Branching random walks on relatively hyperbolic groups

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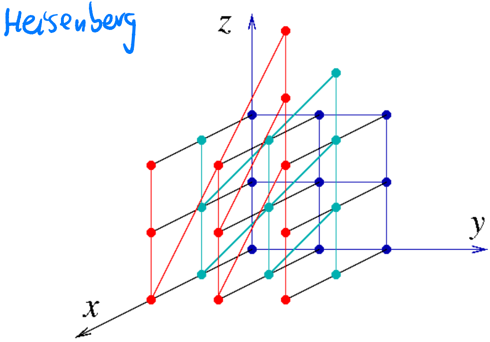
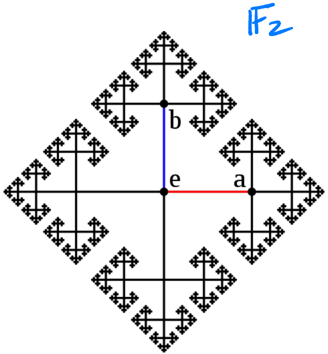
Geometric setup: Cayley graph and word metric

Let G be a group generated by a finite set S with $1 \notin S = S^{-1}$.
The **Cayley graph** $\mathscr{G}(G,S)$ is a graph defined as follows.

1. Vertex set $V := G$,
2. Two vertices $g \longleftrightarrow g'$ iff $g' = g \cdot s$ for some $s \in S$.





which is equipped with combinatorial metric called **word metric** d_S.

Fix a scaling function $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ (eg. $f(n) = \lambda^n$ or $f(n) = n^{-2}$)

- $\sum_{n \geq 0} f(n) < \infty$.
- $\forall n \geq 0 : \lambda \leq \frac{f(n+1)}{f(n)} \leq 1$

Construction of Floyd boundary: fix a basepoint $o \in \mathscr{G}(G, S)$

1. The unit length of each edge e in $\mathscr{G}(G, S)$ is rescaled to be the **Floyd length** $\ell_f(e) := f(n)$, where $n = d_S(o, e)$ is the word distance from edge e to o.

   $\rho_{go}(gx, gy) = \rho_o(x, y)$

2. **Floyd metric** $\rho_o(x, y)$ is the infimum of Floyd lengths of all possible paths between x and y.
3. Let $\overline{G}_f$ be the Cauchy completion of G with respect to ρ_o. The set $\partial_f G := \overline{G}_f \setminus G$ in $\overline{G}_f$ is called **Floyd boundary** of G.

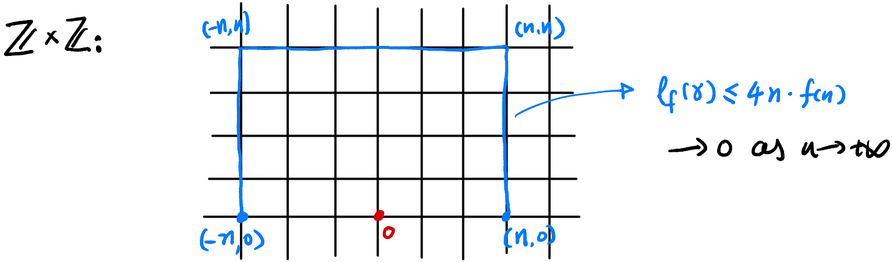
Remark (W. Floyd)

The completion $\overline{G}_f$ is a compact metric space, on which G acts by (bilipschitz) homeomorphisms.

Examples of trivial Floyd boundary: $\sharp\,\partial_f G \le 2$.

1. Finite groups: $\sharp\,\partial_f G = 0$.

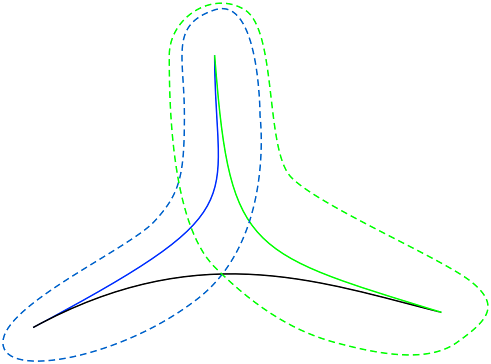
2. $\mathbb{Z}^n$ for $n \ge 2$: $\sharp\,\partial_f G = 1$, but for $\mathbb{Z}$: $\sharp\,\partial_f G = 2$.
3. Product of two infinite groups: $\sharp\,\partial_f G = 1$.
4. Any amenable group.
5. Mapping class groups with closed orientable surfaces of genus ≥ 2.



In the remainder of this talk, we only consider Floyd boundary $\partial_f G = \partial_\lambda G$ defined using scaling function $f(n) = \lambda^n$.

Gromov hyperbolic spaces

1. Let (X, d) be a geodesic metric space.
2. For given $\delta > 0$, a geodesic triangle is called δ-**thin**, if any side is contained in a δ-neighborhood of the other two sides.



3. Then X is called δ-**hyperbolic** if every geodesic triangle is δ-thin.

Hyperbolic groups

Definition

A finitely generated group G is called **hyperbolic** if any Cayley graph is δ-hyperbolic for some $\delta > 0$. Equivalently, if G acts properly and co-compactly on a proper δ-hyperbolic space.

Examples

1. Finite groups,
2. Free groups,
3. closed surface groups,
4. Fundamental groups of compact negatively curved manifolds.

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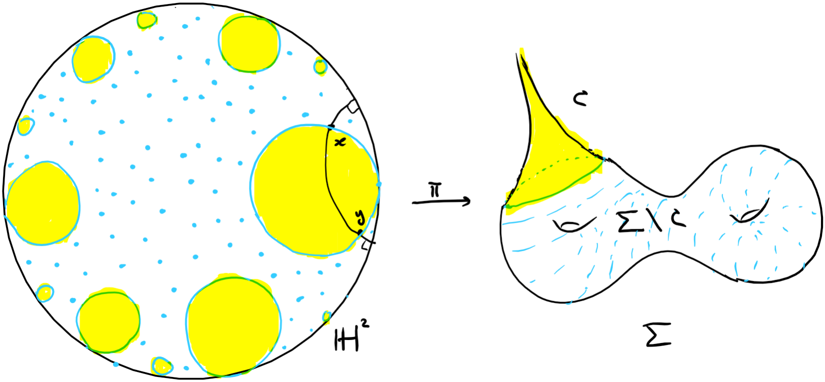
Examples

1. Finite groups,
2. Free groups, ↷ Tree
3. closed surface groups, ↷ $\mathbb{H}^2$
4. Fundamental groups of compact negatively curved manifolds.

What is ... a relatively hyperbolic group

- The fundamental group of hyperbolic manifolds with finite volume



Relatively hyperbolic groups

A finitely generated group G is **relatively hyperbolic** if G acts properly on a proper hyperbolic space X and there exists a G-invariant family of **horoballs** $\mathbb{B}$ such that the action on $X \setminus \cup\{B \in \mathbb{B}\}$ is co-compact.

1. The stabilizers of horoballs are called **maximal parabolic subgroups**.
2. The Gromov boundary of X is called **Bowditch boundary** of the relatively hyperbolic group G.

Examples

1. Hyperbolic groups
2. Infinitely ended groups: free product amalgamation of any two groups over finite subgroups, or HNN extension over finite subgroups [Stallings 1968].
3. Fundamental groups of any finite volume Riemannian manifolds with negatively pinched curvature.

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Boundary comparison: Floyd boundary covers

1. Gromov boundary of hyperbolic groups. $\cong$
2. Ends boundary of groups introduced by Freudenthal.
3. Limits set of geometrically finite Kleinian groups: [Floyd 1980]

Theorem (Floyd, 1980; Gerasimov, Potyagailo-Gerasimov 2012)

Let G be a relatively hyperbolic group with Bowditch boundary ΛG. Then for any $\lambda \in [\lambda_0, 1)$, there exists a continuous and surjective map

$$(\partial_\lambda G, \rho_\lambda) \to (\Lambda G = \{conical\ point\} \bigcup \{parabolic\ points\}, \bar{\rho}_\lambda)$$

such that

1. *The preimage of a conical point in ΛG consists of a single point.*
2. *The preimage of each parabolic point is the same as the limit set of the corresponding parabolic subgroup.*

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Hausdorff dimension of Floyd boundary

Define **growth rate**:

$$\delta_G := \limsup_{n\to\infty} \frac{\log \sharp\{g \in G : d_S(o, go) \le n\}}{n}$$

The following result was conjectured by M. Bourdon.

Theorem (Potyagailo-Y., 2019)

Let G be a relatively hyperbolic group with a finite generating set S. There exists a constant $0 < \lambda_0 < 1$ such that

$$\mathrm{Hdim}_{\rho_\lambda}(\partial_\lambda G) = \mathrm{Hdim}_{\bar{\rho}_\lambda}(\Lambda G) = -\frac{\delta_G}{\log \lambda}$$

for any $\lambda \in [\lambda_0, 1)$, where the Bowditch boundary ΛG is equipped with shortcut metric $\bar{\rho}_\lambda$ induced by Floyd metric ρ_λ.

Probabilistic setup: (branching) random walks on groups

1. Let μ be a symmetric probability measure whose support generates G.
2. The position of the μ-random walk at the time n is a random product ω_n of n independent μ-distributed elements (or steps) s_i for $1 \le i \le n$:

$$\omega_n = \omega_0 \cdot s_1 \cdots s_n.$$

with the law $\mathbf{P}(\omega_n = y, \omega_0 = x) = \mu^{\star n}(x^{-1}y)$, the probability of visiting y starting from x in n-steps.

3. The **spectral radius** $R_\mu := \limsup_{n\to\infty} \mu^{\star n}(x^{-1}y)^{1/n}$ for any $x, y \in G$.

Problem (Dirichlet problem)

*Describe all bounded/positive μ-**harmonic** functions $h : G \to \mathbb{R}$ on a given group G:*

$$h(x) = \sum_{s\in G} \mu(s) h(xs)$$

Via Martin-Poisson representation formula, it is equivalent to determine the Poisson/Martin boundary of μ-random walks.

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Branching random walks

Fix an offspring distribution ν on $\mathbb{N}_{\geq 0}$ with mean $r = \sum_{n \geq 0} n\nu(n) > 0$.

Place a particle at the location $x \in \mathscr{G}(G, S)$ at the time 0.

1. It splits into a ν-random set of particles with offspring mean r.
2. According to the step law μ, each particle steps independently onto a new location $y \in \mathscr{G}(G, S)$ and repeat the step (1) for each particle.

Dichotomy: recurrent/transient BRW

- $r > R_\mu^{-1} \iff$ recurrent BRW: the particles return, with positive probability, to the starting location infinitely often.
- $r \leq R_\mu^{-1} \iff$ transient BRW: the particles eventually leave every finite locations. Equivalently, if the r–**Green function** is finite:

$$G_r(x, y) = \sum_{n \geq 0} \mathbf{P}(\omega_n = y, \omega_0 = x) r^n$$

which is the expected number of particles visiting y from x.

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Martin boundary = Buseman boundary of Green metric

1. Note that $G_r(x,y) = F_r(x,y)G_r(y,y) = F_r(x,y)G_r(e,e)$ where

$$F_r(x,y) = \sum_{n\geq 1} \mathbf{P}(\omega_n = y, \omega_{0\leq i<n} \neq y, \omega_0 = x) r^n$$

is the expected number of particles first visiting y from x.

2. Define the **Green metric**:

$$d_G(x,y) = -\log \frac{G_r(x,y)}{G_r(e,e)} = -\log F_r(x,y)$$

3. We inject all the elements $y \in G$ into the set of normalized Green functions (=1-Lipschitz functions):

$$x \in G \quad \mapsto \quad b_y(x) := \boxed{d_G(x,y) - d_G(e,y) = e^{G_r(e,y)/G_r(x,y)}}$$

The closure $\overline{G}_\mu$ of $\{b_y(x) : y \in G\}$ in $\mathcal{C}(G,\mathbb{R})$ gives a compactification of G, so that $\partial_\mu G := \overline{G}_\mu \setminus G$ is called r-**Martin boundary**.

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Martin boundary covers Floyd boundary

Theorem (Gekhtman-Gerasimov-Potyagailo-Y, 2021)

Let μ be a finitely supported symmetric random walk on a finitely generated group G. Then for any $1 \le r < R_\mu$ the identification $G \to G$ extends to a continuous surjective map

$$\partial_\mu G \to \partial_f G.$$

Moreover, the preimage of each conical point in $\partial_f G$ is a single point.

Past and further works:

1. Martin boundary of virtually abelian groups [Ney-Spitzer, 1968]; Martin boundary for hyperbolic groups [Ancona, 1988, Gouezel-Lalley 2013, Gouezel 2014]
2. Martin boundary for finite volume hyperbolic manifolds groups [Dussaule-Gekhtman-Gerasimov-Potyagailo, 2021]; Stability of Martin boundary at the spectral radius [Dussaule-Gekhtman 2021]

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Key Tool: Relative Ancona Inequality

Lemma (GGPY 2021)

Let $r < R_\mu^{-1}$. There exists a decreasing function $\mathcal{A} : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ with the following property. Let $x, y, z \in G$ such that $\rho_y(x, z) \geq \epsilon > 0$. Then

$$\boxed{A(\epsilon) \cdot G_r(x, y) G_r(y, z) \leq G_r(x, z) G_r(e, e) \leq G_r(x, y) G_r(y, z).}$$

Past and further works

1. If G is a hyperbolic group, then $\rho_y(x, z) \geq \epsilon$ is uniformly bounded below for any triple points x, y, z on a geodesic. This gives the so-called Ancona inequality.
2. Relative Ancona inequality extended up to the spectral radius: [Dussaule-Gekhtman, 2021]
3. Local limit theorems for (relatively) hyperbolic groups [Gouezel 2014; Dussaule 2022]

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The **trace** $\mathcal{P}$ of a BRW consists of the locations that the particles visited.

1. If BRW is recurrent, the trace is the whole graph.
2. If BRW is transient, then $G_r(e,x) = \sum_{n\geq 0} \mathbf{P}(\omega_n = x, \omega_0 = e) r^n$ is finite and consider the volume of Green function over spheres

$$H_r(n) := \sum_{x \in S_n} G_r(e,x)$$

whose growth rate is defined as

$$\omega(r) := \limsup_{n\to\infty} \frac{\log H_r(n)}{n}$$

Problem (Limit behaviour of the trace)

- *The asymptotic behaviour of the trace $\mathcal{P}_n := \mathcal{P} \cap S_n$ and the volume growth $H_r(n)$;*
- *The Hausdorff dimension of the limit set $\Lambda(r)$ of the trace of BRW.*

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Volume growth of BRW trace

Consider a BRW on a relatively hyperbolic group G with underlying symmetric finitely supported μ–random walk and with offspring mean $1 \le r \le R_\mu^{-1}$.

Recall that

$$\delta_G := \limsup_{n\to\infty} \frac{\log \sharp\{g \in G : d_S(o, go) \le n\}}{n}$$

Theorem (Dussaule-Wang-Y. 2022)

1. *The function*
$$r \to \omega(r)$$
is strictly increasing in $[1, R_\mu^{-1}]$, *and continuous in* $[1, R_\mu^{-1})$ *and* $0 < \omega(r) \le \frac{\delta_G}{2}$ *for* $r > 1$.
2. *Almost surely,*
$$\omega(r) = \limsup_{n\to\infty} \frac{\log |\mathcal{P} \cap S_n|}{n}.$$

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Limit set for transient branching random walks

Theorem (DWY 2022)

Let $1 \leq r \leq R_\mu^{-1}$. Let $\Lambda(r)$ denote the limit set of BRW trace in Bowditch boundary with shortcut metric $\bar{\rho}_\lambda$. Then almost surely,

$$\mathrm{Hdim}_{\bar{\rho}_\lambda}(\Lambda(r)) = \frac{\omega(r)}{-\log\lambda} \leq \frac{1}{2}\mathrm{Hdim}_{\bar{\rho}_\lambda}(\Lambda G) = \frac{\delta_G}{-2\log\lambda}$$

Remark

1. This generalizes the work [SWX] of V. Sidoravicius, Longmin Wang, and Kainan Xiang on hyperbolic groups, and resolves their conjecture.
2. The following asymptotic behaviour of $\omega(r)$ was obtained:

$$e^{\omega(R_\mu^{-1})} - e^{\omega(r)} \sim C\sqrt{R_\mu^{-1} - r}, \text{ as } r \to R_\mu^{-1}$$

   for a constant C, in the class of free groups by Hueter and Lalley, hyperbolic groups by [SWX], free products of groups by Candellero, Gilch and Muller.

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Upper bound on Hausdorff dimension

Recall that the trace $\mathcal{P}$ of BRW consists of locations that the particles of BRW visited, and $\Lambda(r)$ is the limit set of the trace at Bowditch boundary.

Theorem (D-W-Y. 2022)

There exists a finite number $\kappa > 0$ such that almost surely, for every conical point $\xi \in \Lambda(r)$,

$$\limsup_{|x|\to\infty} \frac{d(x,\mathcal{P})}{\log|x|} \le \kappa$$

where x is taken over the set of transition points on the geodesic $[o,\xi]$.

Recall that, almost surely, we have

$$\omega(r) = \limsup_{n\to\infty} \frac{\log|\mathcal{P} \cap S_n|}{n}.$$

By a standard argument, we can cover the limit set $\Lambda(r)$ by shadows around the transition points, so $\mathrm{Hdim}_{\bar{\rho}_\lambda}(\Lambda(r)) \le \frac{\omega(r)}{-\log\lambda}$.

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There exists a finite number $\kappa > 0$ such that almost surely, for every conical point $\xi \in \Lambda(r)$,

$$\limsup_{|x|\to\infty} \frac{d(x,\mathcal{P})}{\log|x|} \le \kappa$$

where x is taken over the set of transition points on the geodesic $[o,\xi]$.

Recall that, almost surely, we have

$$\omega(r) = \limsup_{n\to\infty} \frac{\log|\mathcal{P}\cap S_n|}{n}.$$

By a standard argument, we can cover the limit set $\Lambda(r)$ by shadows around the transition points, so $\mathrm{Hdim}_{\bar{\rho}_\lambda}(\Lambda(r)) \le \frac{\omega(r)}{-\log\lambda}$.

Upper bound on Hausdorff dimension

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Theorem (D-W-Y. 2022)

There exists a finite number $\kappa > 0$ such that almost surely, for every conical point $\xi \in \Lambda(r)$,

$$\limsup_{|x|\to\infty} \frac{d(x, \mathcal{P})}{\log |x|} \leq \kappa$$

where x is taken over the set of transition points on the geodesic $[o, \xi]$.

Recall that, almost surely, we have

$$\omega(r) = \limsup_{n\to\infty} \frac{\log |\mathcal{P} \cap S_n|}{n}.$$

By a standard argument, we can cover the limit set $\Lambda(r)$ by shadows around the transition points, so $\mathrm{Hdim}_{\bar{\rho}_\lambda}(\Lambda(r)) \leq \frac{\omega(r)}{-\log \lambda}$.

Thank you for your attention!