The Existence and non-existence results of $\mathbb{Z}_2$ harmonic 1-forms

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The singular $\mathbb{Z}_2$ harmonic 1-form

Let (M,g) be a smooth closed Riemannian manifold, Z be a codimensional 2 closed submanifold of M, let $\mathcal{I}$ be a flat line bundle over $M \setminus Z$ with monodromy -1 along small loop linking Z. A singular $\mathbb{Z}_2$ harmonic 1-forms is a section $v \in \Gamma(\mathcal{I})$ such that

(i) $dv = d \star v = 0$,

(ii) $v \in L^2$.

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(i) $dv = d \star v = 0$,

(ii) $v \in L^2$.

As $\mathcal{I}$ has monodromy -1 along Z, v could also be understood as a two-valued 1-form $\pm v$ defined over $M \setminus Z$ and we usually call Z the singular set.

The singular $\mathbb{Z}_2$ harmonic 1-form

Example

Let $M = \mathbb{C}$ with complex coordinate z, $Z = \{z = 0\}$, let $\mathcal{I}$ be the Möbius bundle over $\mathbb{C} \setminus \{0\}$, then $v = \Re(z^{-\frac{1}{2}} dz)$ is a section of $\mathcal{I}$. As v is a real part of a meromorphic form, v is harmonic. As $|v| \sim |z|^{-\frac{1}{2}}$ along 0, $v \in L^2$. Actually, $v = \Re(z^{-\frac{1}{2}+k} dz)$ for $k \geq 0$ are all $\mathbb{Z}_2$ harmonic 1-forms.

The singular $\mathbb{Z}_2$ harmonic 1-form

The flat bundle $\mathcal{I}$ defines an representation $\rho : \pi_1(M \setminus Z) \to \{\pm 1\}$ and using the kernel of ρ, we could define a double branched covering $p : \tilde{M} \to M$, togather with an involution $\sigma : \tilde{M} \to \tilde{M}$ such that $\sigma^2 = \mathrm{Id}$. The involution induces a decomposition of the cohomology

$$H^k(\tilde{M}; \mathbb{R}) = H^k_-(\tilde{M}) \oplus H^k_+(\tilde{M}),$$

where $H^k_+(\tilde{M}) \cong H^k(M)$.

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where $H^k_+(\tilde{M}) \cong H^k(M)$.
The pull-back $\tilde{v} := p^\star v$ is a 1-form on $\tilde{M}$ will be anti-invariant under the involution $\sigma^\star \tilde{v} = -\tilde{v}$. Moreover, it is harmonic w.r.t. the pull-back singular metric $p^\star g$

$$d\tilde{v} = d \star_{p^\star g} \tilde{v} = 0.$$

By the work of Teleman, also related work of S.Wang, there is L^2 Hodge theorem for this singular metric $p^{\star}g$. Therefore, finding singular $\mathbb{Z}_2$ harmonic 1-forms is purely a topological problem, which could be identified with the space of $H^1_-(\tilde{M})$ for $\tilde{M}$ be any double branched covering of M,

$$\{\text{singular } \mathbb{Z}_2 \text{ harmonic one forms}\} \cong H^1_-(\tilde{M}).$$

Example

Let $K \subset S^3$ be a oriented knot or link, there is a double branched covering $\tilde{M}$ along K. As S^3 has trivial 1st homology, $H^1(\tilde{M}) = H^1_-(\tilde{M})$ are all anti-invariant under the involution. Therefore, the existence of singular $\mathbb{Z}_2$ harmonic 1-form is equivalent to the condition that $H^1(\tilde{M})$ is non-trivial, which is satisfied if and only if the Alexander polynomial $\Delta_K(-1) = 0$.

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However, if K only consists of one component, then $\Delta_K(-1) = \pm 1$. Therefore, the branched set of a singular $\mathbb{Z}_2$ harmonic 1-form over S^3 must have at least two components. This is first observed by Haydys on the study of multi-spinor Seiberg-Witten equations.

$\mathbb{Z}_2$ harmonic 1-forms

Given a singular $\mathbb{Z}_2$ harmonic 1-form v, $|v|$ is well-defined over $M \setminus Z$ and $|v|$ will blow-up at order $-\frac{1}{2}$ along the singular set Z, for example $\Re(z^{-\frac{1}{2}}dz)$.

Definition

A $\mathbb{Z}_2$ harmonic 1-form is a singular $\mathbb{Z}_2$ harmonic 1-form v such that $|v|$ extends continously to a Hölder function on M.

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Example

Let Σ be a Riemann surface of genus ≥ 2, given a quadratic differential $q \in H^0(K^2)$, then any square root of q will defines a $\mathbb{Z}_2$ harmonic 1-form. Near a zero of q, we could write $q = z^k dz \otimes dz$, while $v := \Re(z^{\frac{k}{2}} dz)$ is a $\mathbb{Z}_2$ harmonic 1-form with monodromy -1 along odd zeros.

The main motivation to study the $\mathbb{Z}_2$ harmonic 1-form is coming from low-dimensional topology. Since the 1980s, gauge theory, especially SU(2) gauge theory become a very successful tool in the study of the low dimensional topology. In 2009, Witten proposed possible new invariants for 3 and 4-manifolds which is related to the $SL(2, \mathbb{C})$ connections.

The main motivation to study the $\mathbb{Z}_2$ harmonic 1-form is coming from low-dimensional topology. Since the 1980s, gauge theory, especially SU(2) gauge theory become a very successful tool in the study of the low dimensional topology. In 2009, Witten proposed possible new invariants for 3 and 4-manifolds which is related to the $SL(2,\mathbb{C})$ connections.

One of the most important things to know for Witten's program is that we need to understand the behavior of the flat $SL(2,\mathbb{C})$ connections. Note that the moduli space of the flat $SL(2,\mathbb{C})$ connections can be identified with the character variety. As $SL(2,\mathbb{C})$ is a non-compact group, you expect that the moduli space of flat $SL(2,\mathbb{C})$ connection is also non-compact.

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A singular $\mathbb{Z}_2$ harmonic 1-form is purely topological while you might regard a $\mathbb{Z}_2$ harmonic 1-form as a geometry object. Over S^3, from previous examples we see that there exists a huge amount of singular $\mathbb{Z}_2$ harmonic 1-forms. Moreover, v will satisfies the following Weinzenböck identity

$$\Delta|v|^2 + |\nabla v|^2 + \mathrm{Ric}(v, v) = 0.$$

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For a $\mathbb{Z}_2$ harmonic 1-forms, we found that $\int_{S^3} \Delta|v|^2 = 0$, thus v has to be trivial over the sphere with round metric.

Questions

In today's talk, we will mainly focus on the following two questions:

(1) Could you find examples of $\mathbb{Z}_2$ harmonic 1-forms on a 3-manifold?

(2) Is there any obstruction on the existence of $\mathbb{Z}_2$ harmonic 1-forms besides the non-trivial 1st Betti number?

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Question i: Could you find examples of $\mathbb{Z}_2$ harmonic 1-forms on a 3-manifold?
By the work of Taubes, you might think that $\mathbb{Z}_2$ harmonic 1-forms exist widely, but it is actually very hard to construct examples of them. We will introduce an extra $\mathbb{Z}_3$ symmetry to make a singular $\mathbb{Z}_2$ harmonic 1-form Hölder continous. The $\mathbb{Z}_3$ symmetry is first used by Taubes-Wu to construct examples of $\mathbb{Z}_2$ eigenvalues.
Using the extra symmetry, we could find examples of rational homology spheres that exists a $\mathbb{Z}_2$ harmonic 1-forms.

Let $Z \subset M$ be a codimension 2 submanifold and z be the coordinate on the normal bundle with $Z = \{z = 0\}$, then a $\mathbb{Z}_2$ harmonic 1-form could locally have an expansion

$$v \sim \Re(Az^{-\frac{1}{2}}dz + Bz^{\frac{1}{2}}dz) \quad (1)$$

Suppose there exists an $\mathbb{Z}_3$ action on the normal bundle sending $\sigma : z \to e^{\frac{2\pi}{3}}z$ such that $\sigma^\star v = v$, then

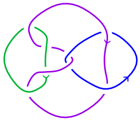
$$z^{-\frac{1}{2}}dz \to \pm e^{\frac{\pi}{3}}z^{-\frac{1}{2}}dz,\ z^{\frac{1}{2}}dz \to \pm z^{\frac{1}{2}}dz$$

Therefore, a $\mathbb{Z}_3$ invariant singular $\mathbb{Z}_2$ harmonic 1-form is actually a $\mathbb{Z}_2$ harmonic 1-form(Hölder continous along Z).

Now, we will give an example which satisfies the extra symmetry. Let M to be a rational homology 3-sphere, L be an oriented link on M, we write M_k to be the k-fold branched covering of M along L. Suppose the Alexander polynomial $\Delta_L(-1) = 0$, then there exists a $\mathbb{Z}_2$ harmonic 1-forms over M_3 w.r.t the pull-back metric.

$$\begin{array}{ccc} M_6 & \xrightarrow{\ p_3\ } & M_3 := M_6/\langle \mathbb{Z}_2 \rangle \\ \downarrow{\scriptstyle p_2} & & \downarrow \\ \alpha \in M_2 := M_6/\langle \mathbb{Z}_3 \rangle & \longrightarrow & M = M_6/\langle \mathbb{Z}_6 \rangle \end{array}$$

One could find examples of links with trivial Alexander polynomial with M_3 be a rational homology 3-sphere. The following link L8n6{0,0} will satisfy the condition. Using connected sum, you could find infinity number of rational homology 3-sphere that admits $\mathbb{Z}_2$ harmonic 1-forms. Moreover, for generic metric, you could make the "B" term of the leading expansion $v \sim \Re(Bz^{\frac{1}{2}}dz)$ nowhere vanishing along Z, which we refer this condition non-degerate.



Theorem

(H.2022) There exists infinity number of rational homology 3-spheres that admit $\mathbb{Z}_2$ harmonic 1-forms.

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The Calabi-Yau manifold

Now, we will explain an application of $\mathbb{Z}_2$ harmonic 1-form to construct deformation of branched immersed special Lagrangians and using recent work of Abouzaid-Imagi to get a non-existence result for $\mathbb{Z}_2$ harmonic 1-forms.

The Calabi-Yau manifold

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Definition

A Calabi-Yau n-fold is a quadruple (X, J, ω, Ω) such that

(i) (X, J, ω) is a n-dimensional Kähler manifold with a Kähler metric g,

(ii) Ω is a nowhere vanishing holomorphic (n,0)-form which satisfies

$$\Omega \wedge \bar{\Omega} = c_n \omega^n,$$

where c_n is a specific constant depends on n.

An Example of Calabi-Yau manifolds

Let $X = \mathbb{C}^n$ with coordinates $(z_1, \cdots, z_n)$, let J be the canonical complex structure,

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i, \ \Omega = dz_1 \wedge \cdots \wedge dz_n,$$

then for the Euclidean metric, $(\mathbb{C}^n, J, \omega, \Omega)$ is a Calabi-Yau n-fold.

The Special Lagrangian Submanifolds

Definition

(Harvey-Lawson)An immersed submanifold $\iota : L \to X$ in a Calabi-Yau (X, J, ω, Ω) is called a special Lagrangian if

(i) $\iota^\star \omega = 0$ (Lagrangian condition),

(ii) $\iota^\star \mathrm{Im}\Omega = 0$ (special condition).

The existence question is a major open problems in general. Some known construction techniques are high symmetrics, gluing constructions and Cartan-Kähler theory.

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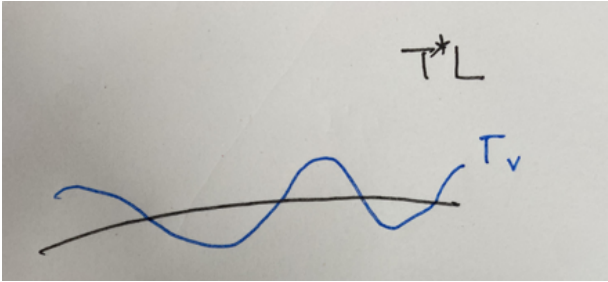
Example

(Bryant, Doice) Let (L, g) be a real analytic Riemannian manifold with $\chi(L) = 0$, then over a neighborhood of the zero section of $T^\star L$, there exists a Calabi-Yau structure with the zero section a special Lagrangian.

McLean's Deformation Theorem

The special Lagrangians have a beautiful local deformation theory due to R.McLean. Let L be a special Lagrangian manifold in a Calabi-Yau, then by the Weinstein neighborhood theorem, a neighborhood of L could be identified with a neighborhood of the zero section in $T^{\star}L$. Therefore, a C^1 deformations of L is given by the graph of a 1-form v on L.



McLean's Deformation Theorem

Theorem

(R.McLean) The C^1 deformation of a special Lagrangian submanifold L is parametrized by the harmonic 1-forms. Especially, suppose $b_1(L) = 0$, then L is rigid.

Sketch of Proof: Over a neighborhood U of the zero section of $T^\star L$, let $\iota_t : L \to U$ be the graph of tv with t a real parameter, then the linearization of the special Lagrangian condition will be

$$\frac{d}{dt}\iota_t^\star \omega = dv, \ \frac{d}{dt}\iota_t^\star \mathrm{Im}\Omega = d \star v. \tag{2}$$

Then by an implict functional argument, harmonic 1-forms parameterized the nearby C^1 special Lagrangians.

Branched Deformation Question

Question

Let L be a special Lagrangian submanifold in a Calabi-Yau (X, J, ω, Ω), does there exist a family of special Lagrangians $\tilde{L}_t$, which are diffeomorphic to a branched covering of L, such that $\tilde{L}_t$ convergence to $2L$?

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(i) Classical problem in minimal surface, multivalued minimal graph.
(ii) Special Lagrangian enumerative invariants.(Joyce)
(iii) First attemps is given by CH.Liu-ST.Yau 11 using gluing arguement and obtain some particial results.

More Comments

The first thing to try is to modified McLean's arguement. Suppose $\iota : L \to X$ is a special Lagrangian, we choose $p : \tilde{L} \to L$ be a branched covering of L, then $\iota \circ p : \tilde{L} \to X$ is a special Lagrangian as

$$(\iota \circ p)^{\star}\omega = (\iota \circ p)^{\star}\Omega = 0,$$

then we apply McLean's theorem to the harmonic 1-forms on $\tilde{L}$, we solve the question.

More Comments

Unfortunately, the above arguement is incorrect. As $\iota \circ p$ is no longer an immersion, the induced metric on $\tilde{L}$ is singular.(cone metric with cone angle 4π)

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Moreover, you don't expect you could use some weighted norm to overcome the singular metric issue. For any weight, you might find the linearization operator $d + d^{\star}$ on 2-valued 1-forms has infinite dimensional cokernel.

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Moreover, you don't expect you could use some weighted norm to overcome the singular metric issue. For any weight, you might find the linearization operator $d + d^{\star}$ on 2-valued 1-forms has infinite dimensional cokernel.
This problem is a free boundary problem which we will explain later. For most of the homology element in the double branched covering, you don't expect to generate a deformation.

An Example of Branched Deformation

Let $\mathbb{C}$ be the complex plane with coordinate z. We identified $T^{\star}\mathbb{C}$ with $\mathbb{C}^2$ and let w be the fiber coordinates.

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Let $v_k = \Re(z^{-\frac{1}{2}+k}dz)$, then the defining equation of the graph of tv_k would be

$$\Gamma_t^k := \{(z,w)|w^2 = t^2 z^{2k-1}\},$$

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which is a family of special Lagrangians.
When $k \geq 1$, we see Γ_t^k convergence to $2\{w=0\}$, and $k=1$, Γ_t^1 is a smooth manifold, which is the graph of the non-degenerate multivalued harmonic 1-form.
When $k=0$, Γ_t^0 is singular.

An Example of Branched Deformations

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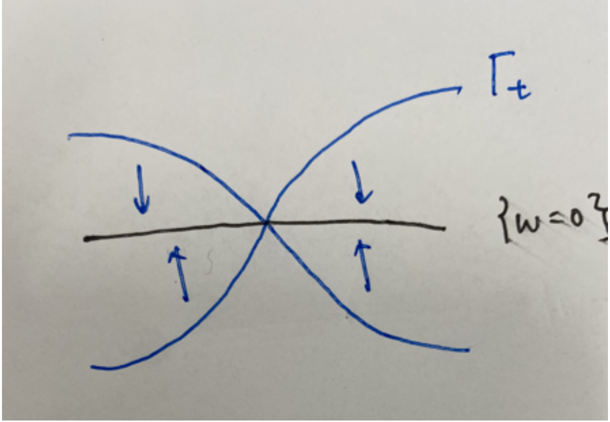

Donaldson's insight

The main idea for this branched deformation is coming from Donaldson, where he studied the deformation problem for $\mathbb{Z}_2$ harmonic 1-forms and the branched deformation problem for sLags might be considered as a non-linear version of Donaldson's theorem.

Donaldson's insight

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Donaldson induced the following idea in his deformation paper: let L be a special Lagrangian submanifold, suppose there exists a nondegenerate multivalued harmonic 1-form on L, does there exist a family of special Lagrangian submanifolds $\tilde{L}_t$, which is diffeomorphic to a branched covering of L, such that $\tilde{L}_t$ convergence to $2L$?

Main Difficulties

There are two main difficulties in solving this problem:
(A) The family $\tilde{L}_t$ has unbounded geometry when $t \to 0$. The Riemannian curvatures, the injective radius all goes to infinity, you need to understand the degenerate behavior.

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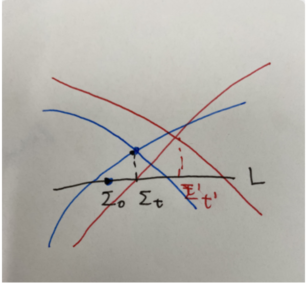
There are two main difficulties in solving this problem:
(A) The family $\tilde{L}_t$ has unbounded geometry when $t \to 0$. The Riemannian curvatures, the injective radius all goes to infinity, you need to understand the degenerate behavior.
(B) The branched deformation problem is a free boundary problem with the branching set itself as a variable that need to be solved. As the deformation theory of multivalued harmonic equation is a free boundary problem, you expect that the deformation problem of special Lagrangians is also a free boundary problem.

Main Difficulties

The previous example in $\mathbb{C}^2$ is in some sense very misleading, as $\mathbb{C}^2$ is hyperKähler. The general picture would be the following:



For different real parameter t, there should be an unique Σ_t and a special Lagrangian L_t which is branching along Σ_t. The problem could be understood as a nonlinear version of Donaldson's deformation theorem.

Branched Deformation Theorem

Theorem

(H. 22') Let L be a special Lagrangian in a Calabi-Yau (X, J, ω, Ω), suppose there exists a multivalued non-degenerate harmonic 1-form v on L, then there exists a family of special Lagrangian submanifold $\tilde{L}_t$ such that

(i) *$\tilde{L}_t$ convergence to $2L$ as currents and as a $\mathcal{C}^{0,\alpha}$ graph, where $0 < \alpha < \frac{1}{2}$.*

(ii) *$\tilde{L}_t$ is close to the graph of tv mod $\mathcal{O}(t^2)$.*

Corollary

Moreover, we find a special Lagrangian with topology rational homology 3-sphere which admits multivalued non-degenerate harmonic 1-form. Therefore, we obtain

Corollary

There exists a special Lagrangian which is rigid in McLean's sense but have branched deformations.

Sketch a Proof

We will explain how could we solve the problem.
Step 1. Using tv, we could construct a family of Lagrangians $\tilde{L}'_t$, even the geometry of $\tilde{L}'_t$ is unbounded, we found that the induced Riemannian metric on $\tilde{L}'_t$ will convergence to a cone metric which gives a uniform lower bound on the first eigenvalue of the Laplacian operator. (The unbounded geometry problem.)

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Step 2, Noting that the problem is a free boundary problem, we move the branching sets of $\tilde{L}'_t$ to make a good approximate solutions $\tilde{L}''_t$ with sufficiently small Lagrangian angle. (The free boundary problem.)

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Step 2, Noting that the problem is a free boundary problem, we move the branching sets of $\tilde{L}'_t$ to make a good approximate solutions $\tilde{L}''_t$ with sufficiently small Lagrangian angle. (The free boundary problem.)
Step 3, We perturb $\tilde{L}''_t$ into a real special Lagrangians using Joyce's nearby special Lagrangian method.

The nearby special Lagrangian theorem

Theorem

(Abouzaid-Imagi 2021) Suppose $\pi_1(L)$ is finite, then all $\mathcal{C}^0$ sufficiently closed unobstructed(in FOOO sense) immersed sLags must be L.

Combing the branched deformation result with the Abouzaid-Imagi's uniqueness theorem, we obtain extra obstruction for the existence of non-degenerate $\mathbb{Z}_2$ harmonic 1-forms.

Suppose (L, g) be a real analytic manifold with $\chi(L) = 0$ with $\pi_1(L)$ finite, then by Bryant, Doice's Calabi-Yau neighborhood theorem, there exists a Calabi-Yau structure in a neighborhood of the zero section of $T^\star L$ such that the zero section is a special Lagrangian. Suppose over L, there exists a non-degenerate $\mathbb{Z}_2$ harmonic 1-forms, then the $\tilde{L}$ we constructed must be obstructed. However, if $b_2(\tilde{L}) = b_2(L)$, then $\tilde{L}$ has to be unobstructed.

Theorem

H. 2022 If there exists a $\mathbb{Z}_2$ harmonic 1-form on L, then $b_2(\tilde{L}) > b_2(L)$.

Applications to $PSL(2,\mathbb{C})$ Gauge Theory

If you could check that every immersed sLag on $T^{\star}L$ is unobstructed, then you could conclude the non-existence of $\mathbb{Z}_2$ harmonic 1-form, which is mission impossible. However, the Calabi-Yau neighborhood has an anti-holomorphic involution by sending $v \to -v$ on $T^{\star}L$ and the sLags we constructed will be preserved under this symmetry. You only need to check the unobstructed condition for the special Lagrangians which is preserved under this extra symmetry. (Soloman, FOOO).

Theorem

(H. 22') Let (L,g) be a real analytic 3-manifold with $\pi_1(L)$ finite, suppose every immersed anti-holomorphic invariant special Lagrangian in the Calabi-Yau neighborhood is unobstructed(in FOOO sense), then there doesn't exist any non-degenerate multivalued harmonic 1-form.

Thank You!

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