Separating Systole for Random Hyperbolic Surfaces of Large Genus

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Table of Contents

Geometric Quantities on Hyperbolic Surfaces

Let $X_g \in \mathcal{M}_g$ be a closed hyperbolic surface of genus g (curvature -1).
By Gauss-Bonnet formula, $\mathrm{Area}(X_g) = 4\pi(g-1)$.
There are many interesting geometric quantities to study. For example:

systole, separating systole, diameter,

inradius, total pants length,

eigenvalues, Cheeger constant $\cdots$

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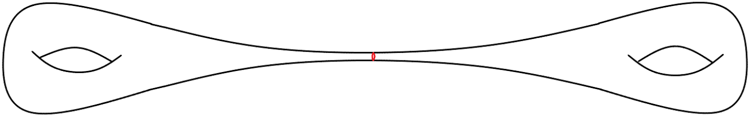
Systole of X_g can be arbitrarily short, and bounded from above by

$$\ell_{\mathrm{sys}}(X_g) \leq 2\log(4g-2).$$



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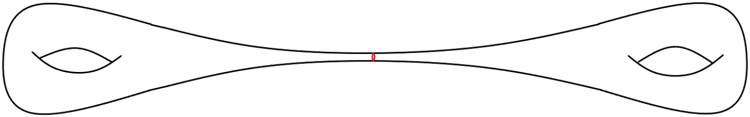

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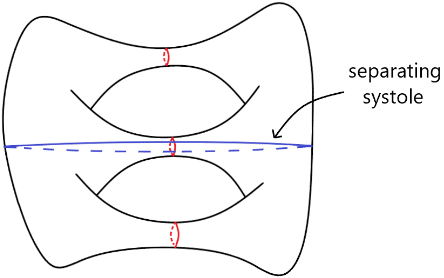
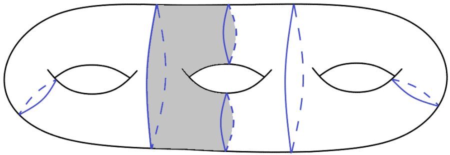


Table of Contents

Random Hyperbolic Surfaces of Weil-Petersson Model

Let $\mathcal{M}_g$ be the moduli space of closed hyperbolic surfaces of genus g. The Teichmüller space $\mathcal{T}_g$ is a universal cover of $\mathcal{M}_g$ and $\mathcal{M}_g = \mathcal{T}_g / \mathrm{Mod}_g$ where $\mathrm{Mod}_g = \mathrm{Homeo}_+(S_g)/\mathrm{Homeo}_0(S_g)$ is the mapping class group.

A pants decomposition $\{\alpha_i\}_{i=1}^{3g-3}$ of the surface are $3g-3$ disjoint simple closed curve that separate the surface into $2g-2$ pants. The Fenchel-Nielsen coordinate $(\ell_{\alpha_i}(X), \tau_{\alpha_i}(X))_{i=1}^{3g-3}$ gives a coordinate for $\mathcal{T}_g$. Here $\ell_{\alpha_i}(X)$ is the length and $\tau_{\alpha_i}(X)$ is the twist along α_i.



Random Hyperbolic Surfaces of Weil-Petersson Model

The Teichmüller space $\mathcal{T}_g$ admits a Mod_g-invariant Riemannian metric called the Weil-Petersson metric, and hence gives an induced metric on the moduli space $\mathcal{M}_g$.

Theorem (Wolpert 1982)

The sympletic form ω_{WP} of Weil-Petersson metric on $\mathcal{T}_g$ is given by

$$\omega_{\text{WP}} = \sum_{i=1}^{3g-3} d\ell_{\alpha_i} \wedge d\tau_{\alpha_i}.$$

The volume form:

$$d\,\mathrm{Vol}_{\text{WP}}(X) := \frac{1}{(3g-3)!}\omega_{\text{WP}} \wedge \cdots \wedge \omega_{\text{WP}} = \bigwedge_{i=1}^{3g-3} (d\ell_{\alpha_i} \wedge d\tau_{\alpha_i}).$$

Random Hyperbolic Surfaces of Weil-Petersson Model

The Weil-Petersson metric on the moduli space $\mathcal{M}_g$ is of finite volume.

Theorem (Mirzakhani-Zograf 2015)

There exists a universal constant $C > 0$ such that

$$\mathrm{Vol}(\mathcal{M}_g) = C \frac{1}{\sqrt{g}} (2g-3)!(4\pi^2)^{2g-3} \left(1 + O(\frac{1}{g})\right).$$

Conjecture: $C = \frac{1}{\sqrt{\pi}}$.

Random Hyperbolic Surfaces of Weil-Petersson Model

The Weil-Petersson metric induces a probability measure $\mathrm{Prob}_{\mathrm{WP}}^{g}$ on $\mathcal{M}_g$.

$$\mathrm{Prob}_{\mathrm{WP}}^{g}(\mathcal{A}) := \frac{1}{\mathrm{Vol}(\mathcal{M}_g)} \int_{\mathcal{M}_g} \mathbf{1}_{\mathcal{A}}(X)\, d\,\mathrm{Vol}_{\mathrm{WP}}(X) = \frac{\mathrm{Vol}(\mathcal{A})}{\mathrm{Vol}(\mathcal{M}_g)}.$$

And the expectation is defined by

$$\mathbb{E}_{\mathrm{WP}}^{g}[f] := \frac{1}{\mathrm{Vol}(\mathcal{M}_g)} \int_{\mathcal{M}_g} f(X)\, d\,\mathrm{Vol}_{\mathrm{WP}}(X).$$

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Say a property P holds for random hyperbolic surfaces, or equivalently say P happens with high probability, if

$$\lim_{g\to\infty} \mathrm{Prob}_{\mathrm{WP}}^{g}(X \in \mathcal{M}_g |\ P \text{ holds for } X) = 1.$$

Other models

Random hyperbolic surfaces of Weil-Petersson model was first studied by Mirzakhani (2013, or her ICM talk 2010). And this is motivated by a model studied by Brooks and Makover, where they constructed surfaces by gluing ideal triangles among random 3-regular graphs.

Other models

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- BM model
- Gluing pants among random 3-regular graphs
- Random covering of a fixed hyperbolic surface

These three models are “discrete”.

Mirzakhani's Integration Formula

Let γ be a simple closed curve. Consider the orbit under Mod_g-action:

$$\mathcal{O}_\gamma = \{h \cdot \gamma |\ h \in \mathrm{Mod}_g\}.$$

Given $F : \mathbb{R}_{\geq 0} \to \mathbb{R}$, we may define a function F^γ on $\mathcal{M}_g$:

$$F^\gamma(X) := \sum_{\alpha \in \mathcal{O}_\gamma} F(\ell_\alpha(X)).$$

Theorem (Mirzakhani 2007)

$$\int_{\mathcal{M}_g} F^\gamma(X) d\,\mathrm{Vol}_{\mathrm{WP}}(X) = C_\gamma \int_{\mathbb{R}_{\geq 0}} F(t) V_g(\gamma, t) t\ dt$$

where the constant $C_\gamma \in (0,1]$ only depends on γ.

Remark: In general, Mirzakhani's integration formula holds for simple closed multi-curves $\Gamma = (\gamma_1, \cdots, \gamma_k)$.

Geometric Quantities for Random Surfaces

Recall that a **systole** is a shortest closed geodesic.
Denote $\ell_{\mathrm{sys}}(X)$ to be the length of a systole of X.

Theorem (Mirzakhani 2013)

There exist universal constants $c_2 > c_1 > 0$ and $r_0 > 0$ such that for any $r < r_0$,

$$c_1 \cdot r^2 \leq \mathrm{Prob}_{\mathrm{WP}}^{g}\left(X \in \mathcal{M}_g \middle| \, \ell_{\mathrm{sys}}(X) < r\right) \leq c_2 \cdot r^2$$

as $g \to \infty$.

Theorem (Mirzakhani-Petri 2017)

$$\lim_{g\to\infty} \mathbb{E}_{\mathrm{WP}}^{g}[\ell_{\mathrm{sys}}(X)] := \lim_{g\to\infty} \frac{\int_{\mathcal{M}_g} \ell_{\mathrm{sys}}(X) dX}{\mathrm{Vol}(\mathcal{M}_g)} = 1.61498\ldots.$$

Let $\mathrm{Emb}(X)$ be the radius of the largest embedded ball in X (also called the inradius of X). For any hyperbolic surface X_g,

$$\mathrm{Emb}(X_g) < \log(4g - 2).$$

Theorem (Mirzakhani 2013)

$$\lim_{g\to\infty} \mathrm{Prob}_{\mathrm{WP}}^{g}\left(X \in \mathcal{M}_g \,\middle|\, \mathrm{Emb}(X) > \frac{1}{6}\log g\right) = 1.$$

Actually she proved that at "most points" on a random hyperbolic surface, the injective radius $> \frac{1}{6}\log g$.

Geometric Quantities for Random Surfaces

Denote $\mathrm{diam}(X)$ to be the diameter of X. For any hyperbolic surface X_g,

$$\mathrm{diam}(X_g) > \log(4g - 4).$$

Theorem (Mirzakhani 2013)

$$\lim_{g\to\infty} \mathrm{Prob}_{\mathrm{WP}}^{g}\left(X \in \mathcal{M}_g \middle| \, \mathrm{diam}(X) < 40 \log g\right) = 1.$$

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$$\lim_{g \to \infty} \mathrm{Prob}_{\mathrm{WP}}^{g} (X \in \mathcal{M}_g | \ \mathrm{diam}(X) < 40 \log g) = 1.$$

this is not a "typical" surface

Geometric Quantities for Random Surfaces

Theorem (Wu-X. 2022)

For any $\epsilon > 0$,

$$\lim_{g\to\infty} \mathrm{Prob}_{\mathrm{WP}}^{g}\left(X \in \mathcal{M}_g \mid \mathrm{diam}(X) < (4+\epsilon)\log g\right) = 1.$$

Geometric Quantities for Random Surfaces

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Recall that:

$$\lim_{g\to\infty} \mathrm{Prob}_{\mathrm{WP}}^{g}\left(X \in \mathcal{M}_g \middle|\ \mathrm{Emb}(X) > \frac{1}{6}\log g\right) = 1.$$

Random hyperbolic surfaces should be "crowded" and "fat".

Reccall that a **separating systole** is a shortest simple closed geodesic which separates the surface into two pieces.
Denote $\ell_{\mathrm{sys}}^{\mathrm{sep}}(X)$ to be the length of a separating systole of X.

Theorem (Mirzakhani 2013)

For any $\epsilon > 0$,

$$\lim_{g\to\infty} \mathrm{Prob}_{\mathrm{WP}}^{g}\left(X \in \mathcal{M}_g \middle|\ \ell_{\mathrm{sys}}^{\mathrm{sep}}(X) > (2-\epsilon)\log g\right) = 1.$$

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Recall:

$$c_1 r^2 \leq \mathrm{Prob}_{\mathrm{WP}}^{g} \left(X \in \mathcal{M}_g \middle| \, \ell_{\text{sys}}(X) < r \right) \leq c_2 r^2.$$

Separating systole behaves very different from systole.

Table of Contents

Main Results

Reccall that $\ell_{\text{sys}}^{\text{sep}}(X)$ is the length of a separating systole of X (a shortest simple closed geodesic which separates the surface into two pieces).

Let $\omega(g)$ satisfies $\lim_{g\to\infty}\omega(g) = +\infty$ and $\lim_{g\to\infty}\frac{\omega(g)}{\log\log g} = 0$.

Theorem (Nie-Wu-X. 2020)

For any fixed $\epsilon > 0$, there exists $\mathcal{A}_g \subset \mathcal{M}_g$ such that $\mathrm{Prob}_{\mathrm{WP}}^g(\mathcal{A}_g) \to 1$ and for any $X \in \mathcal{A}_g$ the following conditions hold.

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(a). $|\ell_{\mathrm{sys}}^{\mathrm{sep}}(X)-(2\log g-4\log\log g)|\leq\omega(g)$;

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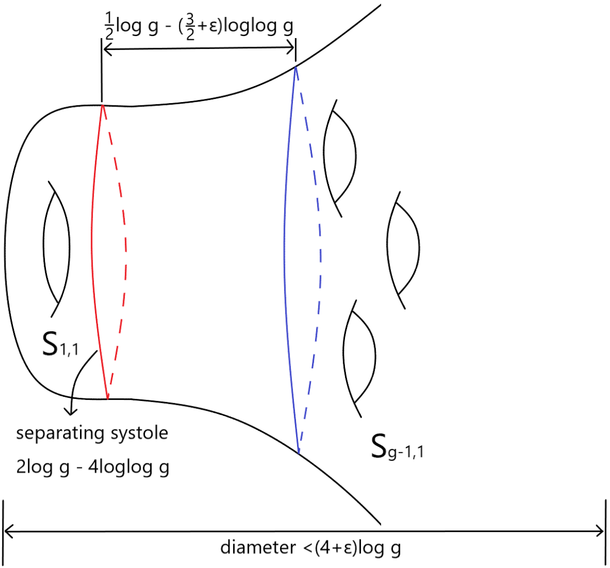
(c). *There is a half-collar in the $S_{g-1,1}$-part of X with width $\frac{1}{2}\log g - (\frac{3}{2} + \epsilon)\log\log g$.*

Main Results



Main Results

Theorem (Parlier-Wu-X. 2020)

$$\lim_{g\to\infty} \frac{\mathbb{E}^g_{\mathrm{WP}}[\ell^{\mathrm{sep}}_{\mathrm{sys}}(X)]}{\log g} \left(:= \lim_{g\to\infty} \frac{\int_{\mathcal{M}_g} \ell^{\mathrm{sep}}_{\mathrm{sys}}(X) dX}{\mathrm{Vol}(\mathcal{M}_g)\cdot \log g} \right) = 2.$$

Remark: $\ell^{\mathrm{sep}}_{\mathrm{sys}}(X)$ is unbounded over $\mathcal{M}_g$.

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Remark: $\ell^{\mathrm{sep}}_{\mathrm{sys}}(X)$ is unbounded over $\mathcal{M}_g$.

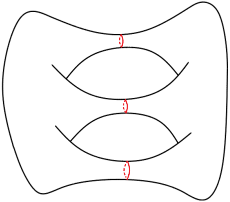
Remark: [Mirzakhani 2013] claimed $c_1 \leq \frac{\mathbb{E}^g_{\mathrm{WP}}[\ell^{\mathrm{sep}}_{\mathrm{sys}}(X)]}{\log g} \leq c_2$, but there is a gap in her paper.

Other Results

Let $\omega(g)$ satisfies $\lim_{g\to\infty}\omega(g)=+\infty$ and $\lim_{g\to\infty}\frac{\omega(g)}{\log\log g}=0$.

$\mathcal{L}_1(X)$: the length of shortest simple separating multi-geodesic of X.

- $\mathcal{L}_1(X)\leq \ell_{\mathrm{sys}}^{\mathrm{sep}}(X)$
- $\mathcal{L}_1(X)\leq C\log g$



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(a). $|\mathcal{L}_1(X)-(2\log g-4\log\log g)|\leq\omega(g)$;

(b). *$\mathcal{L}_1(X)$ is achieved by either a simple closed geodesic separating X into $S_{1,1}\cup S_{g-1,1}$, or three simple closed geodesics separating X into $S_{0,3}\cup S_{g-2,3}$;*

Other Results

Let $\omega(g)$ satisfies $\lim_{g\to\infty}\omega(g)=+\infty$ and $\lim_{g\to\infty}\frac{\omega(g)}{\log\log g}=0$.

$\ell_{\mathrm{sys}}^{\mathrm{ns}}(X)$: the length of shortest non-simple closed geodesic of X.

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(c). $(1-\epsilon)\log g<\ell_{\mathrm{sys}}^{\mathrm{ns}}(X)<2\log g$;

Related Result

Delecroix-Goujard-Zograf-Zorich 2021:

For any closed hyperbolic surface $X \in \mathcal{M}_g$,

$$\frac{\text{“frequency” of simple separating geodesics}}{\text{“frequency” of simple non-separating geodesics}} \sim \sqrt{\frac{2}{3\pi g}} \cdot \frac{1}{4^g}$$

as $g \to \infty$.

Table of Contents

Proof Sketch

Consider

$$N_{1,1}(X,L) := \#\left\{\gamma \subset X \;\middle|\; \begin{array}{l} \gamma \text{ is a simple closed geodesic,} \\ \ell_\gamma(X) \leq L,\ \ X \setminus \gamma \cong S_{1,1} \cup S_{g-1,1} \end{array} \right\}.$$

$N_{1,1}(X,L) = 0$ means each geodesic of such type has length $> L$.
$N_{1,1}(X,L) \geq 1$ means there is a geodesic of such type of length $\leq L$.

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$N_{1,1}(X,L) = 0$ means each geodesic of such type has length $> L$.
$N_{1,1}(X,L) \geq 1$ means there is a geodesic of such type of length $\leq L$.

By Mirzakhani's Integration Formula,

$$\mathbb{E}^g_{\mathrm{WP}}[N_{1,1}(X,L)] = \frac{1}{384\pi^2} L^2 e^{\frac{1}{2}L} \frac{1}{g} \cdot \left(1 + O\left(\frac{1}{L} + \frac{1+L^2}{g}\right)\right).$$

It is reasonable that $L = 2\log g - 4\log\log g$ may be approximate to the smallest length of such type of geodesics.

Similarly, for separating geodesics of other type, the length may be larger.

Proof Sketch

Lower bound $\ell_{\mathrm{sys}}^{\mathrm{sep}}(X) > 2\log g - 4\log\log g - \omega(g)$:

following [Mirzakhani 2013].

$$\begin{aligned} \mathrm{Prob}_{\mathrm{WP}}^{g}\left(N_{1,1}(X,L) \geq 1\right) &\leq \mathbb{E}_{\mathrm{WP}}^{g}[N_{1,1}(X,L)] \\ &\asymp L^2 e^{\frac{1}{2}L}\frac{1}{g} \\ &\to 0. \end{aligned}$$

$N_{1,1}(X,L) = 0$ means each geodesic of such type has length $> L$.

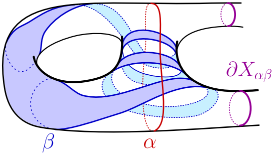
Proof Sketch

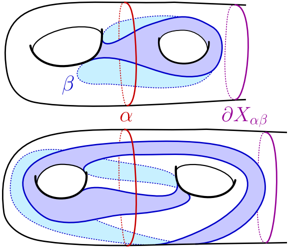
Upper bound $\ell_{\text{sys}}^{\text{sep}}(X) < 2\log g - 4\log\log g + \omega(g)$:
using Chebyshev's Inequality

$$\mathrm{Prob}_{\mathrm{WP}}^{g}(N=0) \le \frac{\mathrm{Var}[N]}{\mathbb{E}_{\mathrm{WP}}^{g}[N]^2} = \frac{\mathbb{E}_{\mathrm{WP}}^{g}[N^2] - \mathbb{E}_{\mathrm{WP}}^{g}[N]^2}{\mathbb{E}_{\mathrm{WP}}^{g}[N]^2}.$$

$N_{1,1}(X,L) \ge 1$ means there exists a geodesic of such type of length $\le L$.

Aim: $\mathrm{Prob}_{\mathrm{WP}}^{g}(N_{1,1}(X,L)=0) \to 0$. $\mathbb{E}_{\mathrm{WP}}^{g}[N^2]$ is the most complicated term. Need to consider how two geodesics intersect with each other.





Proof Sketch

To compute $\mathbb{E}_{\mathrm{WP}}^{g}[\ell_{\mathrm{sys}}^{\mathrm{sep}}(X)]$, use

$$\ell_{\mathrm{sys}}^{\mathrm{sep}}(X) < 2\ell_{\mathrm{sys}}(X) + 4\mathrm{diam}(X),$$

$$\mathrm{diam}(X) \leq 2\left(\frac{\ell_{\mathrm{sys}}(X)}{2} + \frac{1}{h(X)} \cdot \log\left(\frac{2\pi(g-1)}{\mathrm{Area}(B_{\mathbb{H}}(\ell_{\mathrm{sys}}(X)/2))}\right)\right).$$

1. Xin Nie, Yunhui Wu, Yuhao Xue: **Large genus asymptotics for lengths of separating closed geodesics on random surfaces**, *Journal of Topology*, to appear, 64 pages, arXiv:2009.07538.

2. Hugo Parlier, Yunhui Wu, Yuhao Xue: **The simple separating systole for hyperbolic surfaces of large genus**, *Journal of the Institute of Mathematics of Jussieu*, doi:10.1017/S1474748021000190, 10 pages, arXiv:2012.03718.

3. Yunhui Wu, Yuhao Xue: **Random hyperbolic surfaces of large genus have first eigenvalues greater than $\frac{3}{16} - \epsilon$**, *Geom. Funct. Anal.* **32** (2022), no. 2, 340-410.

Thank You!