Almost complex manifolds with prescribed Betti numbers

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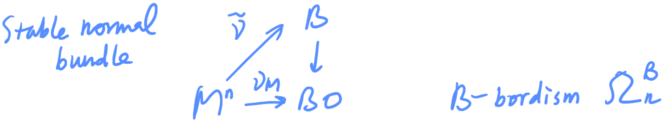
General Question

Prescribing a sequence of nonnegative integers $b_0, b_1, \dots, b_n$, does there exist any closed orientable manifold M^n that realizes b_i as its Betti numbers?

$$b_i = b_i(M^n) = \dim H_i(M^n; \mathbb{Q})$$

★ We restrict to **simply connected closed orientable** manifold, $b_1 = 0$, $b_0 = b_n = 1$.

- Poincaré duality $\Longrightarrow b_i = b_{n-i}$.
- connecting sum $M^n \# S^k \times S^{n-k}$ realizes $b_k + 1, b_{n-k} + 1$.
- realization by B-manifold, $B = BSO, BU, BSU, BSpin, \dots$



$b_0 = b_n = 1$, rational homology sphere

smooth manifolds

- dim $n = 2, 3, 4$, every simply-connected $\mathbb{Q}$ homology sphere is homeomorphic to S^n.
- dim $n \geq 5$, $\exists$ $\mathbb{Q}$ homology sphere $\not\approx S^n$.

  (e.g., $n = 5$, Wu manifold $SU(3)/SO(3)$ with $H_* = \mathbb{Z}, 0, \mathbb{Z}_2, 0, 0, \mathbb{Z}$.)

almost complex manifolds

$J: TM \to TM \quad J^2 = -Id$

[Borel & Serre, 53'] S^n *admits an almost complex structure iff* $n = 2, 6$.

Obstructions: divisibility of Chern number and Hirzebruch signature equation. (work for $S^n_{\mathbb{Q}}$)

- $S^{2k}_{\mathbb{Q}}$, $k > 3$ can not admit almost complex structure.

  Chern character $ch(E^n) = n + \frac{c_n(E)}{(n-1)!}$

  $\langle ch(E), [S^{2n}] \rangle \in \mathbb{Z}$
  $\langle c_n(E), [S^{2n}] \rangle = \chi(S^{2n}) = 2$ $\Big\} \Rightarrow (n-1)! \mid 2 \Rightarrow n \leq 3$, S^2 ~~S^4~~ S^6

  $\mathbb{C}P^1 \quad \mathbb{H}P^1$

- $S^{4k}_{\mathbb{Q}}$ can not admit almost complex structure.

  S^{4k}: Pontryagin class $0 = p_1 = -2c_2$ but $c_2[S^{4k}] = \chi = 2$

  or signature eqn $0 = \sigma = \langle s_k p_k, [S^{4k}] \rangle = 2(-1)^k s_k \cdot \chi \neq 0$ (with $p_k = (-1)^k 2 c_{2k}$)

“simplest" nontrivial case $b_0 = b_{n/2} = b_n = 1$

well-known smooth examples

In dim $n = 4, 8, 16$, complex, quaternionic, Cayley(octonionic) projective planes: $\mathbb{CP}^2$, $\mathbb{HP}^2$, $\mathbb{OP}^2$.

$$M^{2k} = S^k \cup_\phi D^{2k}$$

where $\phi : S^{2k-1} \to S^k$ has Hopf invariant $h(\phi) = 1$. $(\alpha_k \cup \alpha_k = h(\phi)\beta_{2k})$

[**Hopf Invariant One Theorem** (Adams ’60)] *The only such maps ϕ with Hopf invariant one are the Hopf fibrations $S^{2k-1} \to S^k$ where $k = 1, 2, 4$ or 8.*

$\Longrightarrow \nexists$ close manifold M^n in dim $n > 16$ with $H^*(M^n; \mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^3)$.

rational examples in higher dimensions?

Question. In dim $n > 16$, does there exist closed manifold M^n with Betti numbers $b_0 = b_{n/2} = b_n = 1$, otherwise $b_i = 0$?

$n = 4k+2$ skew symmetric $b_{n/2}$ even

- intersection form $\Rightarrow$ dim of such M must be $n = 4k$.
- the rational cohomology ring is $H^*(M^{4k}; \mathbb{Q}) = \mathbb{Q}[\alpha]/(\alpha^3)$, $|\alpha| = 2k$.

rational projective planes ($b_0 = b_{n/2} = b_n = 1$)

A **smooth** manifold M^{4k} with $H^*(M;\mathbb{Q}) = \mathbb{Q}[\alpha]/(\alpha^3)$ will be called a $\mathbb{Q}$P.P.

Theorem [Su (Thesis) '09, Su '14]

- $\exists$ $\mathbb{Q}$P.P. in dim $n = 32$.
- $\nexists$ $\mathbb{Q}$P.P. in dim $n = 4k$ with k odd, or between dim $16 < n < 32$.

Theorem [Fowler-Su '16]

$\mathbb{Q}[x]/\langle x\rangle^m$

- If $\exists$ $\mathbb{Q}$P.P. in dim n, n must be $8(2^a)$, or $8(2^a + 2^b)$ with $a < b$.
- $\nexists$ $\mathbb{Q}$P.P. between dim $32 < n < 128$ or $128 < n < 256$.

Theorem [Kennard-Su '18]

- $\exists$ $\mathbb{Q}$P.P. in dim $16 < n \leq 512$ iff $n \in \{32, 128, 256\}$.
- $\nexists$ $\mathbb{Q}$P.P. between dim $512 < n < 2^{13}$ except five possible exceptions.
- Nonexistence in ∞ many dim of the form 2^a.

Kreck and Zagier, communicated '17-'18

With computer program, they can show nonexistence between dim $256 < n < 8(10^4)$ except 13 exceptions. The likelihood of existence besides dim 32, 128, 256 is "slim".

(Non-existence) obstruction from:

1. Hirzebruch signature Theorem. $\langle \mathcal{L}_k(p(\tau_M)), [M] \rangle = \sigma(M^{4k})$
2. Integrality of the Pontryagin numbers of $4k$-dim smooth manifold.

(Existence) Sullivan-Barge rational surgery realization theorem:

The above necessary conditions are also sufficient for the realization.

Proof of nonexistence

What are some “obvious" necessary conditions of a realization?

If M^{4k} is a smooth manifold with $H^*(M^{4k};\mathbb{Q}) = \mathbb{Q}[\alpha]/(\alpha^3)$,
The only possibly nontrivial Pontryagin classes:

$$p_{k/2} \in H^{2k}(M;\mathbb{Q}) \quad \text{and} \quad p_k \in H^{4k}(M;\mathbb{Q})$$

The only possible nontrivial Pontryagin numbers are:

$$\langle p_{k/2}^2(\tau_M), [M] \rangle = x^2 \quad \text{and} \quad \langle p_k(\tau_M), [M] \rangle = y, \qquad x, y \in \mathbb{Z}$$

1. The **Hirzebruch signature formula**:

$$\langle \mathcal{L}_k(p), [M] \rangle = s_{\frac{k}{2},\frac{k}{2}}\, x^2 + s_k\, y = 1$$

Example. In dim 8,

$$-\frac{1}{45}x^2 + \frac{7}{45}y = 1$$

Proof of nonexistence (continued)

$$\langle \mathcal{L}_k(p), [M] \rangle = s_{\frac{k}{2},\frac{k}{2}}\, x^2 + s_k\, y = 1$$

the coefficient

$$s_k = \frac{2^{2k}(2^{2k-1}-1)|B_{2k}|}{(2k)!}, \quad s_{k,k} = \frac{1}{2}(s_k^2 - s_{2k})$$

B_{2k} is the even Bernoulli sequence $B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, \ldots.$

If $n = 4k$ with k odd, the signature equation becomes

$$s_k y = 1$$

by some number theoretic property of the Bernoulli numbers
$\Rightarrow$ 2-adic order $\nu_2(s_k) > 0 \Rightarrow$ no solution $\Rightarrow$ dim must be $n = 8k$.

• dim $n = 24$.
The quadratic residue problem has no solution.

2. Integrality of the Pontryagin numbers of $4k$-dim smooth manifold.

Hattori-Stong (smooth case): $B_*^{SO} = \mathrm{Im}\,(\tau : \Omega_*^{SO}/\mathrm{Tor} \to H_*(BSO;\mathbb{Q}))$ consists of classes a that satisfy the integrality relations:

$$\begin{array}{rcl} \langle p_I(\gamma), a\rangle & \in & \mathbb{Z} \\ \langle ph(KO(BSO)) \cdot \mathcal{L}, a\rangle & \in & \mathbb{Z}[1/2] \end{array}$$

which gives all the integrality relations characterizing the Pontryagin numbers of smooth manifolds:

$$\begin{array}{rcl} \langle p_I, \mu\rangle & \in & \mathbb{Z} \\ \langle \mathbb{Z}[e_1, e_2, \ldots] \cdot \mathcal{L}(p), \mu\rangle & \in & \mathbb{Z}[1/2] \end{array}$$

(e_j is the j-th elementary symmetric functions of the variables $e^{x_i} + e^{-x_i} - 2$ such that the total Pontryagin class is formally written as $p = \prod_i (1 + x_i^2)$.)

Example. In dim 8, these relations are equivalent to

$$\begin{cases} x^2, y \in \mathbb{Z} \\ -\frac{1}{45}x^2 + \frac{7}{45}y \in \mathbb{Z}[1/2] \end{cases}$$

⋆ In higher dimensions, solution to the signature equation does not necessarily satisfy the Hattori-Stong integrality conditions.

Proof of existence

These necessary conditions are also sufficient for the realization.

Rational surgery realization theorem (Barge, Sullivan '76)

Given a Sullivan minimal model X under the prescribed $\mathbb{Q}$ cohomology ring which is 1-connected and satisfies Poincaré duality, there exists a $4k$-dim orientable smooth closed manifold realizing the algebraic data iff

There exist $p_1, \ldots, p_k \in \prod H^{4i}(X; \mathbb{Q})$ and $\mu \in H_n(X; \mathbb{Q})$ that satisfy:

(i). Hirzebruch signature Theorem. $\langle \mathcal{L}_k(p), \mu \rangle = \sigma(X)$

(ii) The intersection form is equivalent to $\oplus\langle \pm 1 \rangle$ over $\mathbb{Q}$.

(iii) Integrality conditions characterizing the Pontryagin numbers of $4k$-dim smooth manifold.

Example. In dim 8, any integer solution x^2, y to the signature equation

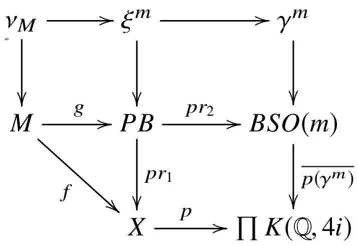
$$-\frac{1}{45}x^2 + \frac{7}{45}y = 1$$

corresponds to the Pontryagin numbers of a realizing manifold.

Input data:

- $p_1, \ldots, p_k \in \prod H^{4i}(X;\mathbb{Q})$, i.e., a map $p : X \to BSO_0 \simeq \prod K(\mathbb{Q}, 4i)$.
- a choice of fundamental class $\mu \in H_n(X;\mathbb{Q})$.

In the degree 1 normal map,



condition (iii) guarantees that there exist a "correct" class in $\pi_{n+m}(T\xi^m)$ to perform Thom-Pontryagin construction, so that the fundamental class of M is mapped to μ.

Question. ∃? almost complex $\mathbb{Q}$P.P. besides the rational $\mathbb{CP}^2$'s?

Fact. Rational $\mathbb{HP}^2$ does not admit any almost complex structure.

(Hirzebruch): an 8-dim almost complex manifold with $b_2 = 0$ must have Euler characteristic divisible by 6. $(c_1 c_3 + 2c_4)[M^8] \equiv 0 \mod 12$, $c_4[M^8] = \chi$.

[Albanese-Milivojević, '18-'19] The dim of any almost complex manifold with sum of Betti numbers three must be a power of 2. Smooth mfld $n = 8(2^a), 8(2^a + 2^b)$

proof. obstruction from signature equation using $p_k = 2(-1)^k c_{2k}$, $p_{2k} = c_{2k}^2 + 2c_{4k}$, and $c_{4k}[M] = \chi = 3$

(Su, announced 18') In dim $n > 4$, there does not exist any almost complex $\mathbb{Q}$P.P.

proof. obstruction from the signature equation and the integrality condition that $\frac{c_{2k}^2[M^{8k}]}{[(2k-1)!]^2} \in \mathbb{Z}$.

(Hu Jiahao, arXiv.2108.06067) If M is a closed almost complex manifold with sum of Betti numbers three, then $\dim M = 4$ and M is complex cobordant to $\mathbb{CP}^2$.

proof. obstruction from the signature equation and the integrality of Todd genus.

Rational surgery realization theorem for almost complex manifolds

[Su,arXiv:2204.04800], [Milivojević, thesis, '21]

Let $H^* = \bigoplus_{i=0}^{n} H^i$ be a 1–connected $\mathbb{Q}$ Poincaré duality algebra of dim $n = 4k$ with $k > 1$.
There exists a simply-connected, closed, almost complex manifold M^n such that
$H^*(M;\mathbb{Q}) \cong H^*$ iff
There exist $c = 1 + c_1 + \cdots + c_{2k} \in \bigoplus_{i=0}^{2k} H^{2i}$ and $\mu \in H_0 = (H^n)^* \cong \mathbb{Q}$ that satisfy:

(i) The intersection form (H^{2k}, λ_μ) is isomorphic to $a\langle +1\rangle \oplus b\langle -1\rangle$ over $\mathbb{Q}$.

(ii) Hirzebruch signature equation. $\langle L_k(p_1(c), \ldots p_k(c)), \mu\rangle = \sigma(H^{2k}, \lambda_\mu)$, where
$p_i(c) = (-1)^i \sum_{j=0}^{2i} (-1)^j c_j c_{2i-j}$.

(iii) Riemann-Roch integrality conditions among Chern numbers of $4k$-dim almost complex manifolds. $\langle \mathbb{Z}[e_1^c, e_2^c, \ldots]\cdot\mathrm{Td}, \mu\rangle \in \mathbb{Z}$ (Stong, $\mathrm{Im}(\tau : \Omega_n^U \to H_n(BU;\mathbb{Q}))$

(iv) If $n \equiv 4 \pmod 8$,
case 1. If $c_1 \neq 0$, no additional condition.
case 2. If $c_1 = 0$, integrality conditions among Chern numbers of $4k$-dim SU manifolds. $\langle \mathbb{Z}[e_1^{p(c)}, e_2^{p(c)}, \ldots]\cdot\hat{A}, \mu\rangle \in 2\mathbb{Z}$ (Stong, $\mathrm{Im}(\tau : \Omega_n^{SU} \to H_n(BSU;\mathbb{Q}))$

(v) The number $\langle c_{2k}, \mu\rangle = \chi(H^*) = \sum_{i=0}^{n} (-1)^i \dim(H^i)$.

almost complex manifolds $b_{n/2} \geq 1, b_0 = b_n = 1$

[Su,arXiv:2204.04800]

Theorem

An $n = 4k(k > 1)$–dimensional closed almost complex manifold with Betti number $b_i = 0$ except $b_0 = b_n = 1$,$b_{n/2} \geq 1$ must have even signature σ and even Euler characteristic χ, i.e., the middle Betti number $b_{n/2}$ must be even.

proof. The only nonzero Chern numbers are $x = c_k^2[M]$ and $\chi = c_{2k}[M]$

- When $n = 8k$,

$$\begin{cases} L_{2k}[M] = (2s_k^2 - s_{2k})x + 2s_{2k}\chi = \sigma \ \text{ with } \ s_k = \frac{2^{2k}(2^{2k-1}-1)|B_{2k}|}{(2k)!} \\ \mathrm{Td}[M] = \frac{1}{2}(t_{2k}^2 - t_{4k})x + t_{4k}\chi \in \mathbb{Z} \ \text{ with } \ t_k = \frac{B_k}{k!} \\ e_1^c \cdot \mathrm{Td}[M] = \left[\frac{-t_{2k}}{(2k-1)!} + \frac{1}{2(4k-1)!}\right]x - \frac{\chi}{(4k-1)!} \in \mathbb{Z} \\ e_1^c e_1^c \cdot \mathrm{Td}[M] = \frac{x}{[(2k-1)!]^2} \in \delta_k\mathbb{Z} \ \text{ with } \delta_k = \begin{cases} 1 & k=1 \\ 2 & k>1 \end{cases} \end{cases}$$

$$\nu_2(\sigma) \geq 4k - 2\nu_2(k) - 3,$$
$$\nu_2(\chi) \geq 4k - 2\nu_2(k) - 2\mathrm{wt}(k) - 2.$$

For example, in dim $n = 8$, $\sigma \equiv 0 \bmod 2$,
in dim $n = 16$, $\sigma \equiv 0 \bmod 2^3$ and $\chi \equiv 0 \bmod 2^2$,
in dim $n = 24$, $\sigma \equiv 0 \bmod 2^9$ and $\chi \equiv 0 \bmod 2^6$.

Existence Results

Let (H^*, σ, χ) denote a $n = 4k$–dim rational cohomology ring with $b_i = 0$ except $b_{n/2} \geq 1, b_0 = b_n = 1$, signature σ, and Euler characteristic χ. Does there exist any closed almost complex manifold realizing (H^*, σ, χ)?

- (intersection form) need to assume the intersection form (H^{2k}, λ_μ) is isomorphic to $a\langle 1\rangle \oplus b\langle -1\rangle$ with $a, b \geq 0$ for some fundamental class $\mu \in H_0 \cong \mathbb{Q}$. prescribe $a = \frac{\chi+\sigma-2}{2} \geq 0, b = \frac{\chi-\sigma-2}{2} \geq 0$.

proposition

In dim 8, (H^*, σ, χ) is realizable if and only if $\sigma \equiv 0 \mod 2$, $\chi \equiv 0 \mod 6$, $3\sigma - \chi \equiv 0 \mod 48$, and $a = \frac{\chi+\sigma-2}{2} > 0, b = \frac{\chi-\sigma-2}{2} > 0$.

$$\begin{cases} \langle L_2, \mu\rangle = \frac{1}{45}(3x + 14\chi) = \sigma \\ \langle \mathrm{Td}, \mu\rangle = \frac{1}{720}(3x - \chi) \in \mathbb{Z} \\ \langle e_1^c \cdot \mathrm{Td}, \mu\rangle = -\frac{1}{6}\chi \in \mathbb{Z} \\ \langle e_1^c e_1^c \cdot \mathrm{Td}, \mu\rangle = x \in \mathbb{Z} \end{cases} \Longleftrightarrow \begin{cases} 3x + 14\chi = 45\sigma \\ 3x - \chi = 720m \text{ for } m \in \mathbb{Z} \\ \chi = 6s \text{ for } s \in \mathbb{Z}^+ \end{cases}.$$

$\#_a \mathbb{HP}^2 \#_b \overline{\mathbb{HP}^2}$

$\chi = 6, \sigma = 2; \quad (a, b) = (3, 1);$

$\chi = 12, \sigma = 4; \quad (a, b) = (7, 3);$

$\chi = 18, \sigma = -10, 6; \quad (a, b) = (3, 13), (11, 5);$

$\chi = 24, \sigma = -8, 8; \quad (a, b) = (7, 15), (15, 7);$

$\chi = 30, \sigma = -22, -6, 10, 26; \quad (a, b) = (3, 25), (11, 17), (19, 9), (27, 1);$