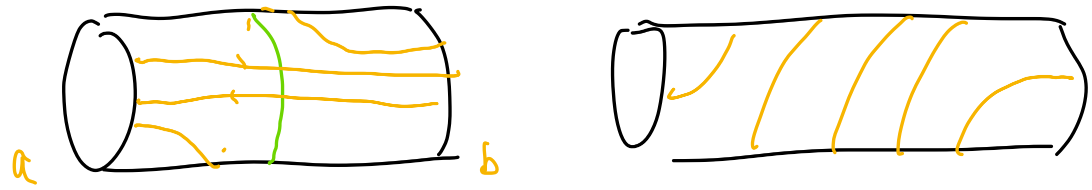
Dehn–Thurston coordinates for $\mathcal{C}_0(\Sigma) = \{\text{simple curves on } \Sigma\}$

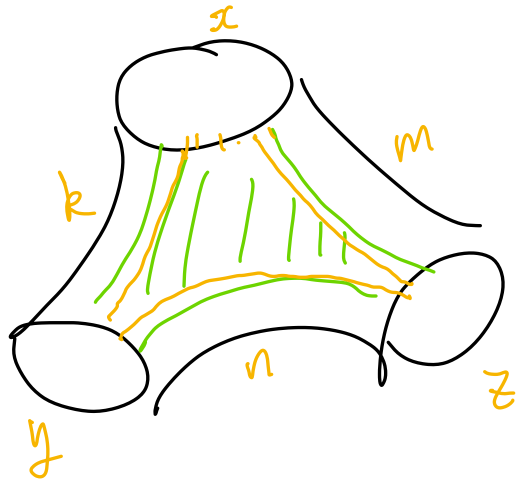
Key idea:
1. pants-cylinder decomposition
2. check in each pants
3. check in each cylinder.

pair of pants decomp: maximal collection of simple curves - disjoint.
- non homotopic

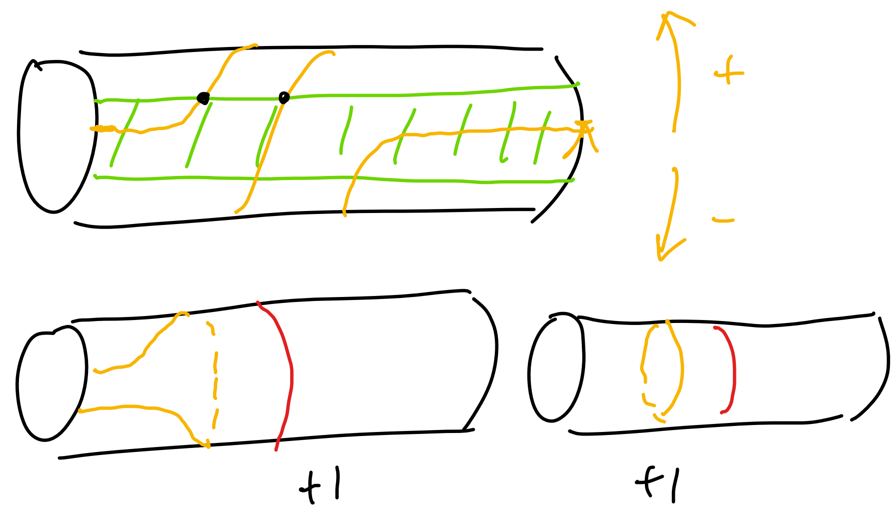








$(k,m,n)=(x,y,z)$



$k=0$

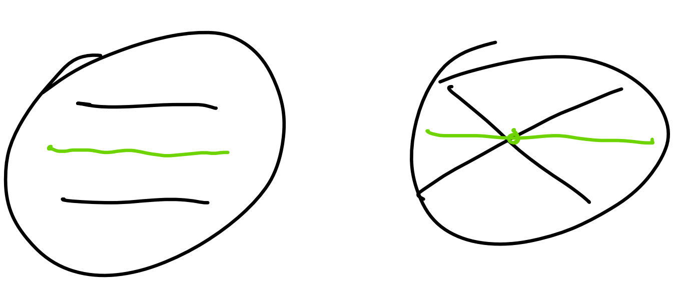
$\exists \{\gamma_1, \ldots, \gamma_n\} \subset \mathcal{C}_0(\Sigma)$ s.t. $\forall \alpha, \beta \in \mathcal{C}_0(\Sigma)$
if $\forall \gamma_i$ $i(\alpha,\gamma_i)=i(\beta,\gamma_i)$, then $\alpha \sim \beta$.

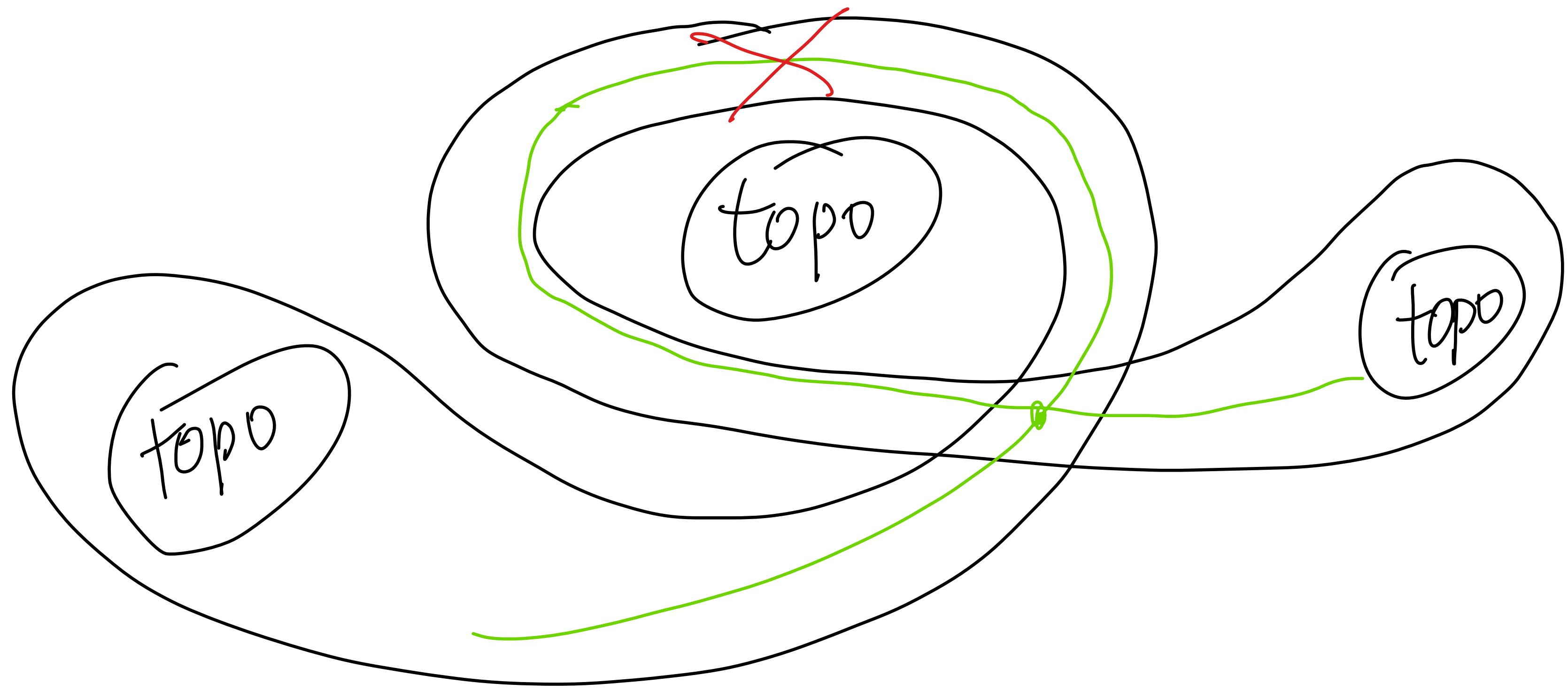
General Case $k>0$

Q: ① Is D-T coord's still work? NO.

② adding more simple curves? NO

Counter-ex





Obstruction: let $\mathcal{C}_k(\Sigma) := \{k\text{-curves on } \Sigma\}/\sim$

$\forall k\in\mathbb{N}$

α, β curves in Σ

- If $\forall \gamma \in \mathcal{C}_k(\Sigma)$, $i(\alpha,\gamma) = i(\beta,\gamma)$

we say α and β are k-equivalent.

$$\boxed{\alpha \sim_k \beta}$$

Main Result (Parlier – X.)

① $\forall k \in \mathbb{N}$, $\alpha \sim_k \beta \Rightarrow \alpha \sim_0 \beta$.

② $\forall K = \{k_1, \dots, k_m\} \subset \mathbb{N}_{>0}$ $\quad K \cap K' = \phi$

$K' = \{k_1', \dots, k_n'\} \subset \mathbb{N}_{>0}$

$\exists \alpha, \beta \in \mathcal{C}(\Sigma)$, s.t. $\alpha \sim_{k_j} \beta \quad \forall 1 \leq j \leq m$

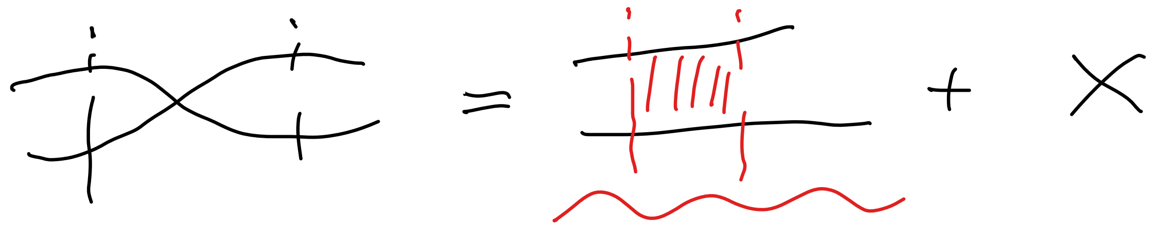
$\alpha \not\sim_{k_j'} \beta \quad \forall 1 \leq j \leq n.$

③ $\forall \alpha, \beta$ curves on Σ $\quad \exists \infty$ k's $\in \mathbb{N}$ s.t. $\alpha \not\sim_k \beta$

non-homotopic.

Key Point:

① Curves with self-intercections can always be carried by some train tracks.

② 

$\boxed{\text{Erlandsson–Souto}}$

③ Good news: ① $\forall \alpha, \beta \in \mathcal{C}(\Sigma)$ non-homotopy $\exists \gamma \in \mathcal{C}(\Sigma)$ s.t. $i(\alpha,\gamma) \neq i(\beta,\gamma)$.

② $\forall k \in \mathbb{N}$, $\forall \alpha, \beta \in \mathcal{C}_{\leq k}$ non-homotopic $\exists k' \leq 12k$, $\alpha \sim_{k'} \beta$

Bad news need ∞ curves to build a "coord sys".

finitely many to tell the track track + ∞ for testing the locations for crossing (MCG orbit)