Positive scalar curvature and exotic aspherical manifolds

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Overview

- Scalar curvature
- Exotic aspherical manifolds
- Locally CAT(0)-manifolds
- Enlargeable length-structure

Scalar curvature

Given a smooth Riemannian n-manifold (M^n, g), the scalar curvature Sc_g can be defined as

$$\mathrm{Vol}_g(B_r(x)) = \mathrm{Vol}_{\mathbb{E}^n}(B_r)[1 - \frac{Sc_g(x)}{6(n+2)} r^2 + o(r^4)],$$

as $r \to 0$.

Example: For smooth Riemannian surfaces $(\Sigma, g) \Longrightarrow K_g = 2Sc_g$.

$K_g \in C^\infty(\Sigma, g)$,
$Sc_g \in C^\infty(M^n, g) \rightsquigarrow Sc_g \approx K$ in higher dimensions,

i.e. the scalar curvature can be considered as a higher dimensional generalization of the Gauss curvature.

The scalar curvature is interesting not only in analysis, geometry, and topology but also in physics. For example,

- General Relativity:

  I. Einstein field equations : $Ric_g - \frac{1}{2}Sc_g g + \Lambda g = \kappa T$.

  II. Positive Mass Theorem $\Longleftrightarrow$ T^3 does not admit a Riemannian metric g with $Sc_g > 0$. (Schoen-Yau 1979, Gromov-Lawson 1980, E. Witten 1982, Weighted version, D. 2021)

- Seiberg-Witten Theory (1994)
  Assume that a closed oriented smooth 4-manifold M admits a Riemannian metric with positive scalar curvature, then the Seiberg-Witten invariants of M are zero. (E. Witten 1994, Weighted version, D. 2021)

Closed (Σ, g) with $K_g > 0 \xrightarrow{\text{Gauss-Bonnet}} \Sigma = S^2$.

Question: Which closed (smooth) manifold admits a Riemannian metric with positive scalar curvature (PSC-metric)?
i.e. A closed $M^n \rightsquigarrow \exists g$ s.t. $Sc_g > 0$?

- What about when $Sc_g < 0$?

Answer: For $h \in C^\infty(M^n)$, $n \geq 3$, $\exists p \in M^n$ s.t. $h(p) < 0$
$\Longrightarrow \exists g$ s.t. $Sc_g = h$. (Kazdan-Warner 1975)

- How about when $Sc_g \geq (\neq)\ 0$?

Answer: By Ricci-flow arguments $\Longrightarrow \exists g_0$ s.t. $Sc_{g_0} > 0$.

Closed oriented 3-manifold M admitting PSC-metrics $\Longrightarrow M =$ spherical 3-manifolds, $S^2 \times S^1$, or spherical 3-manifolds$\# \sum S^2 \times S^1$. (Schoen-Yau 1979, Gromov-Lawson 1980, Thurston 1982, Perelman 2003.)

For simply connected smooth closed n-manifold M^n ($n \geq 5$),

- if M^n is not spin, then it carries PSC-metrics, (Gromov-Lawson 1980);
- if M^n is spin, then it carries PSC-metrics iff $\alpha(M^n) = 0$. (Stolz 1990) (M^n is spin $\iff w_2(M) = 0$.)

Inspired by Schoen-Yau's positive mass theorem (which was proved with minimal hypersurface method)(1979), Gromov and Lawson (1983) show that non-positively curved manifolds carry no PSC-metric according to the index theory.

Conjecture A: A closed aspherical manifold (K(π,1)-manifold) does not admit a PSC-metric.

There are several approaches to proving this conjecture:

- Farrell-Jones conjecture $\Longrightarrow$ Conjecture A;
- Strong Novikov conjecture $\Longrightarrow$ Conjecture A;
- Injective Baum-Connes map (conjecture) $\Longrightarrow$ Conjecture A;
- Using tools from Noncommutative Geometry, KK-theory (Kasparov-Skandalis 2003), Metric Geometry (D. 2021), ect.

Exotic aspherical manifolds

Gromov's question: Are aspherical manifolds covered by Euclidean spaces?

Recall: In dimensions$\geq$ 4, a necessary and sufficient condition for a contractible manifold to be homeomorphic to the Euclidean space is that it is simply connected at infinity (Stallings for dim$\geq$5 (1962), Freedman for dim$=$ 4 (1986)).

Answer: Using the reflection group trick, Michael W. Davis (1983) constructed aspherical n-manifolds ($n \geq 4$) that are not covered by Euclidean spaces (exotic aspherical manifold).

Davis-Januszkiewicz-Lafont (DJL) (2012) constructed a closed smooth manifold M^4, whose universal cover $\widetilde{M^4}$ is diffeomorphic to R^4, but $\pi_1(M^4)$ is not isomorphic to the fundamental group of any compact Riemannian manifold with non-positive sectional curvature.

Theorem 1 (D. 2021)

The connected sum of DJL's manifold and a closed manifold does not admit a PSC-metric.

Remark: The methods of using the results from Farrell-Jones conjecture, Strong Novikov conjecture, and Injective Baum-Connes map only work for K(π, 1)-manifolds.

The proof of this theorem uses the tool from metric geometry, since DJL's manifold is a locally CAT(0)-manifold.

A metric space (X, d) is a length metric space if the distance between each pair of points equals the infimum of the lengths of curves joining the points.

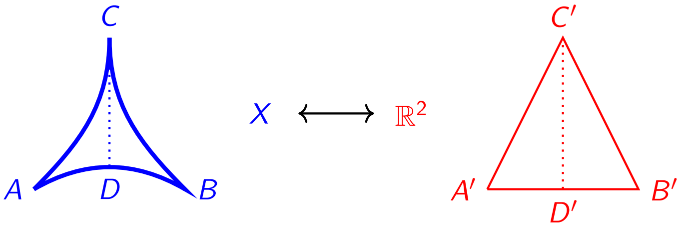
Examples:

- If $\pi : \hat{X} \to X$ is a covering map, every length metric d on X lifts to a unique length metric $\hat{d}$ for which the covering map is a local isometry.
- Metrics induced by smooth Riemannian metrics on a closed smooth manifold are length metrics. Any two Riemannian metrics on a smooth closed manifold are bi-Lipschitz equivalent.

However, two length metrics may not be bi-Lipschitz equivalent in general, if one of them is induced by a Riemannian metric and the other by the Finsler metric.

Locally CAT(0)-manifolds

Let (X, d_X) be a length space. A geodesic triangle $\triangle$ in X with geodesic segments as its sides is said to satisfy the CAT(0)-inequality if it is slimmer than the comparison triangle in the Euclidean plane.



$$d_X(A, D) = \|A'D'\| \Rightarrow d_X(C, D) \leq \|C'D'\|$$

A length metric d on X is said to be a locally CAT(0)-metric if every point in X has a geodesically convex neighborhood, in which every geodesic triangle satisfies the CAT(0)-inequality. A locally CAT(0)-manifold is a topological manifold endowed with a locally CAT(0)-metric. It is an aspherical manifold (Gromov's generalized Hadamard-Cartan theorem 1985).

Riemannian metrics with non-positive sectional curvature are locally CAT(0)-metrics. If a manifold of dimension 2 or 3 admits a locally CAT(0)-metric, then it also admits non-positive curvature metrics, according to the classic surface theory and Thurston-Perelmann Geometrization Theorem.

But there is a difference between locally CAT(0)-manifolds and non-positive curved manifolds in dimensions$\geq$ 4.

Locally CAT(0) VS Non-positive curved

Aravinda and Farrell (1994) showed that the existence of non-positive curvature metric is not a homeomorphism invariant in general, but the existence of a locally CAT(0)-metric is homeomorphism invariant.

The existence of a non-positive curvature metric depends on that of the smooth structure. Furthermore, locally CAT(0)-manifolds that do not support a smooth structure in dimensions≥ 5 were constructed by Davis and Hausmann (1989).

In dimensions≥ 5, Davis and Januszkiewicz (1991) constructed a topological locally CAT(0)-manifold, whose universal cover is not homeomorphic to $\mathbb{R}^n$.

They also constructed a smooth locally CAT(0)-manifold whose universal cover is homeomorphic to $\mathbb{R}^n$, but the boundary at infinity is distinct from S^{n-1}.

Product construction: Let M^n ($n \geq 5$) be a locally CAT(0)-manifold with center-free fundamental group, whose universal cover is distinct from $\mathrm{R^n}$, as introduced previously, and N be an arbitrary closed locally CAT(0)-manifold with center-free fundamental group, then the product $M^n \times N$ is a locally CAT(0)-manifold, which does not support any non-positive curvature metrics.

The product construction is based on the following two theorems:

(1). Farrell-Jones theorem (1991): Borel conjecture holds for non-positively curved manifolds;

(2). Gromoll-Wolf/Lawson-Yau's splitting theorem (1971): Let M be a compact non-positively curved manifold and $\pi_1(M)$ is a direct product $A \times B$ with a trivial center, then M is isometric to a Riemannian product $M_1 \times M_2$ with $\pi_1(M_1) = A$ and $\pi_1(M_2) = B$.

Furthermore, using Davis's construction, Sapir (2020) firstly created closed aspherical topological n-manifolds ($n \geq 4$), whose fundamental groups coarsely contain expanders and the aspherical n-manifold can be smooth if $n \geq 5$.

Thus, Sapir's aspherical manifolds have infinite asymptotic dimension, are not coarsely embeddable into a Hilbert space, do not satisfy G. Yu's property A, and do not satisfy the Baum-Connes conjecture with coefficients.

Using Davis's construction and Sapir's techniques, Osajda (2021) constructed closed aspherical topological n-manifolds ($n \geq 4$), whose fundamental groups contain coarsely embedded expanders.

Locally CAT(0) VS Non-positive curved

M^n $(n \geq 4)$	Locally CAT(0)	Non-positive curved
Homeomorphism invariant	Yes	No, in general
$\exists$ Smooth structures	No, in general	Yes
$\widetilde{M^n}$	Not necessarily R^n	R^n
π_1	$\exists$ π_1 is not isomorphic to the fundamental group of any compact Riemannian manifold with non-positive curvature.	

Enlargeable Length-structure

A topological manifold X endowed with a complete length metric is called *ε-hyperspherical* if it admits a continuous map f to S^n ($n = \dim(X)$) that is constant at infinity, of non-zero degree and such that

$$\mathrm{Lip}(f) := \sup_{\substack{a \neq b \\ a,b \in X}} \frac{d_{S^n}(f(a), f(b))}{d_X(a,b)} < \varepsilon.$$

Here *constant at infinity* means that there is a compact subset such that f maps the complement of the compact subset to a point in S^n and S^n is endowed with the standard round metric d_{S^n}.

Definition 2 (D. 2021)

A length metric d on a closed orientable n-dimensional topological manifold X^n is said to be enlargeable if for each $\varepsilon > 0$ there is an oriented covering manifold $\widetilde{X^n}$ endowed with the induced metric $\widetilde{d}$ that is ε-hyperspherical.
An enlargeable length-structure on X^n is a bi-Lipschitz equivalent class of an enlargeable metric.

Properties of the enlargeable metric:

(1) If a closed n-manifold X carries an enlargeable length-structure and Y is an arbitrary closed n-manifold, then $X \# Y$ still carries an enlargeable length-structure.

(2) The product of two enlargeable metrics is an enlargeable metric.

Example: the length metric induced by a Riemannian metric with non-positive sectional curvature on a closed manifold is an enlargeable length metric.

Remark: Gromov-Lawson (1980) introduced enlargeability as an obstruction based on the index theory. Later, they (1984) relaxed the spin assumption in dimensions less than 8. Inspired by Schoen-Yau's results (2017), Cecchini-Schick (2021) show that a closed enlargeable manifold cannot carry any PSC-metric.

Both enlargeabilities mentioned above are defined on Riemannian metrics and need at least C^1-smoothness for the maps to prove their results.

The enlargeable length-structure works for length metric spaces and only require the maps to be continuous such that we can give a new obstruction to the existence of PSC-metrics on a closed manifold.

Theorem 3 (D. 2021)

Let X^n $(2 \leq n \leq 8)$ be a closed orientable smooth manifold, then X^n carries no PSC-metrics in its enlargeable length-structures.

Remark: the proof of this Theorem is based on Gromov's Spherical Lipschitz Bound Theorem (2018), which holds for dimensions from two to eight.

Schoen-Yau Theorem (2017 arXiv): $T^n \# M$ does not admit a PSC-metric.

Based on Schoen-Yau's (2017) method of the "handling" of the singular subsets, then Theorem 3 also holds for all higher dimensions.

• The existence of an enlargeable length-structure is an obstruction of the existence of "scalar curvature bounded below" on the topological manifolds. (D. 2021)

Lemma: A locally CAT(0)-metric that is bi-Lipschitz equivalent to a Riemannian metric on a closed smooth manifold M^n is an enlargeable metric.

Proof: Let (M^n, d) be a closed n-dimensional smooth locally CAT(0)-manifold, then its universal cover $(\widetilde{M^n}, \tilde{d})$ is a globally CAT(0)-manifold according to Gromov's Theorem.

(1.) Consider the map

$$f_t : \widetilde{M^n} \to \widetilde{M^n} \quad x \to \gamma_x(t\tilde{d}(x, x_0)),$$

where x_0 is a fixed point in $\widetilde{M^n}$, $t \in (0, 1]$, and γ_x is the unique geodesic segment from x to x_0. It is well defined by the property of globally CAT(0)-manifolds and is a proper map such that the degree of f_t is non-zero.

(2.) Applying the CAT(0)-inequality to the geodesic triangle with endpoints x, y and x_0, one gets

$$\tilde{d}(f_t(x), f_t(y)) \leq t d_{\mathbb{R}^2}(x, y) = t d(x, y)$$

for $x, y \in \widetilde{M^n}$. Therefore, $\mathrm{Lip}(f_t) \leq t$.

(3). Thus, $\pi \circ f_t : \widetilde{M^n} \to S^n$ has non-zero degree and $\mathrm{Lip}(\pi \circ f_t) \leq tC$. Here $\pi : (\widetilde{M^n}, \tilde{d}) \to S^n$ is a collapse map around x_0, the degree of π is 1 and $\mathrm{Lip}(\pi) \leq C$ by the smoothness of π. $\square$

Remark: DJL's manifold M^4 is a locally CAT(0)- manifold M^4 and the locally CAT(0)-metric is bi-Lipschitz equivalent to a Riemannian metric on the aspherical 4-manifold.

Therefore, combining Theorem 3, Lemma, and Properties (1), one has Theorem 1.

However, combining Stone theorem (1976) with Siebenmann-Sullivan theorem (1979), we know that the locally CAT(0)-metric on a closed manifold, that its universal cover is not homeomorphic to the appropriate $\mathbb{R}^n$, is not bi-Lipschitz equivalent to a Riemannian metric on the manifold.

Therefore, some of Davis's exotic aspherical manifolds do not come from the locally $\mathrm{CAT}(0)$-manifolds, where the metric is bi-Lipschitz equivalent to a Riemannian metric on it.

Question: Does the connected sum of the locally CAT(0)-manifold, which is not bi-Lipschitz equivalent to a Riemannian manifold, and a closed manifold admit a PSC-metric or not?

Thank you for your attention!