Hyperbolic surfaces as singular flat surfaces

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Topology seminar
Tsinghua University, June 2022

Part I

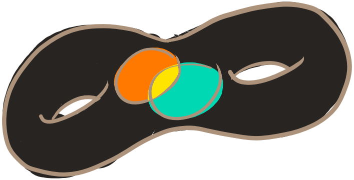
Geometry

Hyperbolic surface

Modeled on hyperbolic plane,
with isometries as symmetries.

Uniform negative curvature.

hyperbolic
plane



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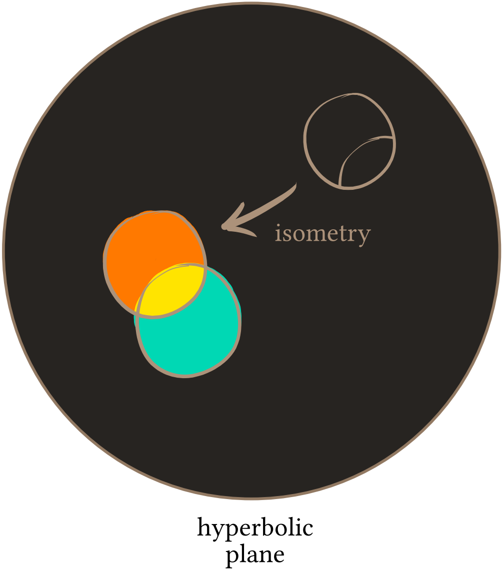
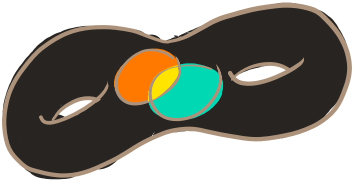
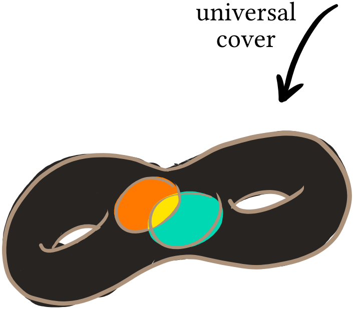


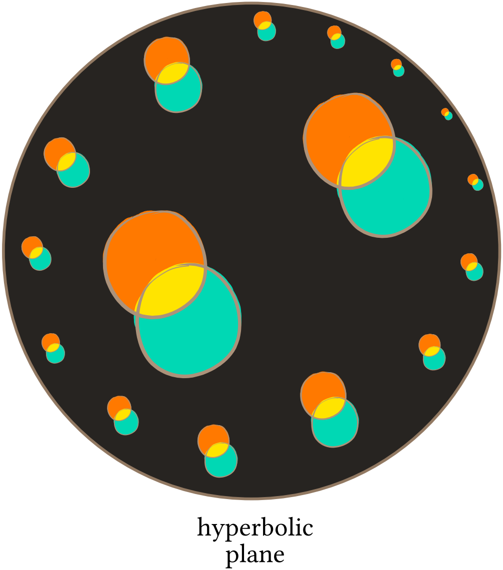


Hyperbolic surface

Universal cover is isometric to hyperbolic plane.

Convenient for visualization.

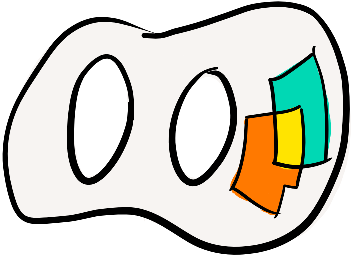


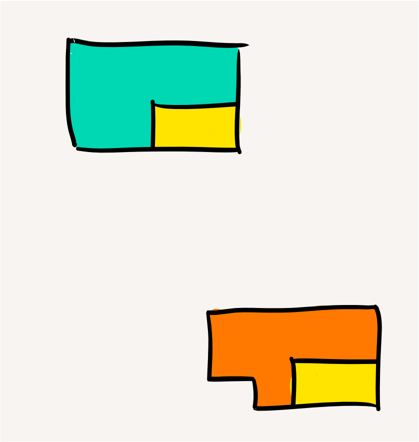


Half-translation surface

Modeled on the euclidean plane, with translations and 180° flips as symmetries.

Curvature concentrated at conical singularities.





euclidean plane

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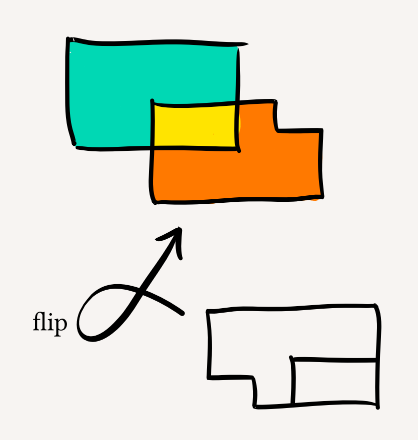
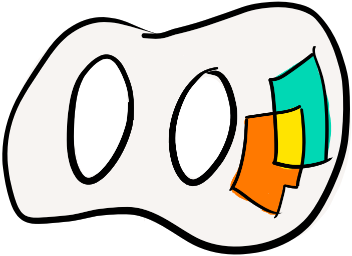


euclidean plane

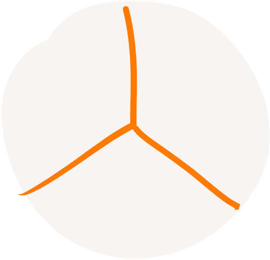


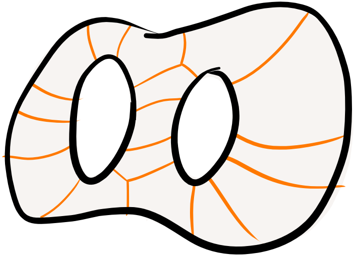
Half-translation surface

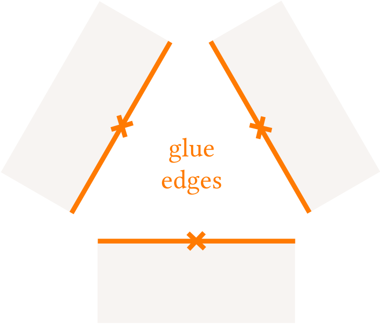
We'll only use the simplest kind of conical singularity.

It looks like three half-planes glued along their edges.

The angle around it is 3π.



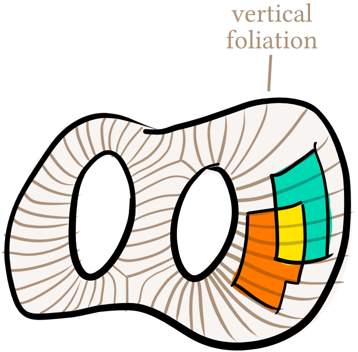


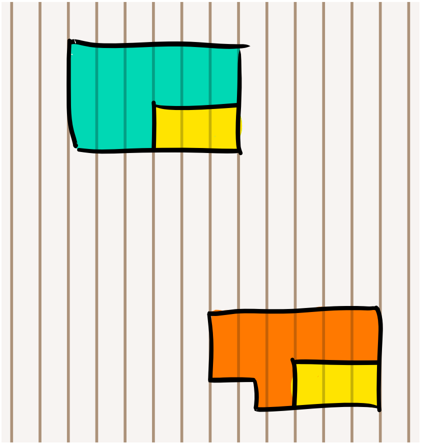


Half-translation surface
with its vertical foliation

The foliations of the charts by vertical lines fit together into a foliation of the surface.

Horizontal distance gives a local measure on swaths of leaves.



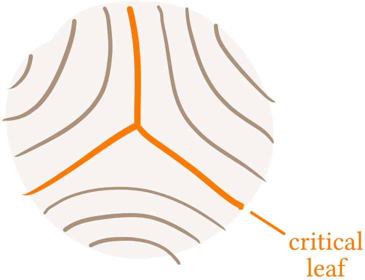


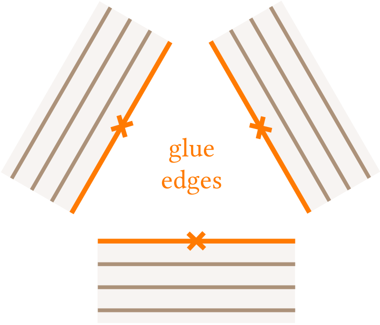
euclidean plane

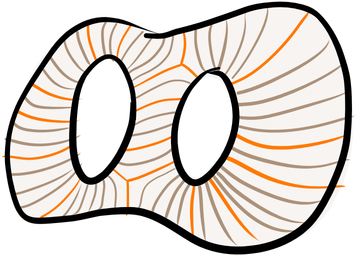
Half-translation surface
with its vertical foliation

At a conical singularity, three vertical leaves meet.

The vertical leaves that hit singularities are called *critical*.



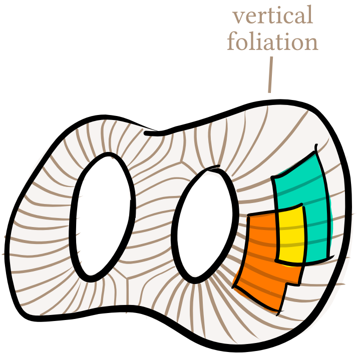


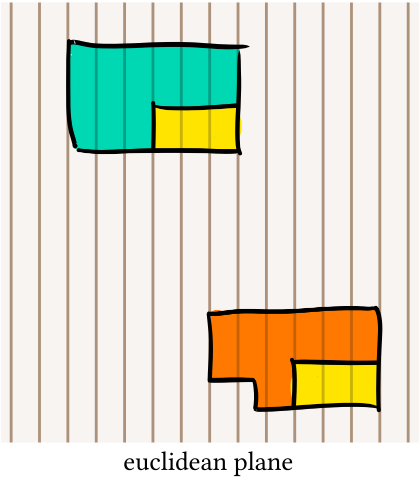


Half-translation surface
with its vertical foliation

The vertical foliation makes half-translation surfaces different from hyperbolic surfaces.

It also hints at a similarity.

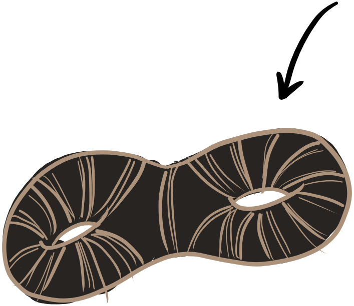




Hyperbolic surface
with a geodesic lamination

The closest thing to a geodesic foliation is a maximal set of non-intersecting geodesics.

Can give it a measure, which assigns a “thickness” to each swath of leaves.



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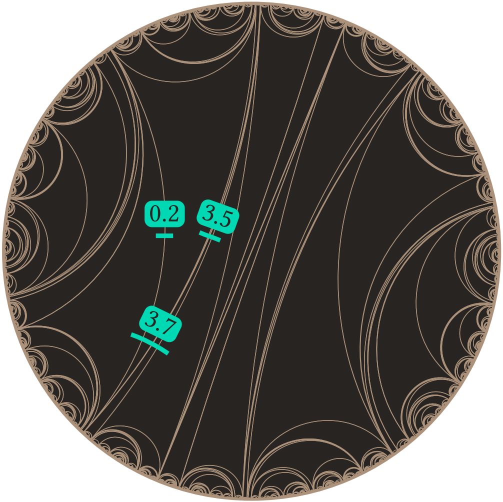
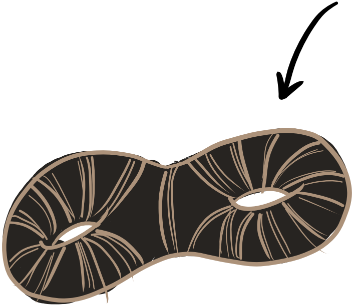
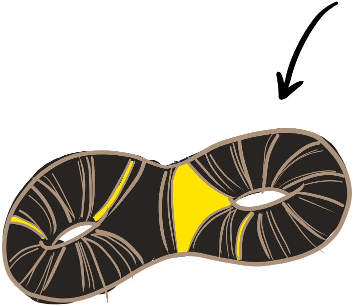




Hyperbolic surface
with a geodesic lamination

Its complement is a finite set of ideal triangles.



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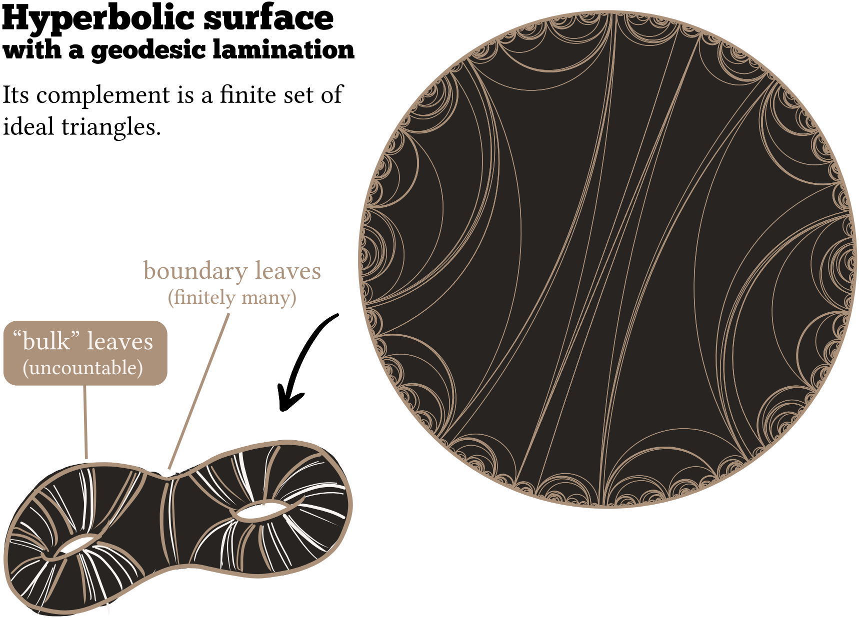
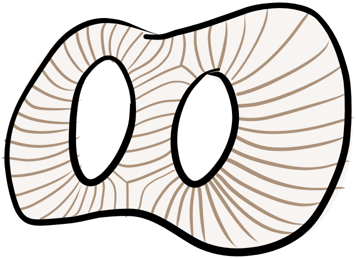


Analogy



hyperbolic surface

Chosen maximal geodesic lamination

Chosen measure

half-translation surface

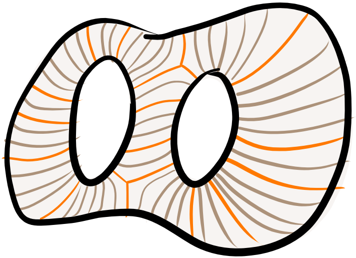
Vertical foliation

Horizontal distance measure

Analogy

hyperbolic surface

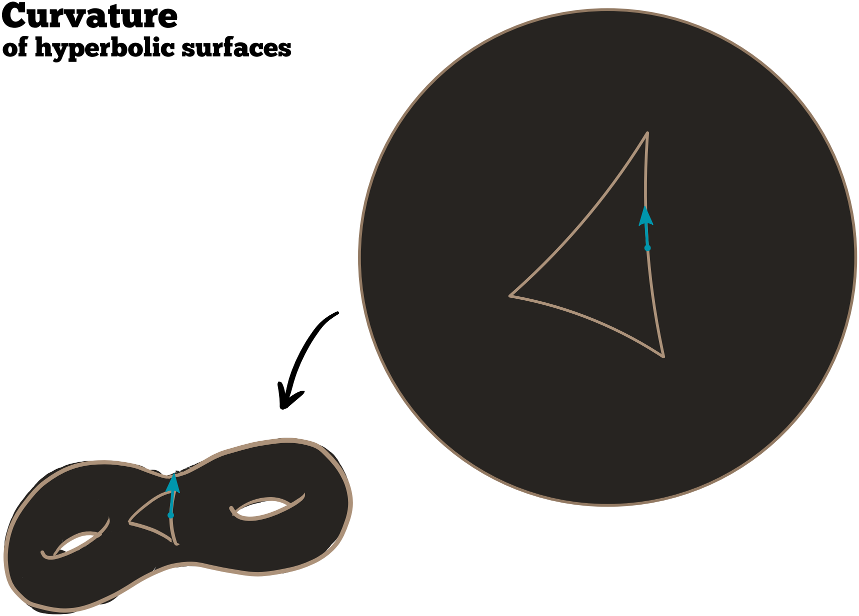
Chosen maximal geodesic lamination
- Chosen measure
- Boundary leaves
- Bulk leaves



half-translation surface

Vertical foliation
- Horizontal distance measure
- Critical leaves
- Non-critical leaves

Curvature
of hyperbolic surfaces



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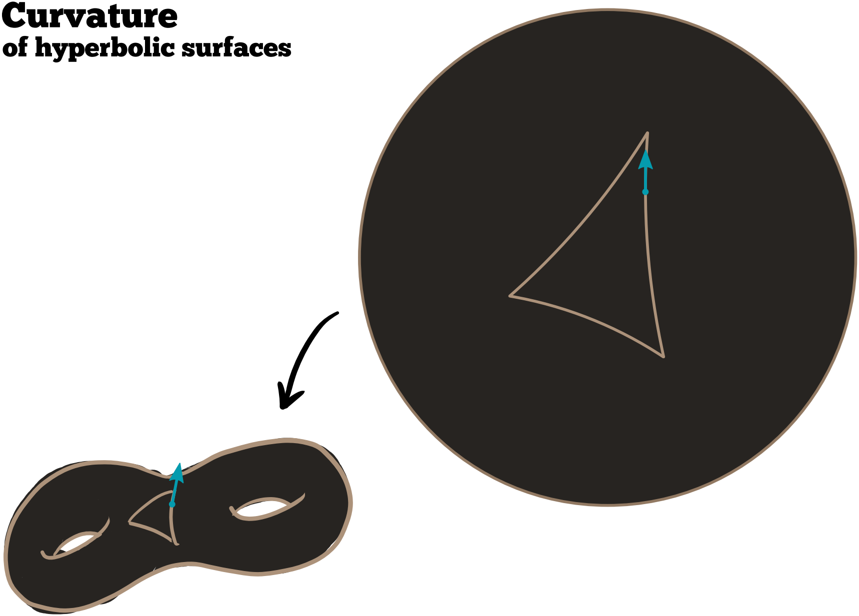

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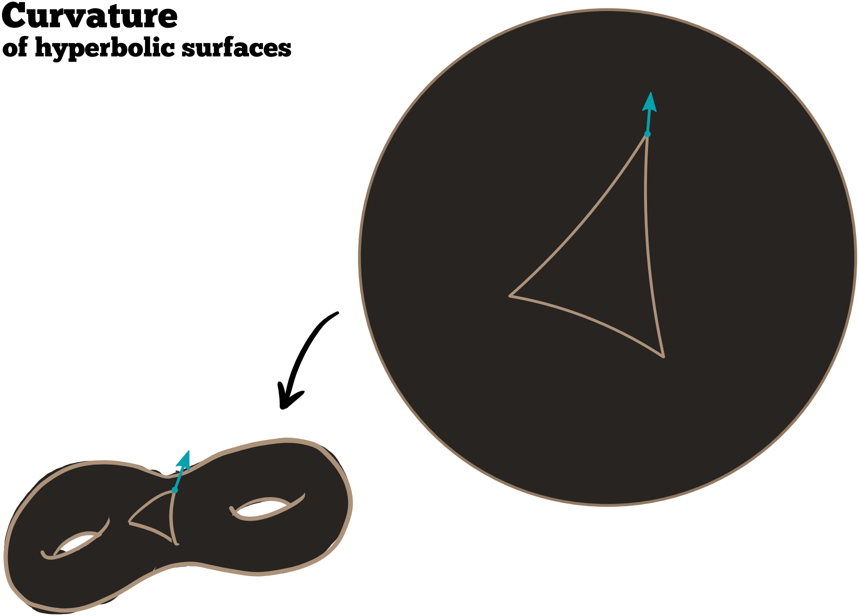

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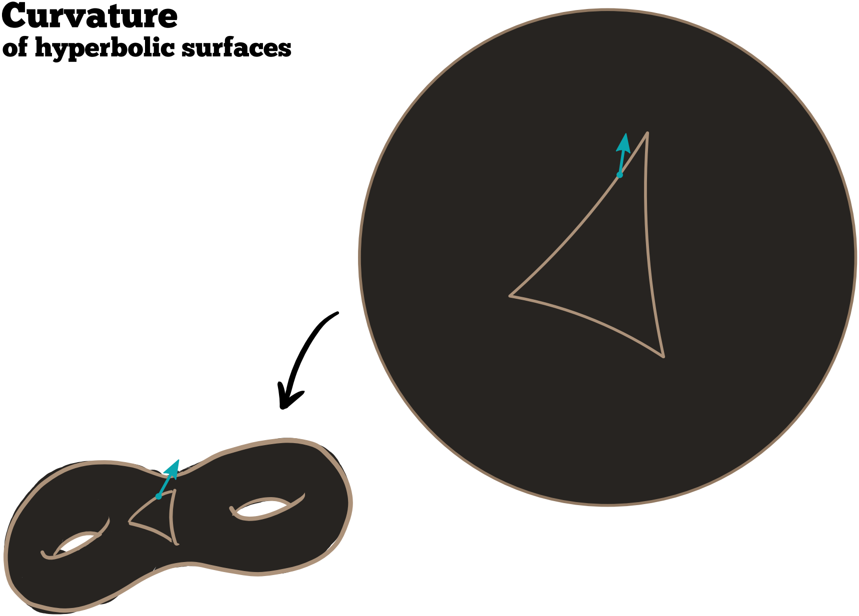

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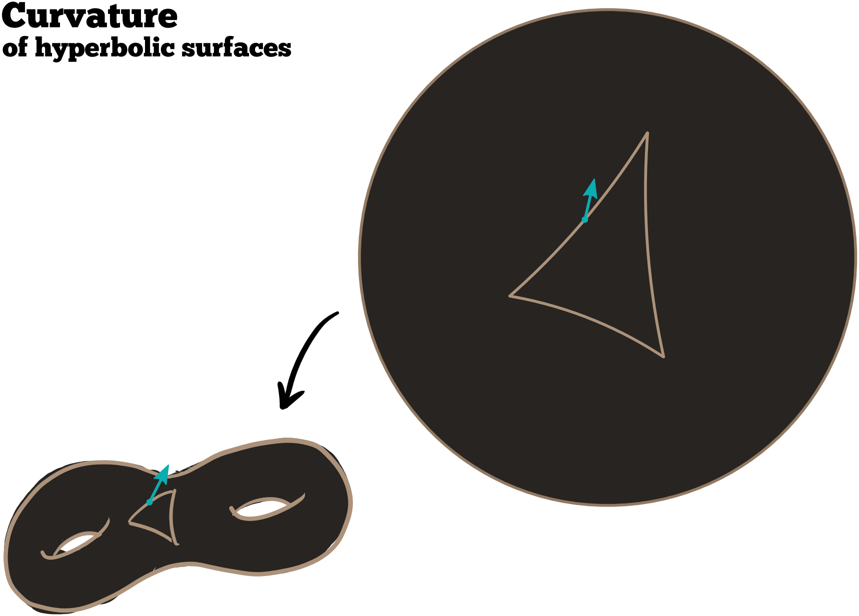

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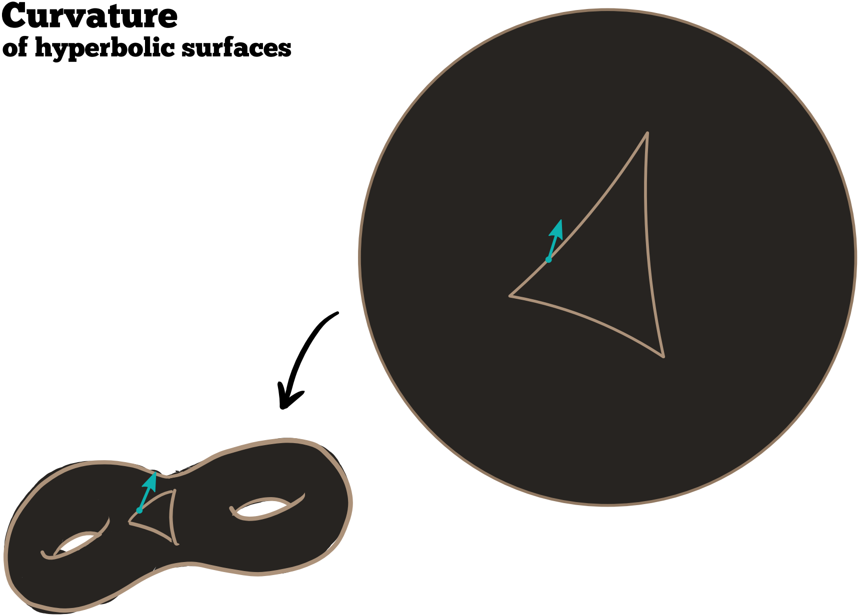

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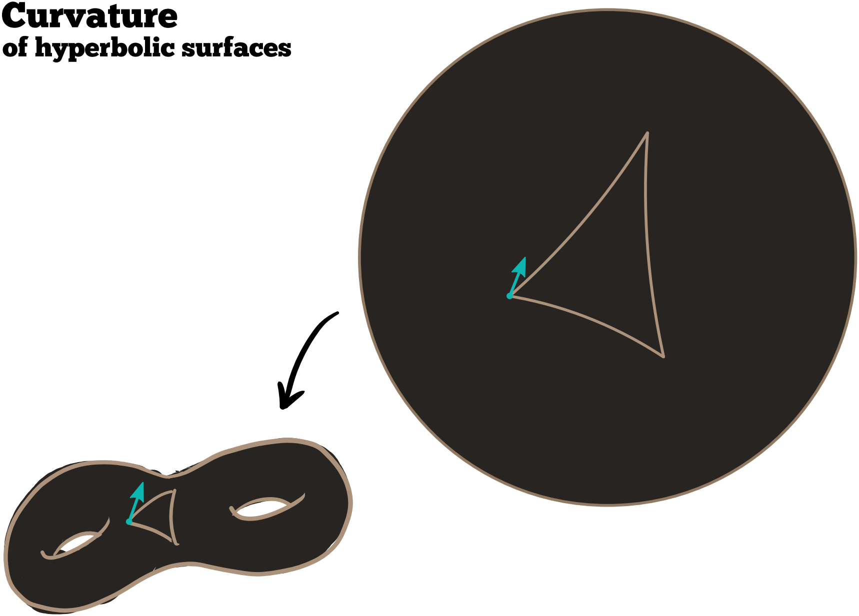

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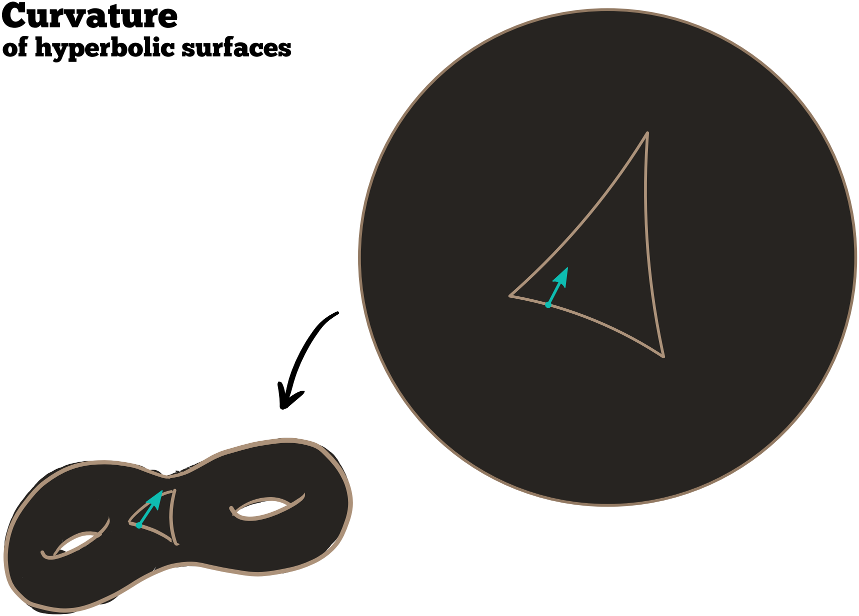

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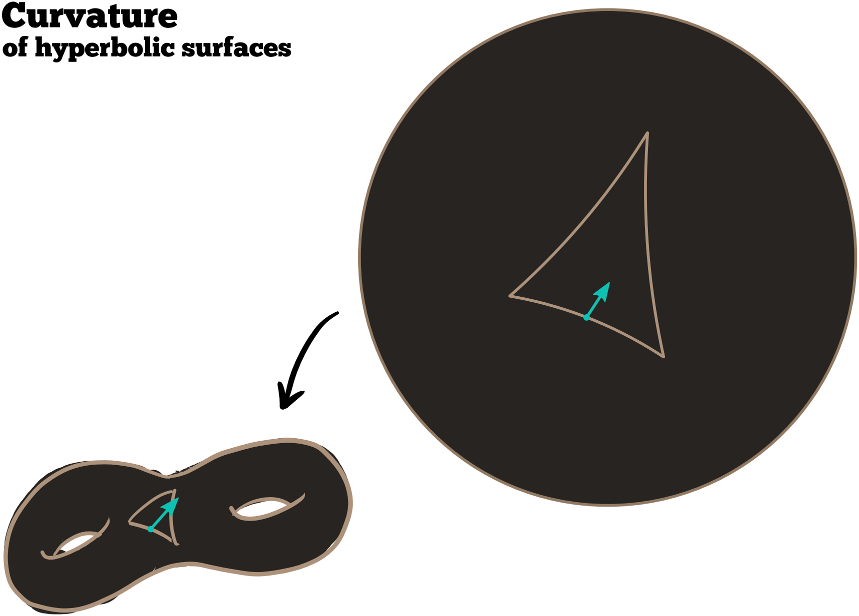

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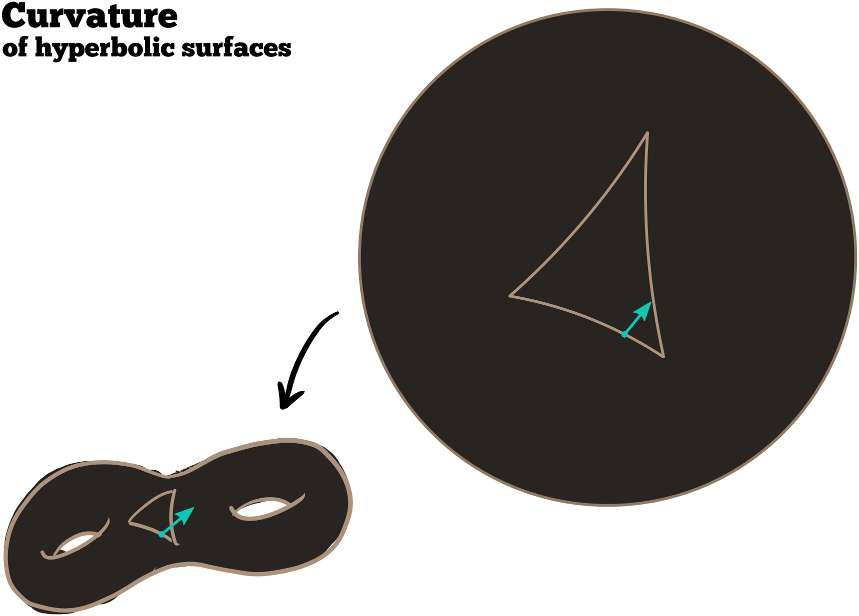

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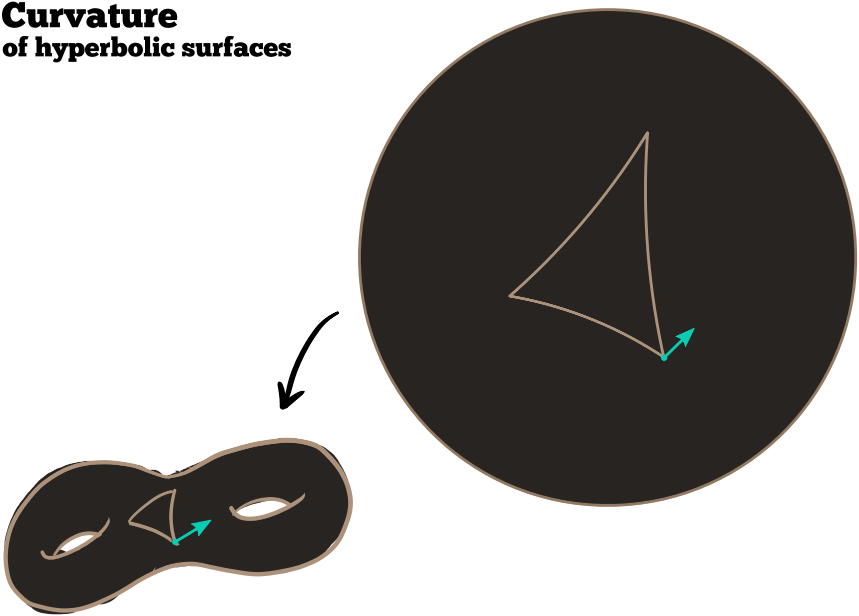

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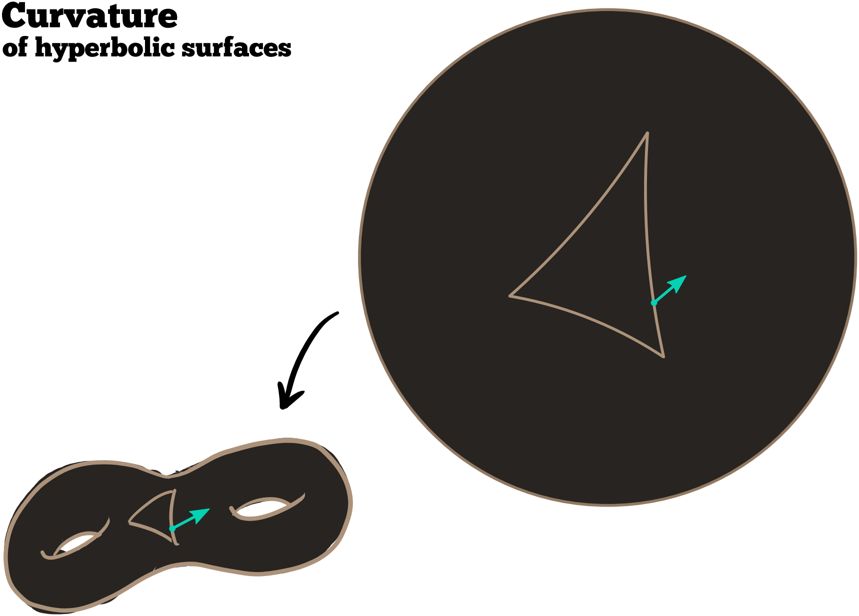

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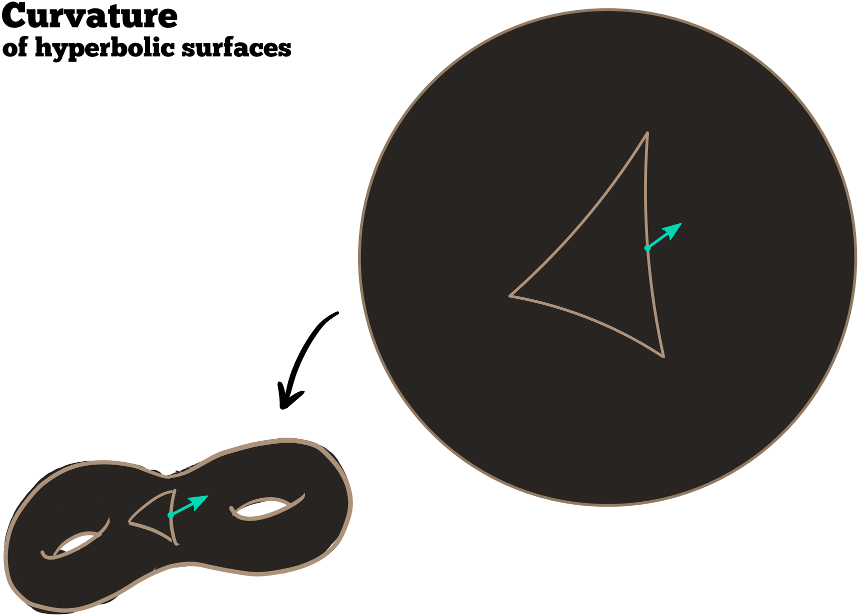

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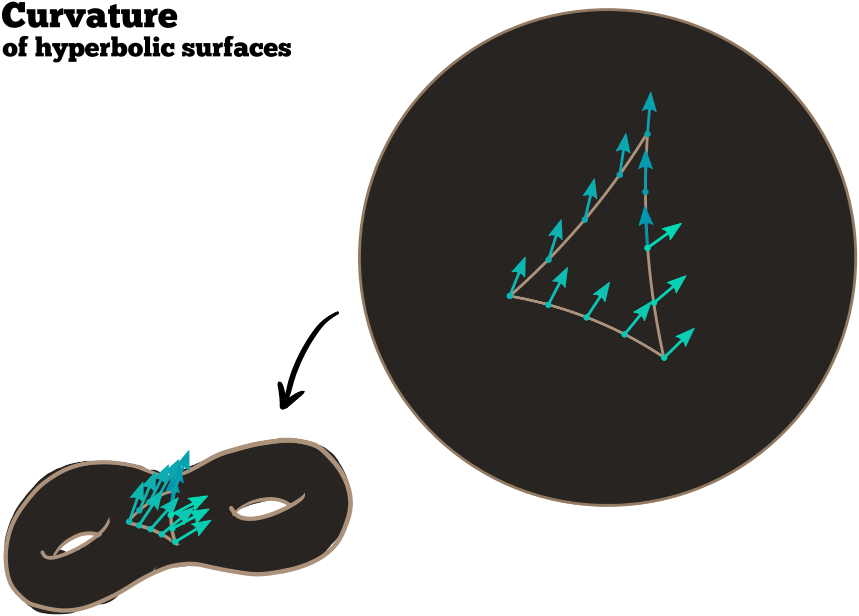

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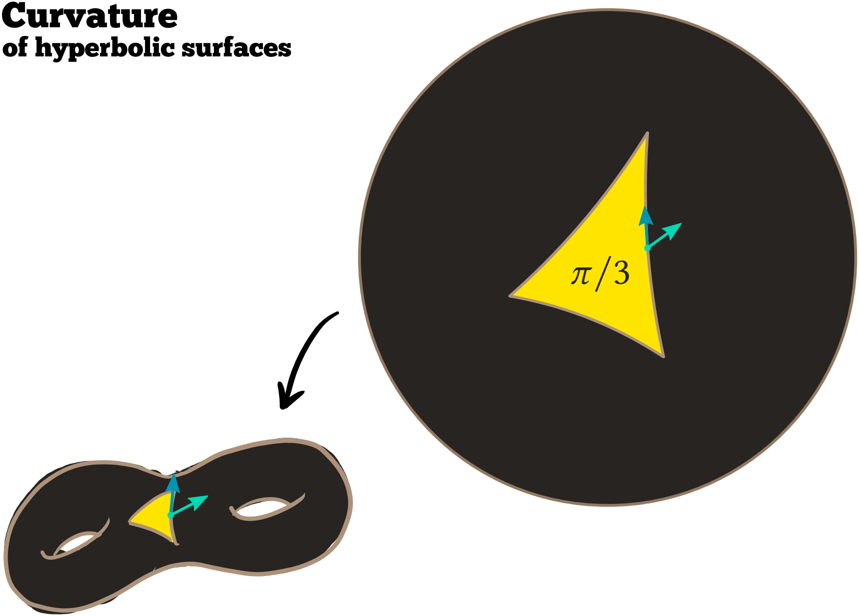

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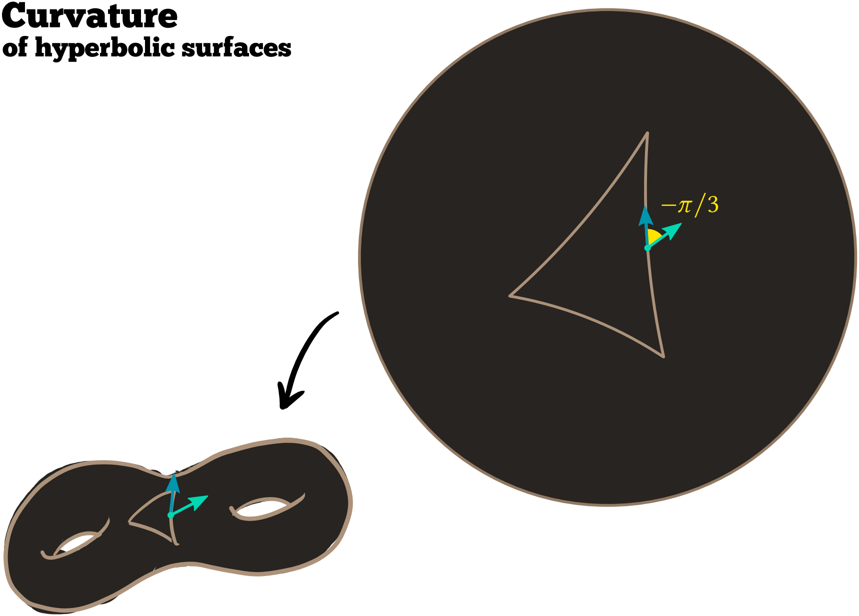

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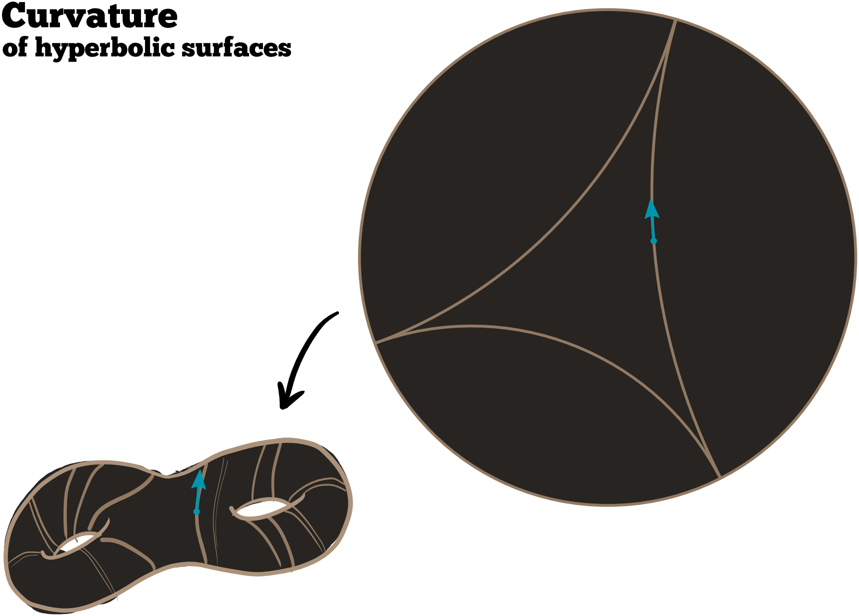

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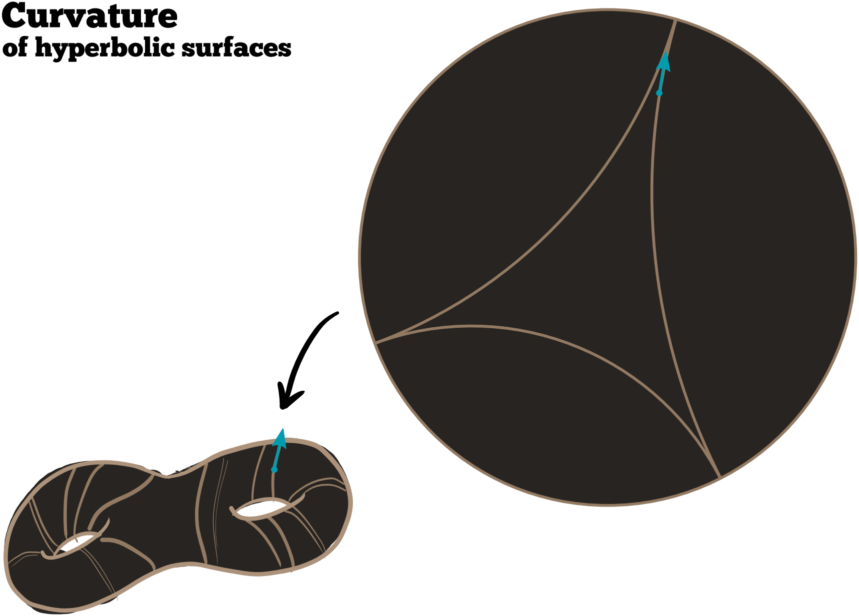

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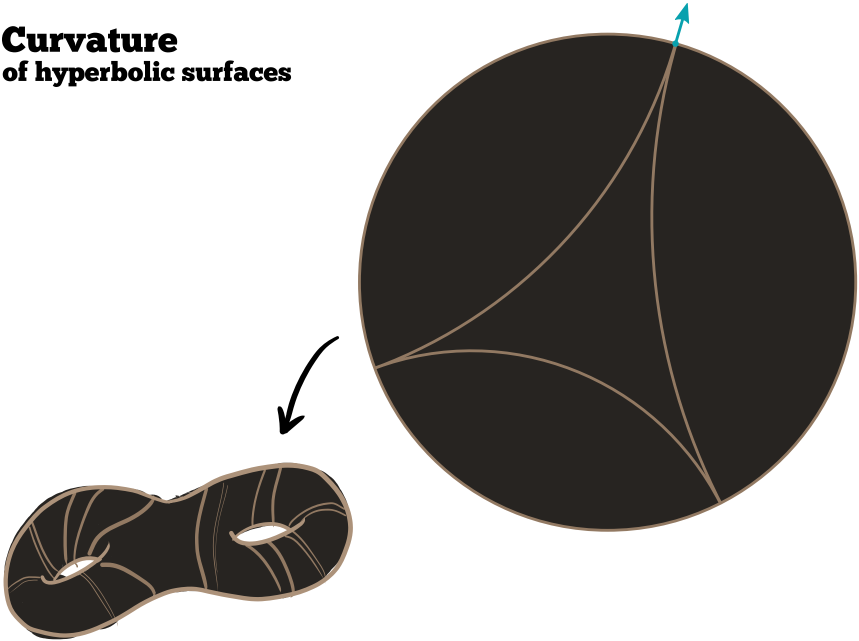

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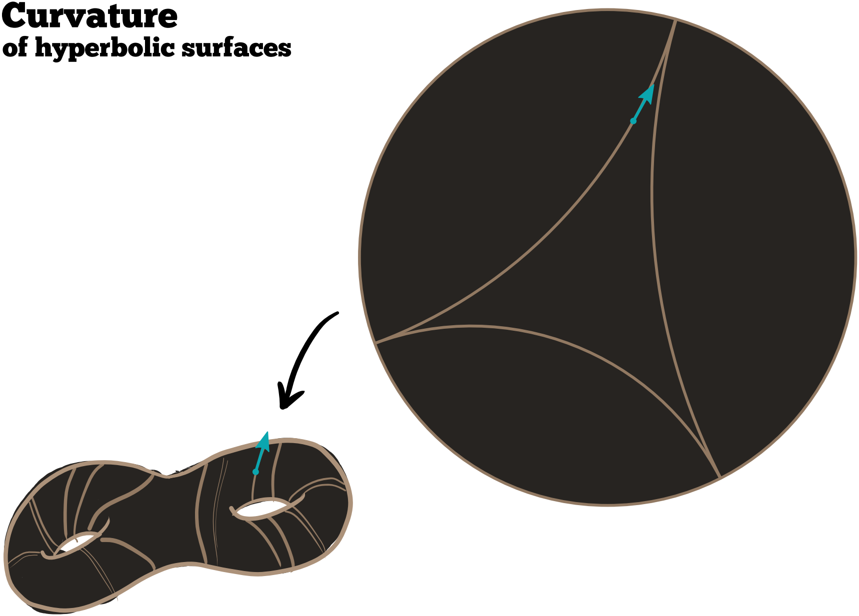

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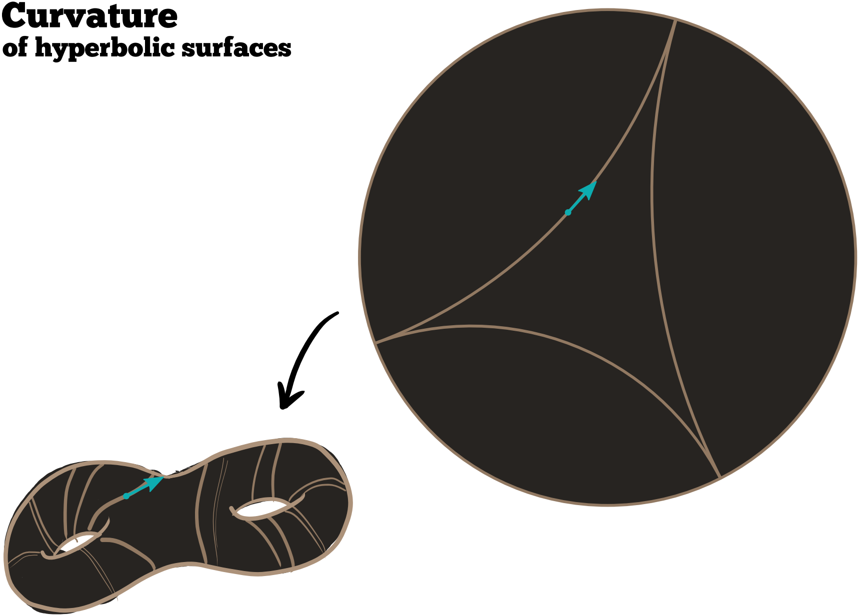

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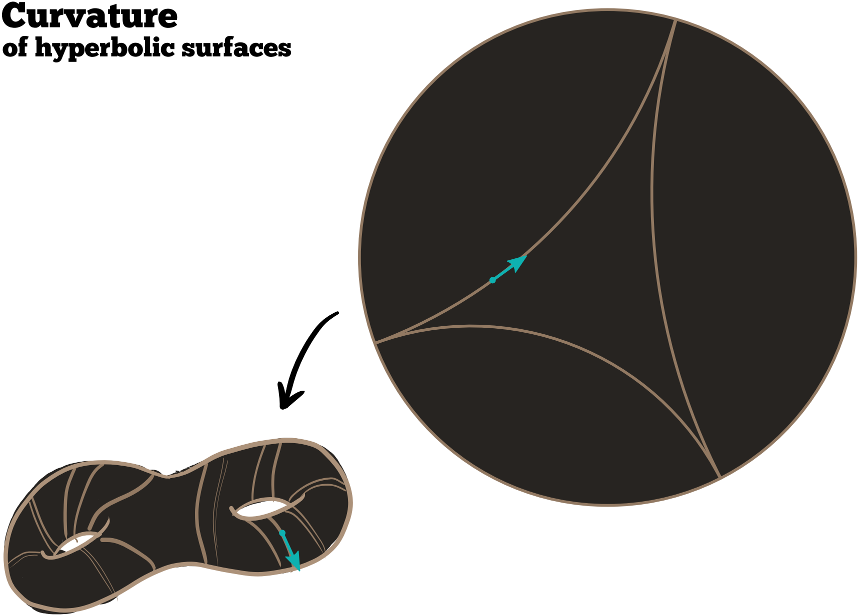

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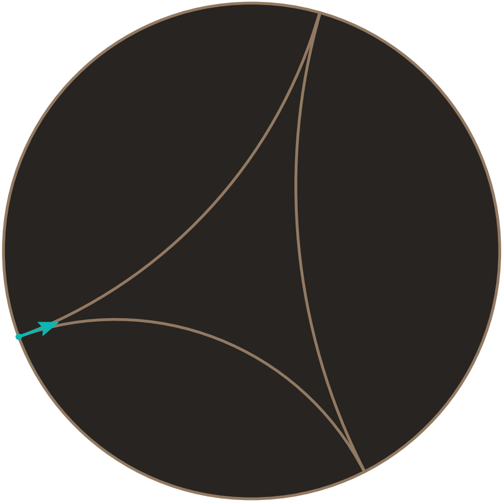

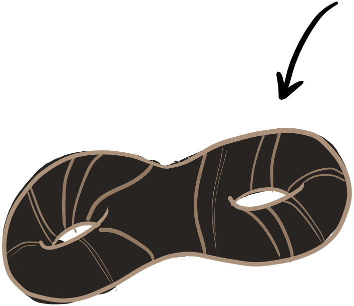

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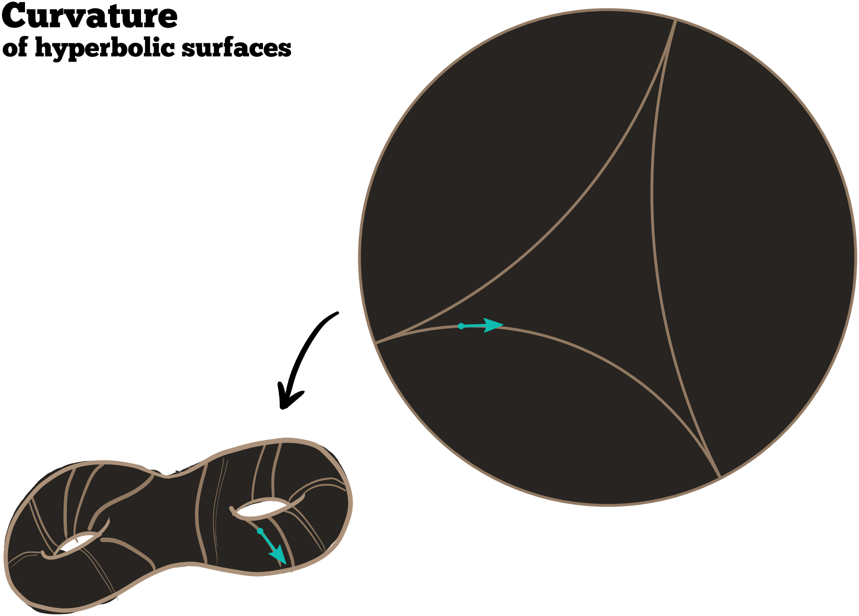

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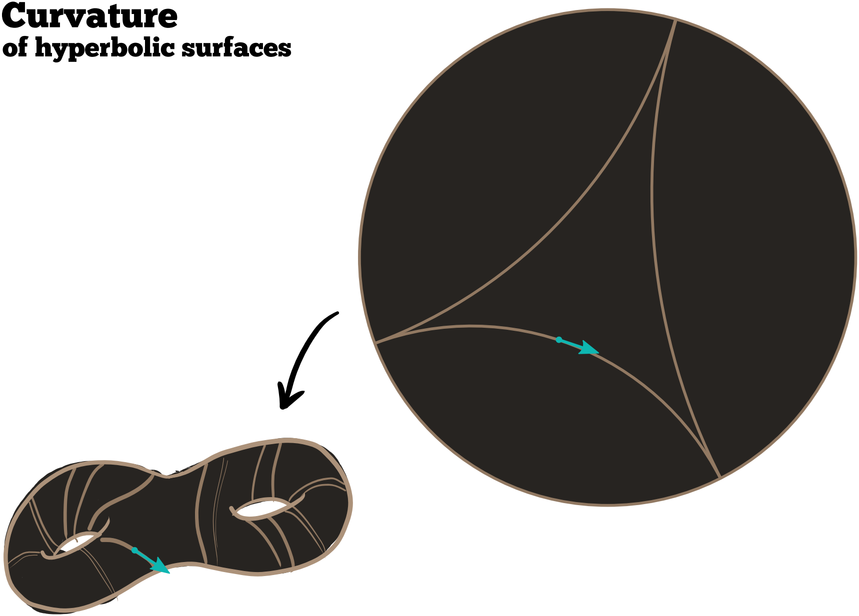

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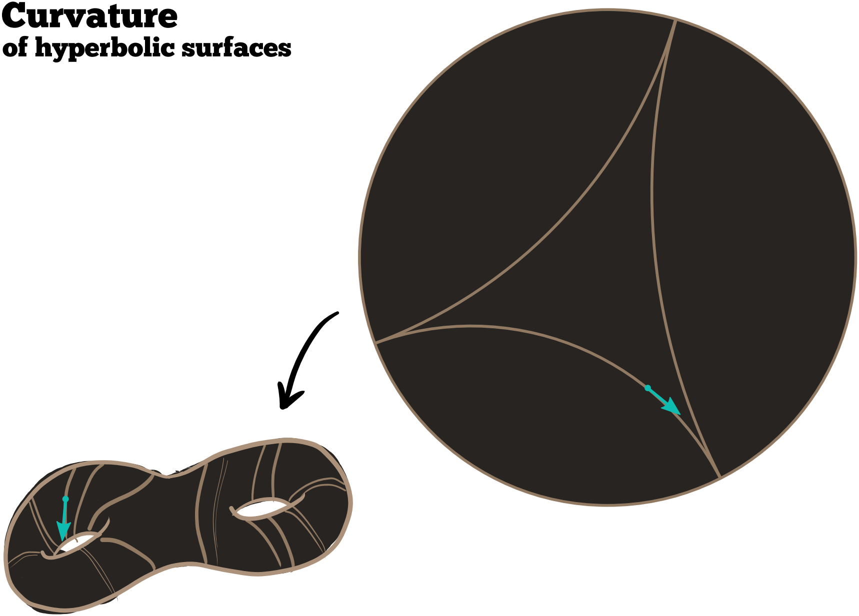

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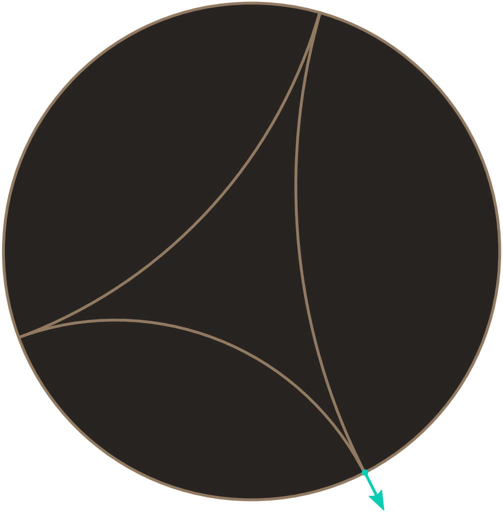

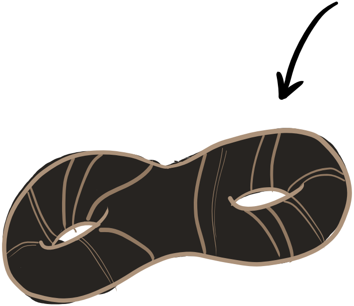

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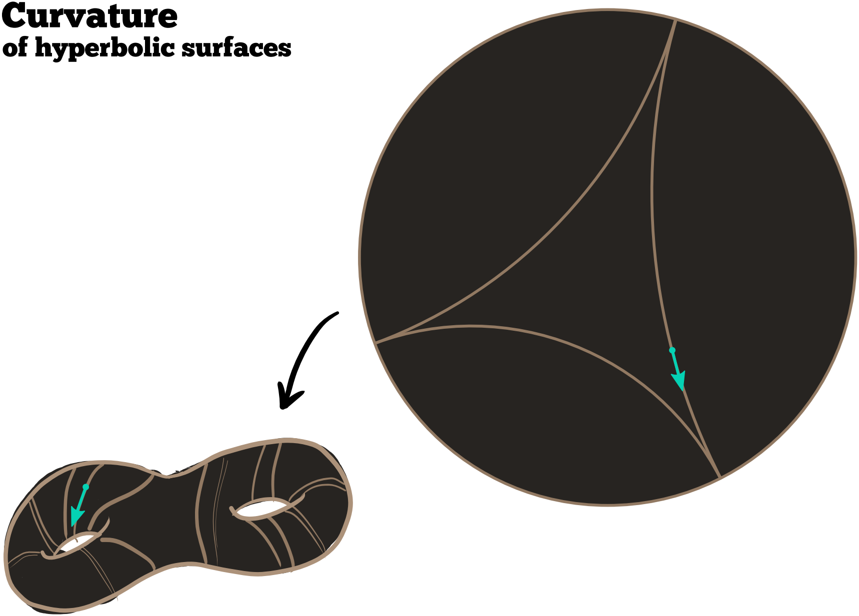

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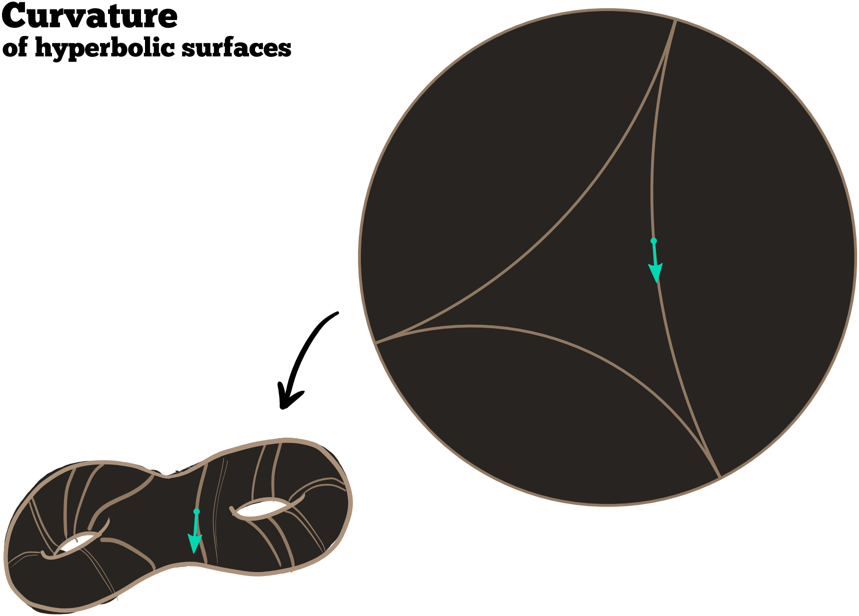

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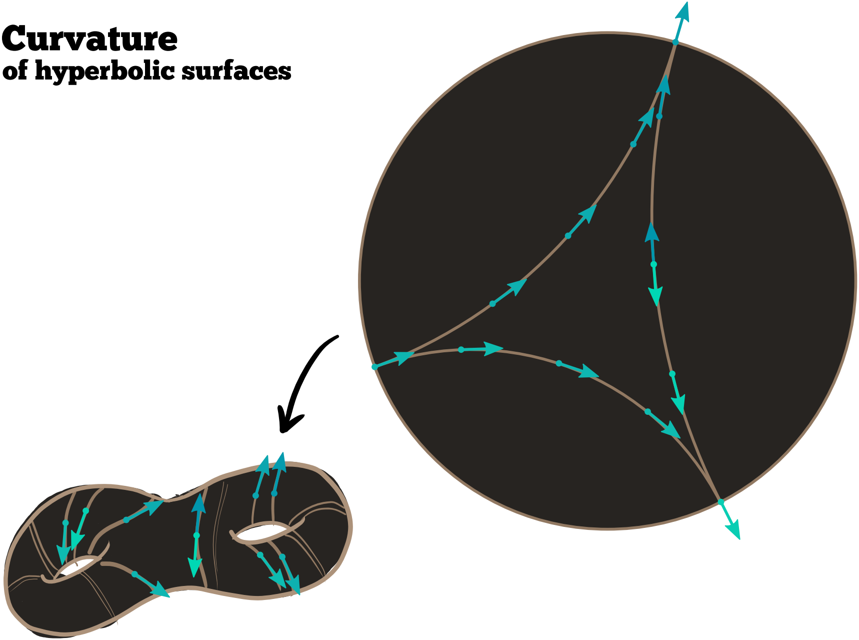

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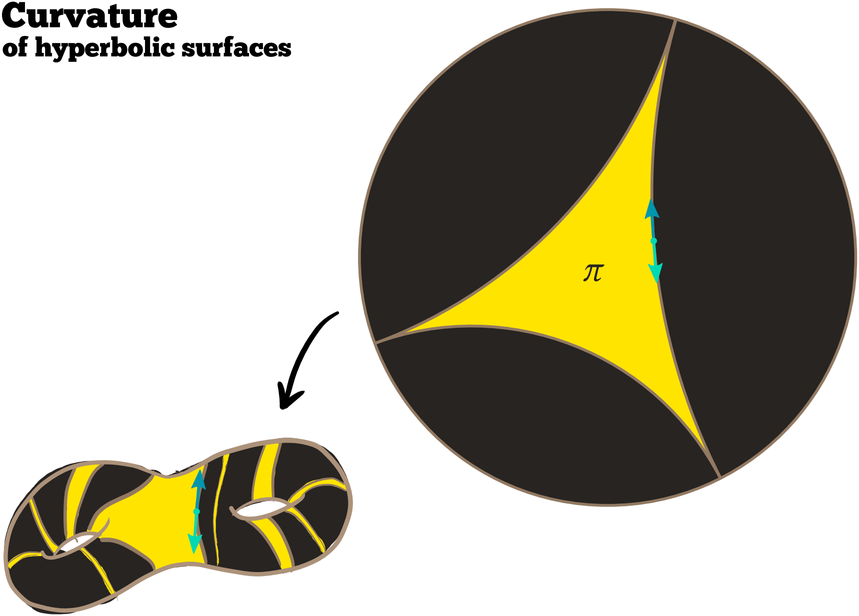

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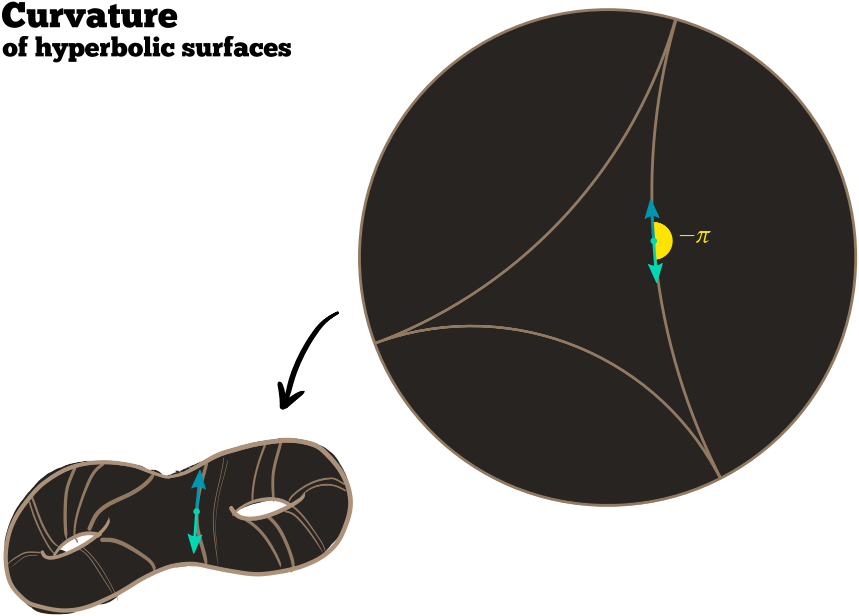
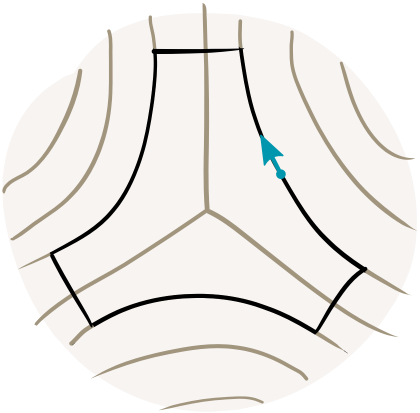
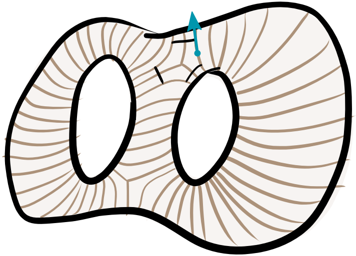


Curvature
of half-translation surfaces





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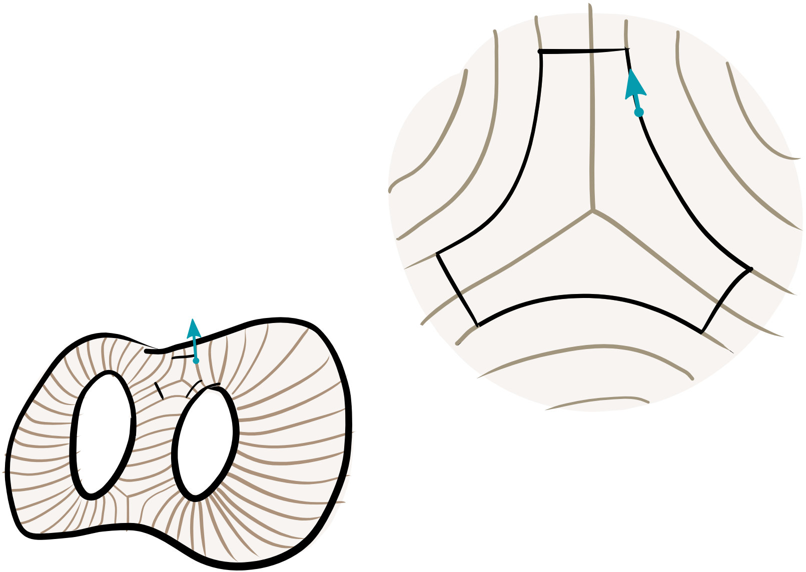

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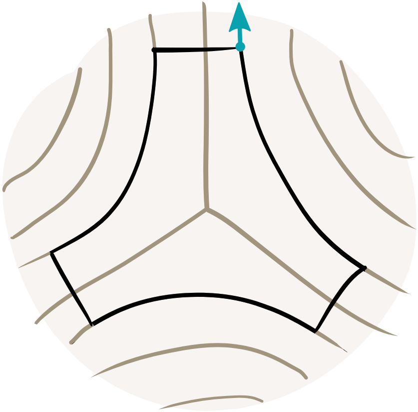

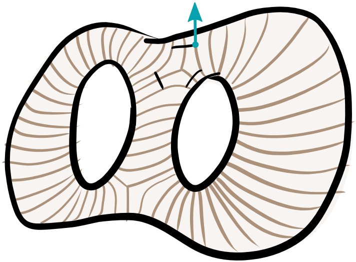

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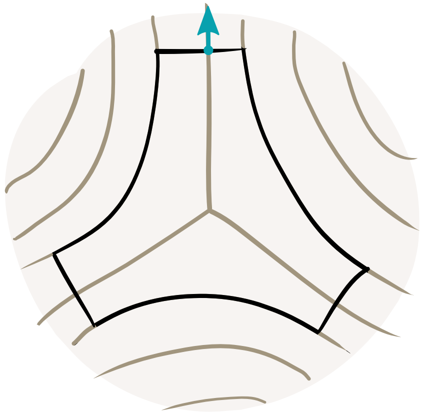

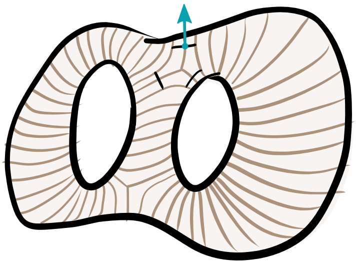

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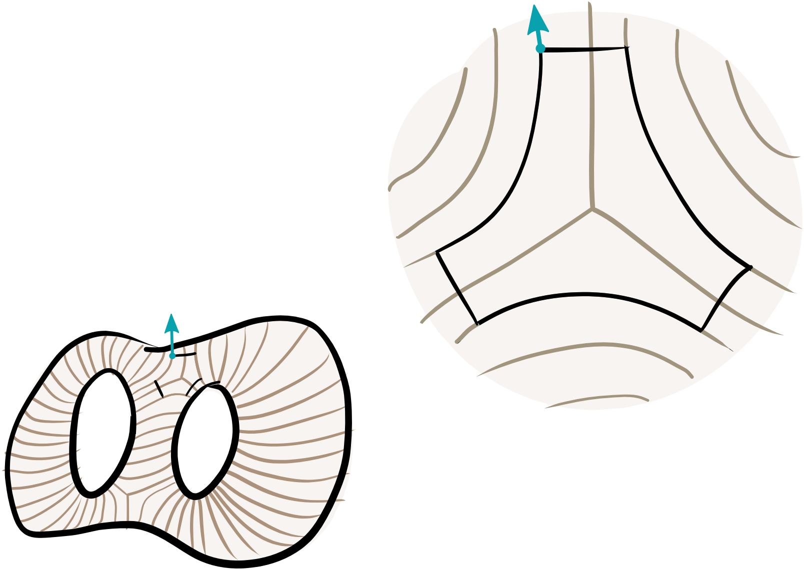
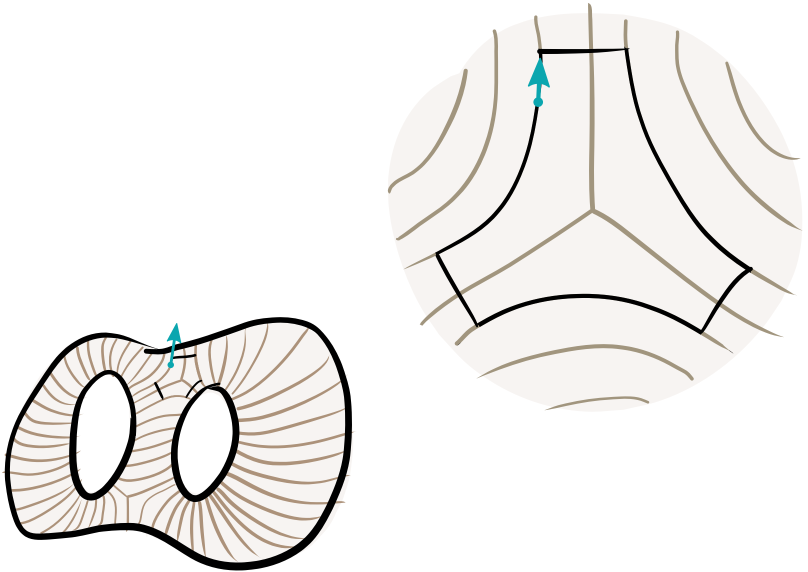
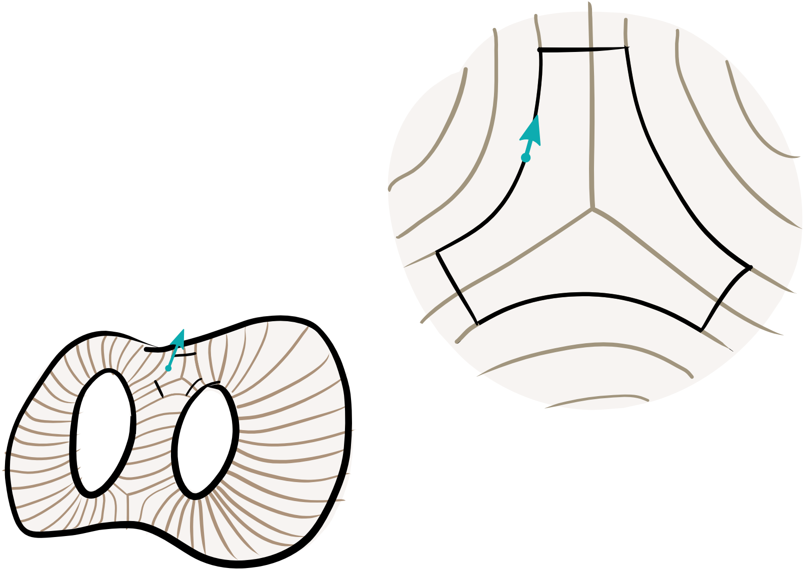


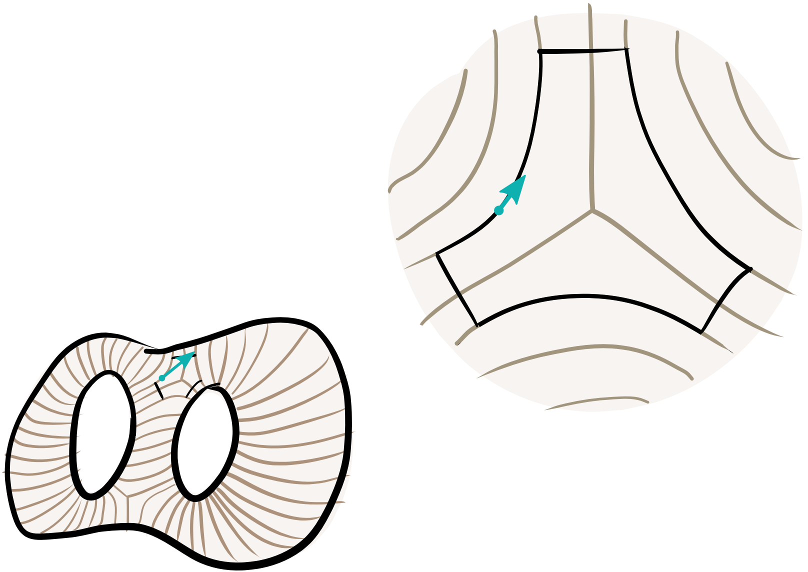
Curvature
of half-translation surfaces



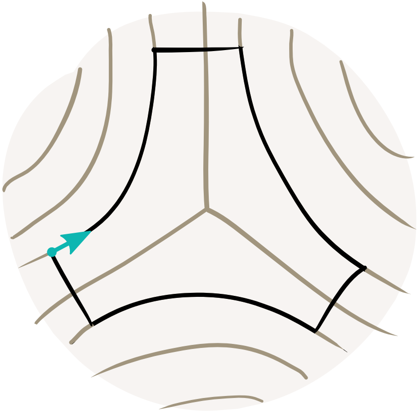
Curvature
of half-translation surfaces

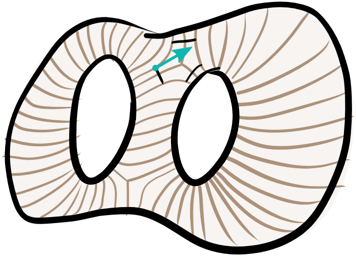


Curvature
of half-translation surfaces

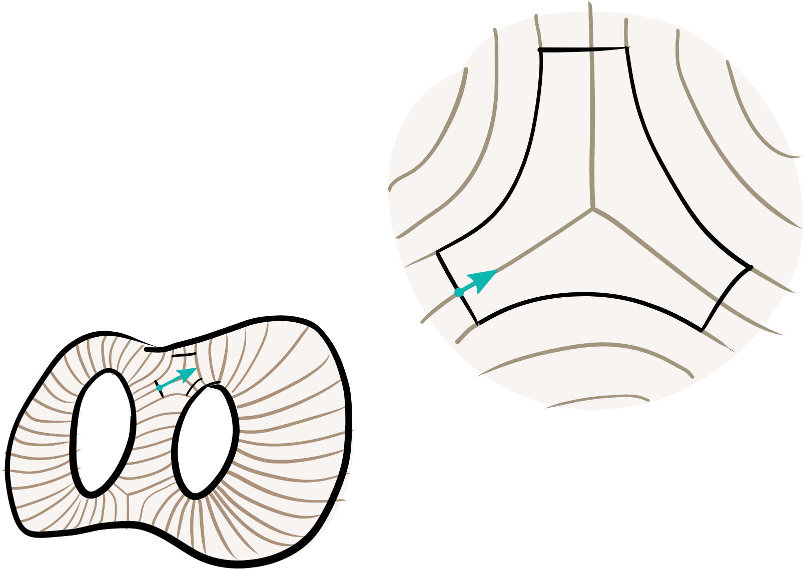


Curvature
of half-translation surfaces

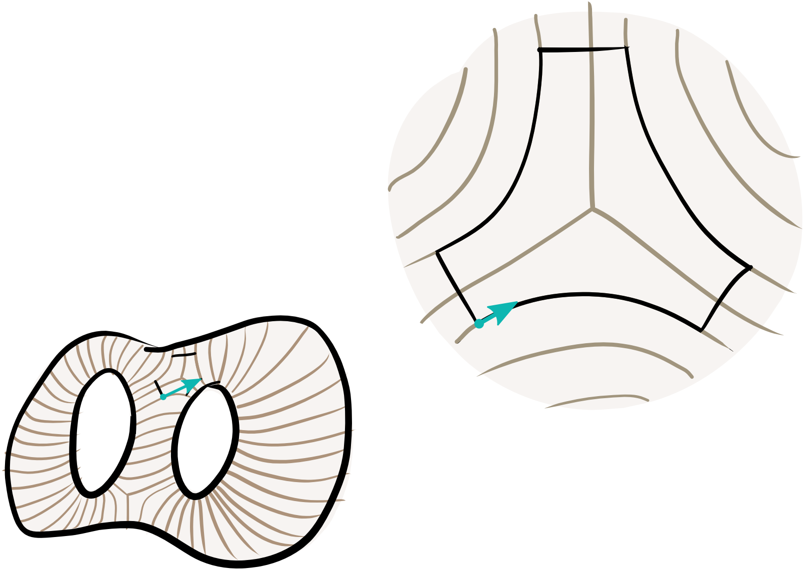




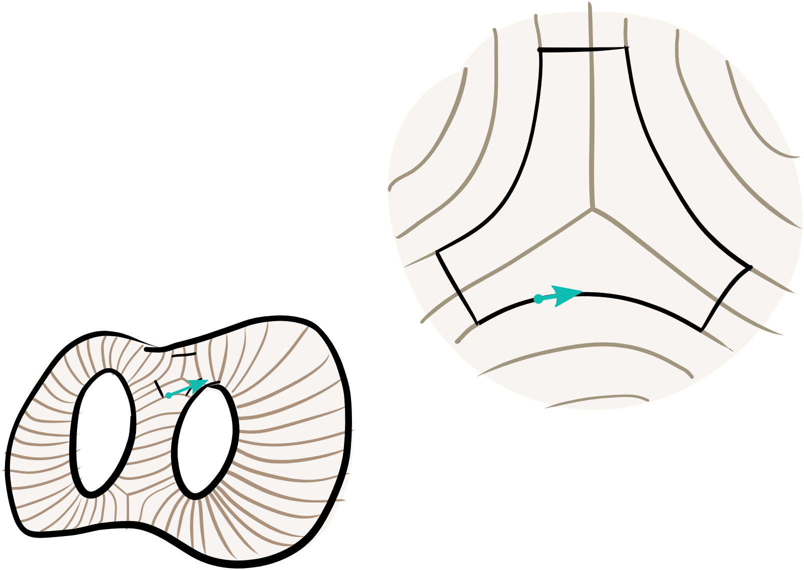
Curvature
of half-translation surfaces



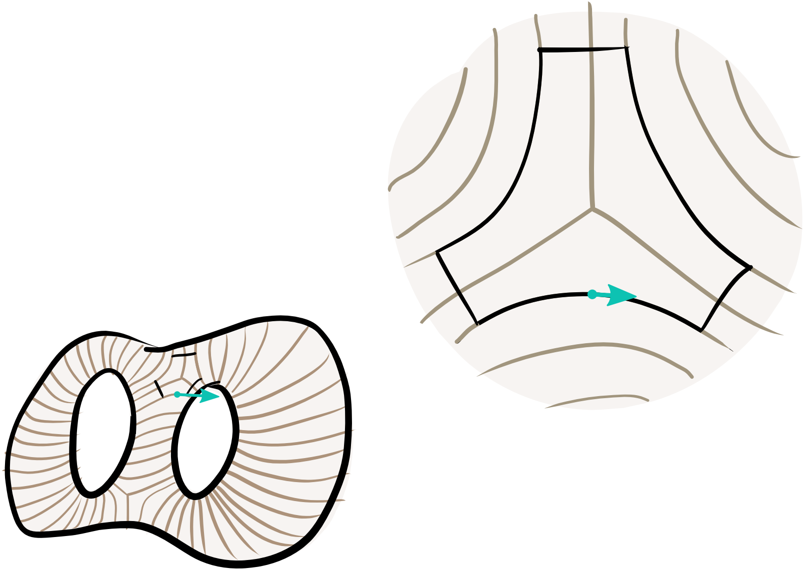
Curvature
of half-translation surfaces



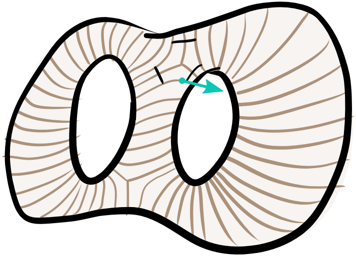
Curvature
of half-translation surfaces

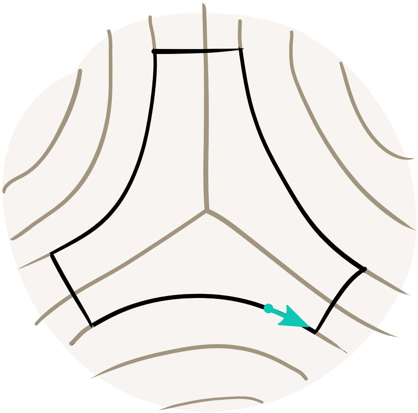


Curvature
of half-translation surfaces

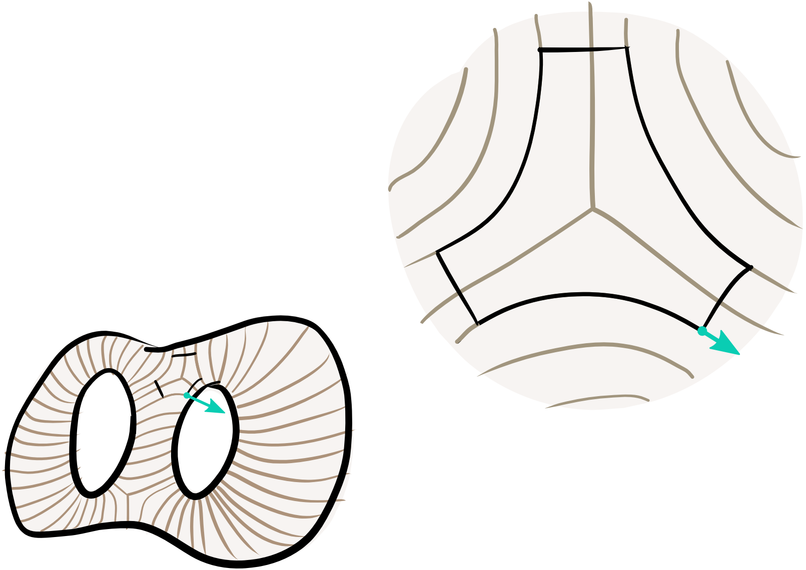


Curvature
of half-translation surfaces

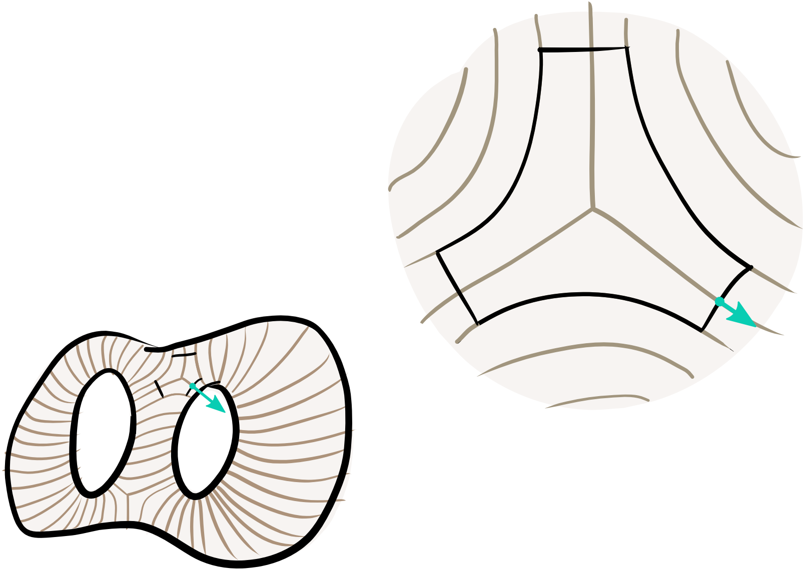




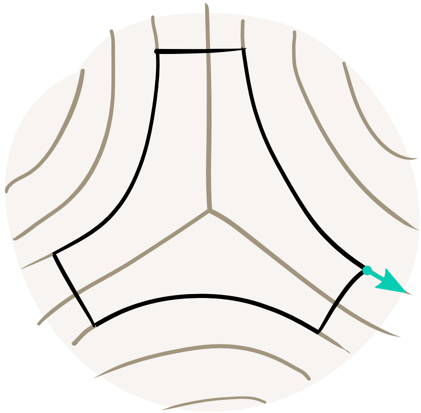
Curvature
of half-translation surfaces

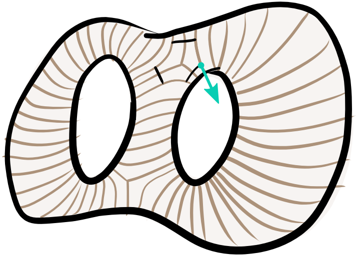


Curvature
of half-translation surfaces

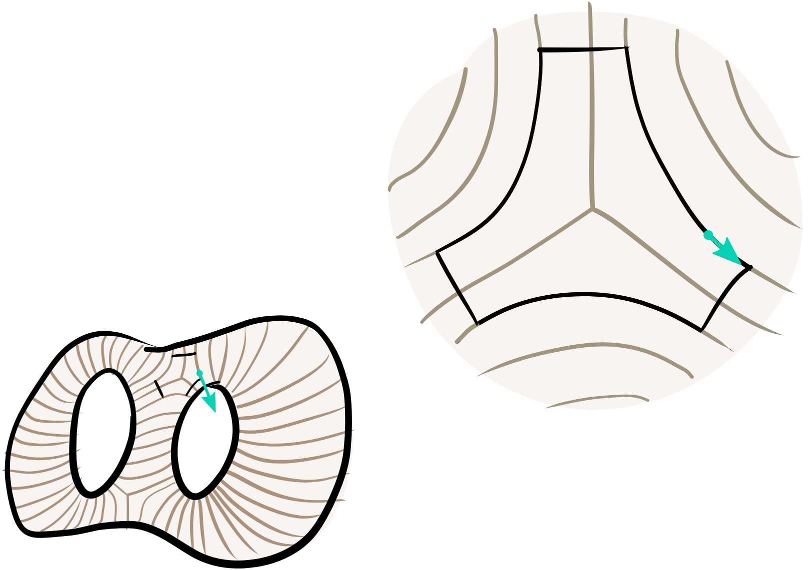


Curvature
of half-translation surfaces

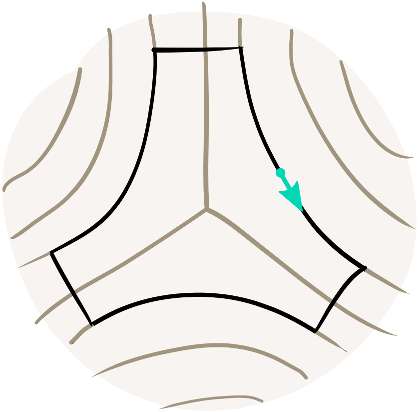


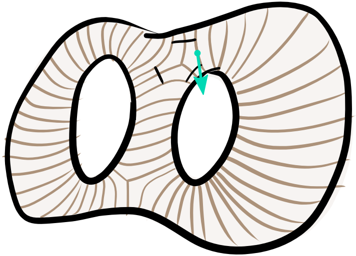


Curvature
of half-translation surfaces

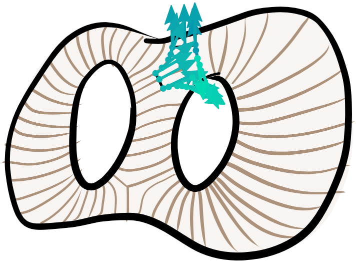


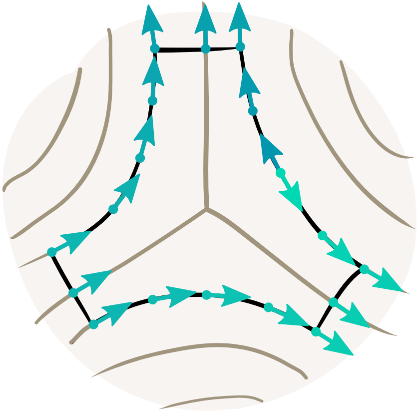
Curvature
of half-translation surfaces



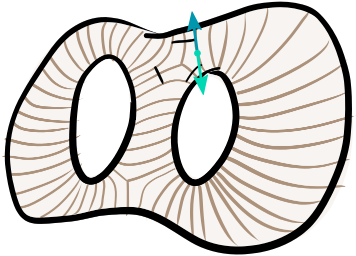


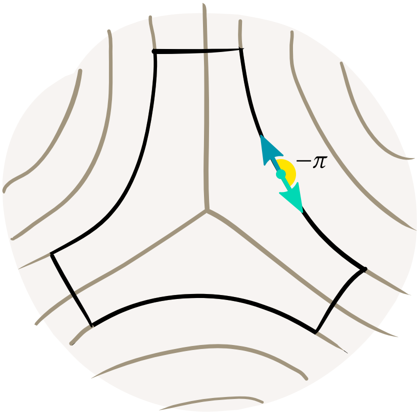
Curvature
of half-translation surfaces



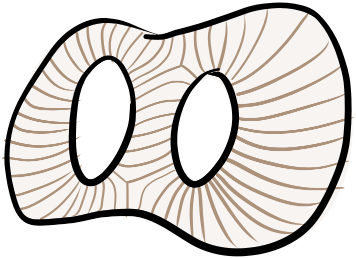


Curvature
of half-translation surfaces



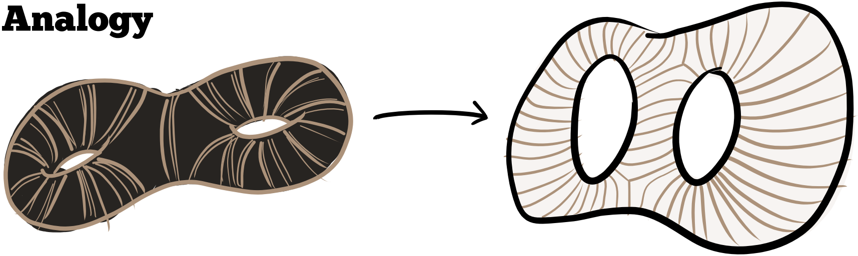


Analogy



hyperbolic surface	**half-translation surface**
Chosen maximal geodesic lamination	Vertical foliation
Chosen measure	Horizontal distance measure
Boundary leaves	Critical leaves
Bulk leaves	Non-critical leaves
Complementary ideal triangle	Tripod of critical leaves
Curvature $-\pi$ within triangle	Curvature $-\pi$ at singularity

Analogy



hyperbolic surface

Chosen maximal geodesic lamination

Complementary ideal triangle

half-translation surface

Vertical foliation

Tripod of critical leaves

Gupta's *collapsing* process makes this analogy concrete.

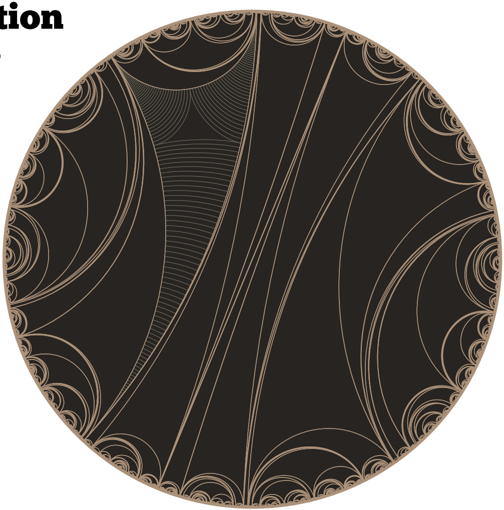
It links each hyperbolic surface to a half-translation surface through a quotient map that lines up analogous features.

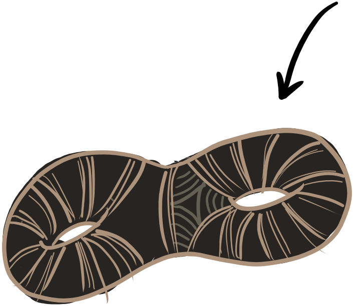
(Gupta 2014; Mirzakhani 2008; Bonahon 1987; Casson, Bleiler 1982.)

The horocyclic foliation
from a geodesic lamination

An ideal triangle comes with a foliation by horocycles.

A surface with a maximal geodesic lamination gets a foliation by horocycles.





The horocyclic foliation
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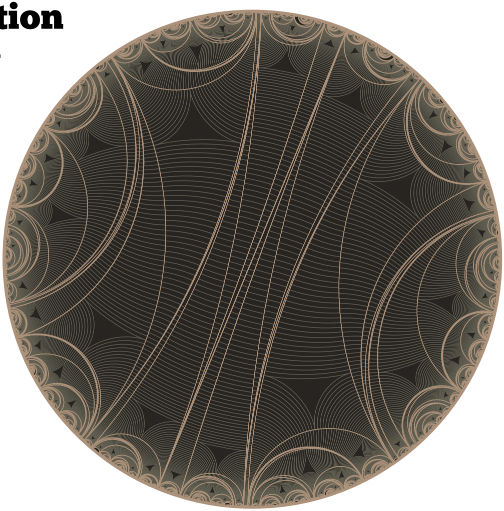
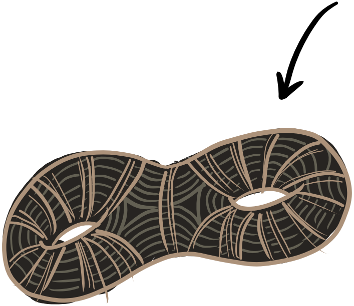
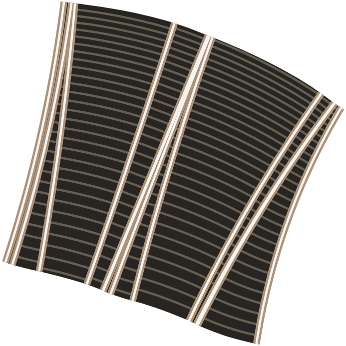




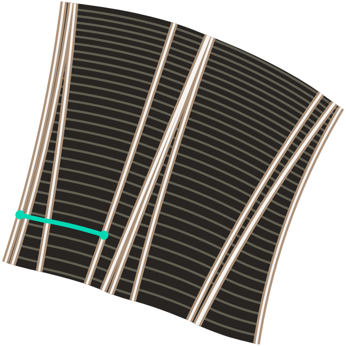
Collapsing
hyperbolic surfaces



Horizontal distance: measure of the geodesic lamination.

Vertical distance: metric distance perpendicular to horocyclic foliation.

Collapsing
hyperbolic surfaces



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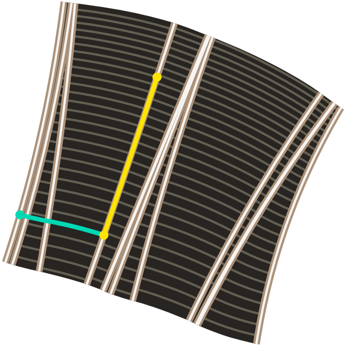
Vertical distance: metric distance perpendicular to horocyclic foliation.

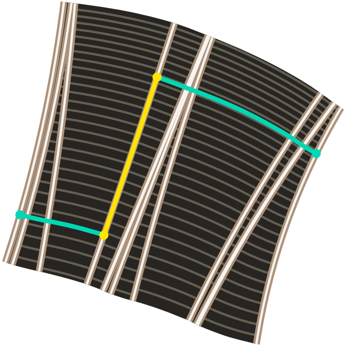
Collapsing
hyperbolic surfaces



Horizontal distance: measure of the geodesic lamination.

Vertical distance: metric distance perpendicular to horocyclic foliation.

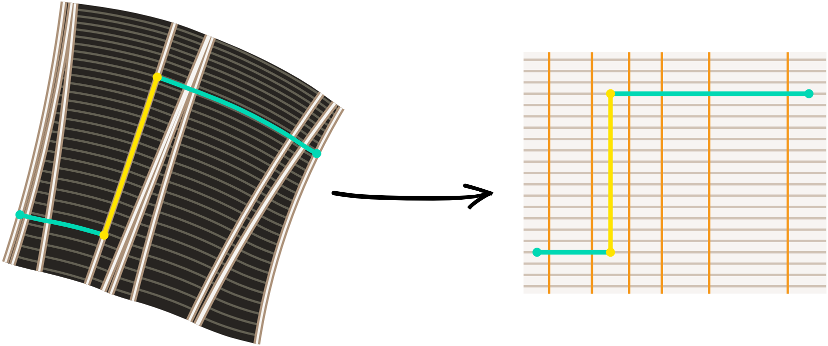
Collapsing
hyperbolic surfaces



Horizontal distance: measure of the geodesic lamination.

Vertical distance: metric distance perpendicular to horocyclic foliation.

Collapsing
hyperbolic surfaces

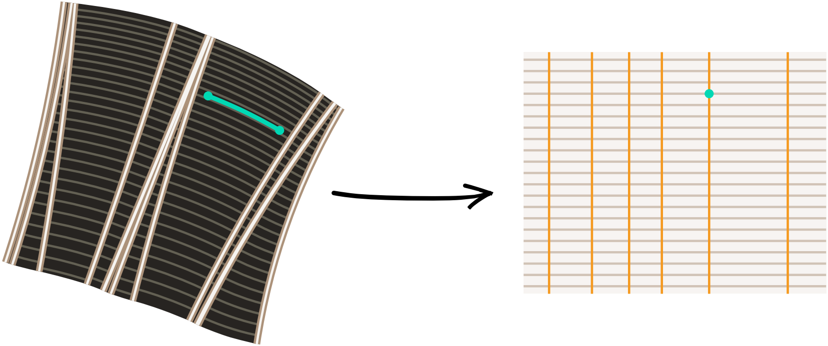


Collapsing charts: maps to $\mathbb{R}^2$ preserving vertical and horizontal distances.

They straighten the geodesic lamination and the horocyclic foliation.

They collapse the complementary triangles of the geodesic lamination.

Collapsing
hyperbolic surfaces



Collapsing charts: maps to $\mathbb{R}^2$ preserving vertical and horizontal distances.

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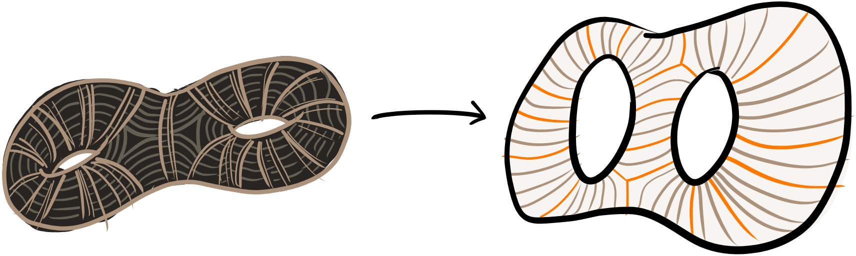
Collapsing
hyperbolic surfaces

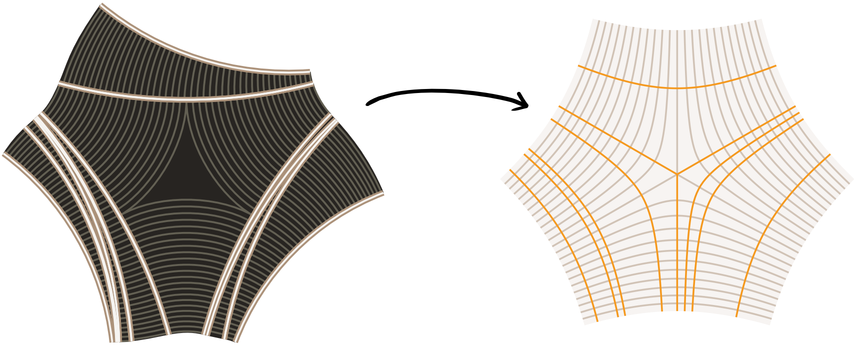


Collapsing charts are related by translations and 180° flips.

Their images fit together into a half-translation surface.

They fit together into a quotient map, which should also be a homotopy equivalence (by Edmonds 1979).

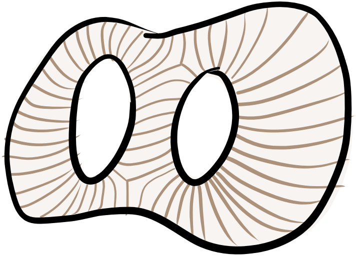
Collapsing
hyperbolic surfaces



Each complementary triangle collapses to a tripod of critical leaves.

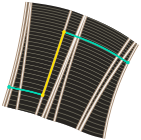
The unfoliated *contact triangle* in the middle collapses to the singularity.

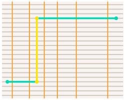
Analogy



hyperbolic surface → **half-translation surface**

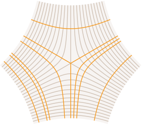
Chosen maximal geodesic lamination





Vertical foliation

Complementary ideal triangle



Tripod of critical leaves

Part II
Representation theory

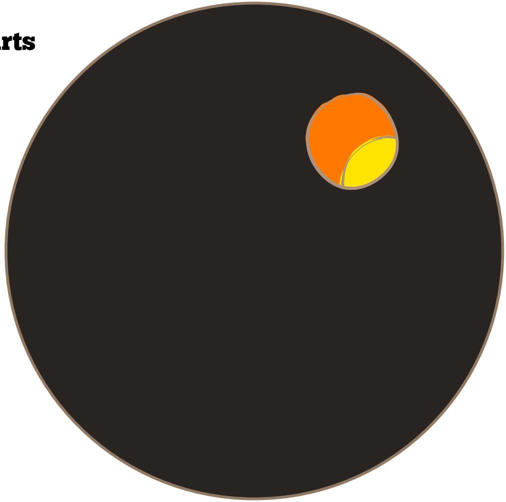
Hyperbolic surface
with its local system of charts

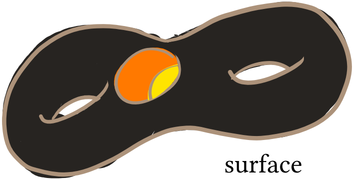
Each open subset comes with a set of charts.

Charts can be restricted and glued, so they form a sheaf.

Charts extend uniquely, so the sheaf is locally constant.

The action of $\mathrm{Isom}^+ \mathbb{H}^2$ makes the sheaf a local system.





Hyperbolic surface
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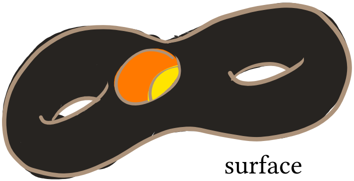

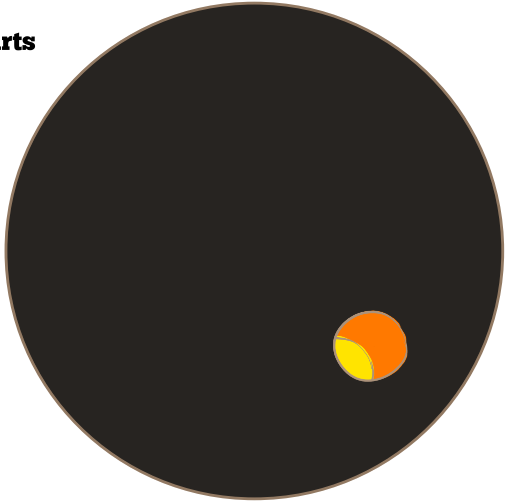

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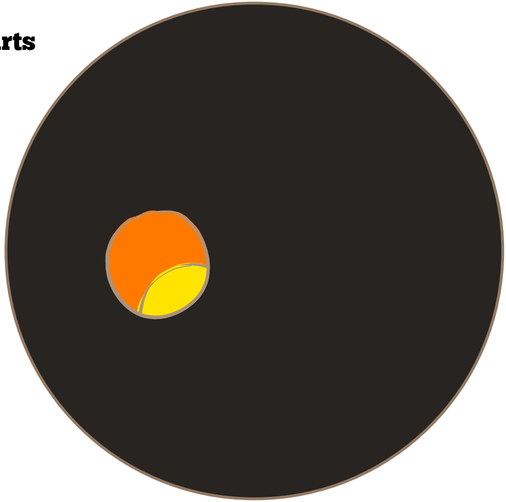
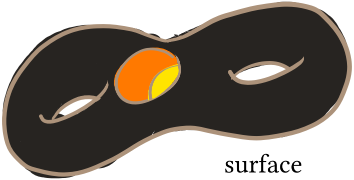


hyperbolic plane



surface

Hyperbolic surface
with its local system of charts

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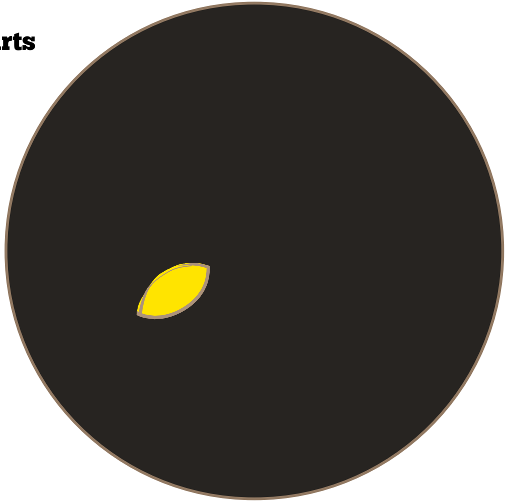
surface



hyperbolic plane

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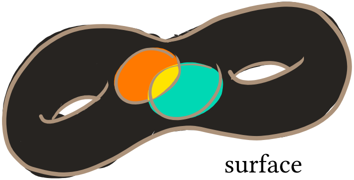
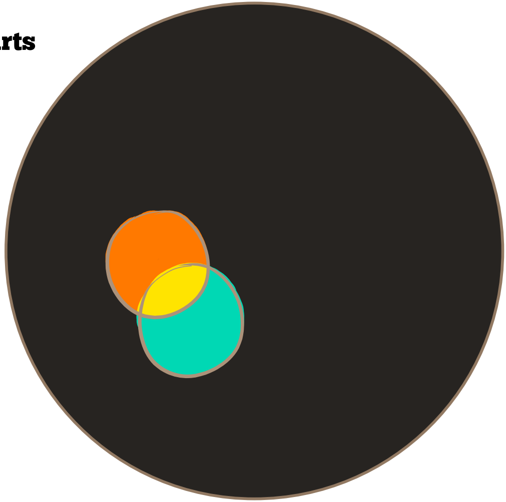


surface



hyperbolic plane

Hyperbolic surface
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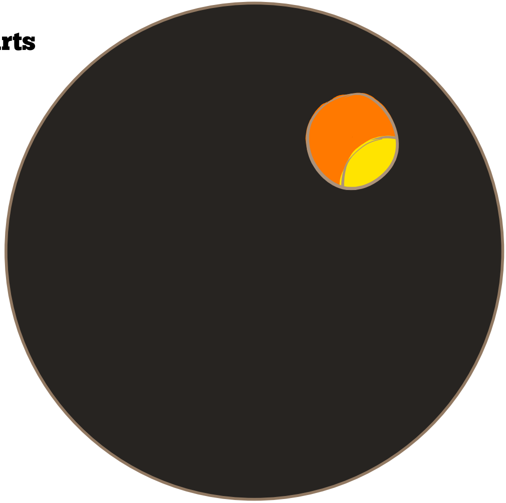
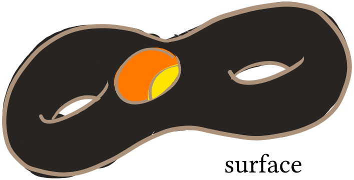


hyperbolic plane



surface

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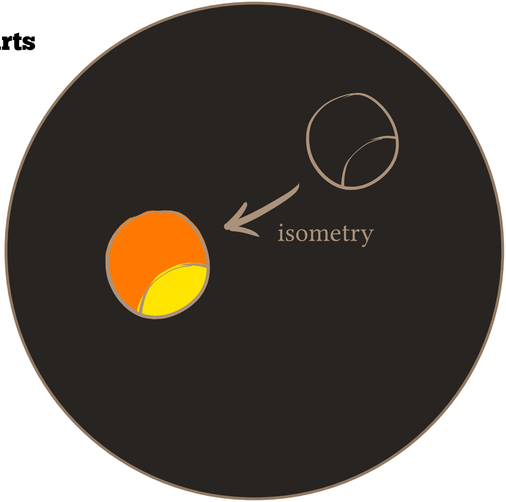
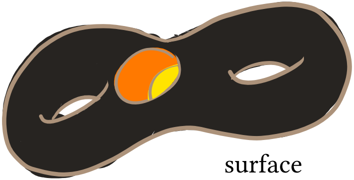


hyperbolic
plane



surface

Hyperbolic surface
with its spin charts

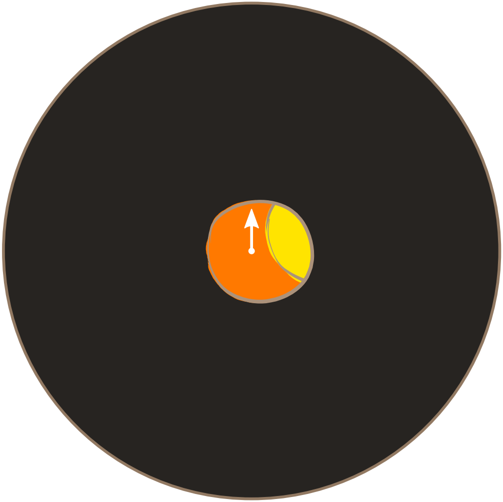
Over the unit tangent bundle, the local system of charts trivializes canonically.

Hence, it lifts canonically to a $\mathrm{SL}_2\,\mathbb{R}$ local system along the double covering

$$\mathrm{SL}_2\,\mathbb{R} \longrightarrow \mathrm{Isom}^+\,\mathbb{H}^2$$

I'll call its lift the *local system of spin charts*.

unit tangent bundle

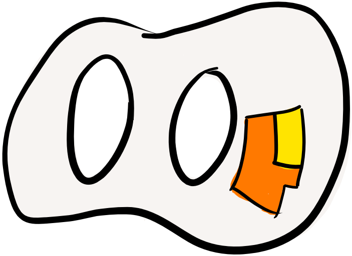


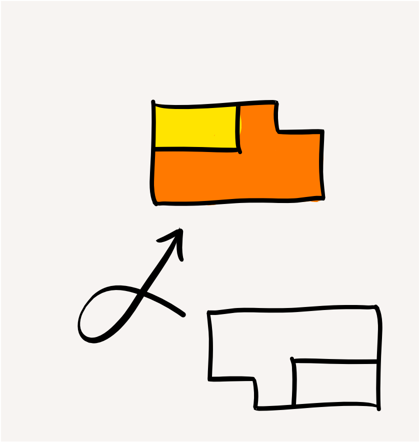
hyperbolic plane

Half-translation surface
with its local system of charts

A half-translation surface's local system of charts is acted on by translations and flips.

Over the vertical unit tangent bundle, it lifts canonically to a translation local system.





euclidean plane

Half-translation surface
with its local system of charts

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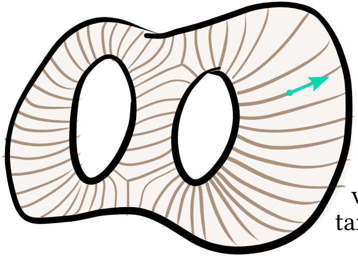
Over the vertical unit tangent bundle, it lifts canonically to a translation local system.

euclidean plane



vertical unit tangent bundle

Half-translation surface
with its local system of charts

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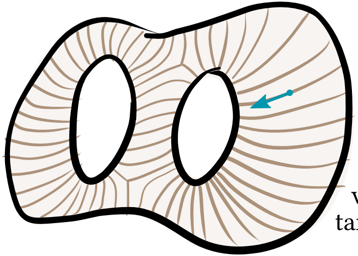
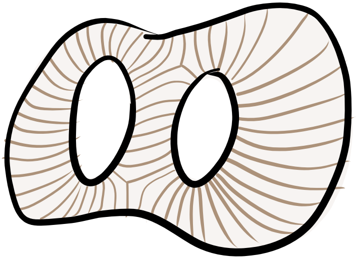
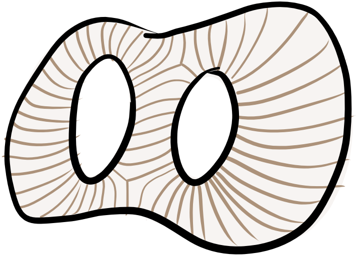


Analogy



hyperbolic surface	**half-translation surface**
Chosen maximal geodesic lamination	Vertical foliation
Complementary ideal triangle	Tripod of critical leaves
Local system of spin charts	Local system of vertical charts
Structure group $\mathrm{SL}_2\,\mathbb{R}$	Structure group $\mathrm{diag}^+\,\mathrm{SL}_2\,\mathbb{R}$

Analogy

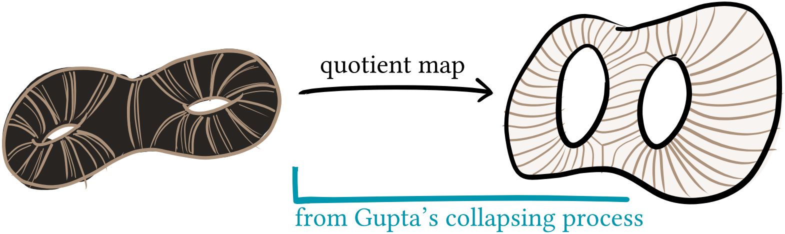


hyperbolic surface	half-translation surface
Chosen maximal geodesic lamination	Vertical foliation
Complementary ideal triangle	Tripod of critical leaves
Local system of spin charts	Local system of vertical charts

Gaiotto, Hollands, Moore, and Neitzke's *abelianization* process extends the collapsing process to include the analogy between local systems of charts.

Abelianization

Abelianization



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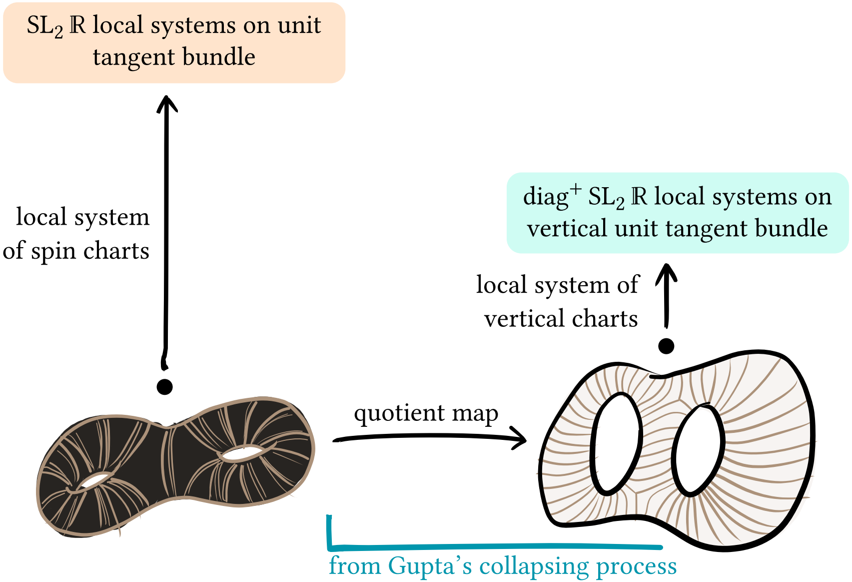

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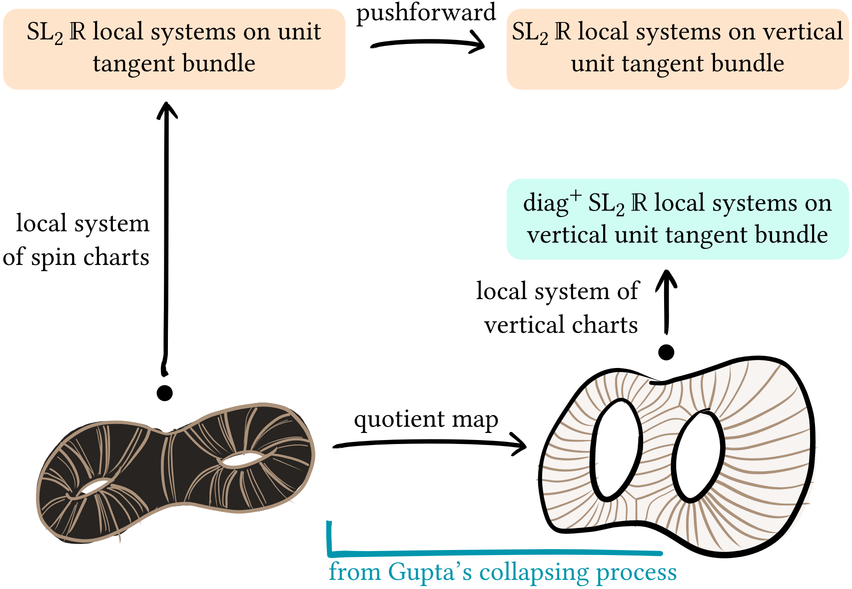

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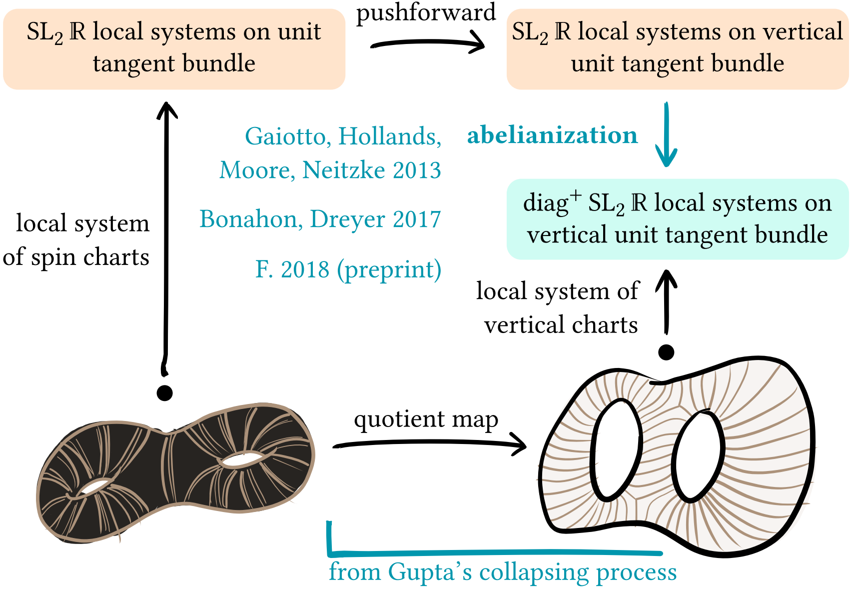


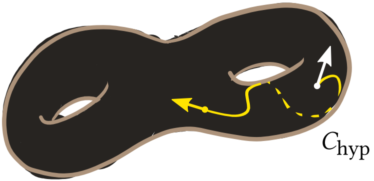
Hyperbolic charts
via the bundle of paths

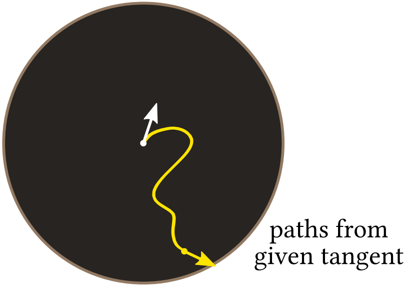
Take smooth paths up to:

- Homotopies fixing starting and ending unit tangent vectors.
- Removing loops.

Projection to starting tangent gives bundle $M \to UC_{\text{hyp}}$.

Each fiber is the unit tangent bundle of a universal cover of C_{hyp}.





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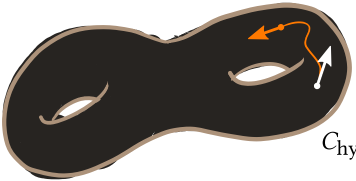

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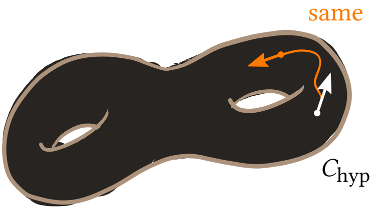

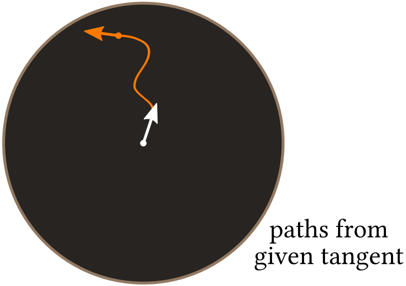

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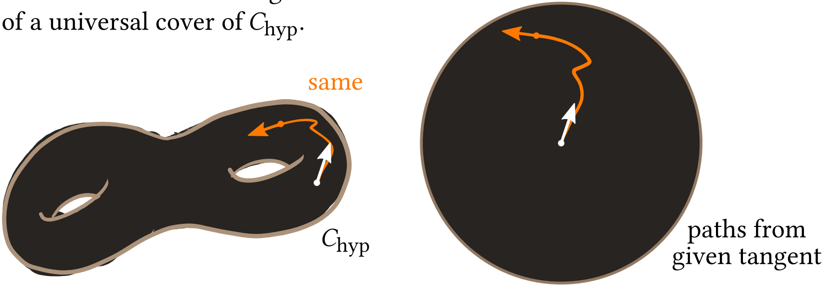

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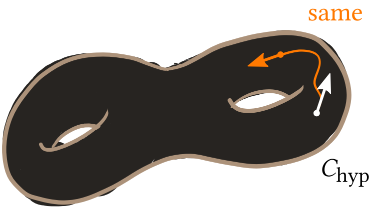

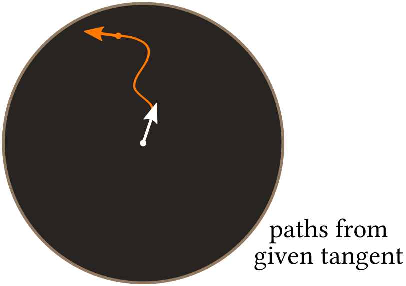

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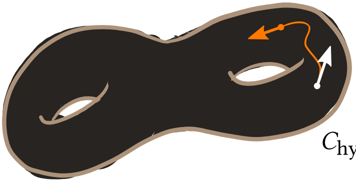

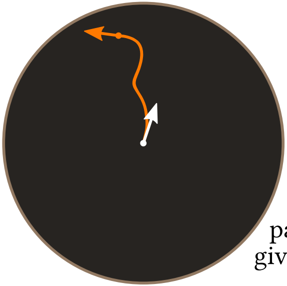

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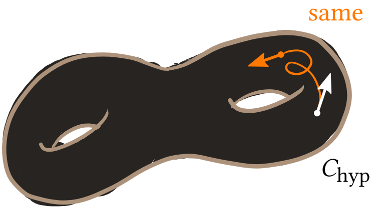

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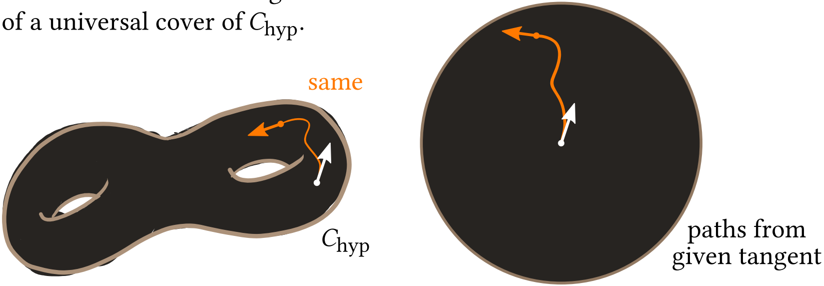


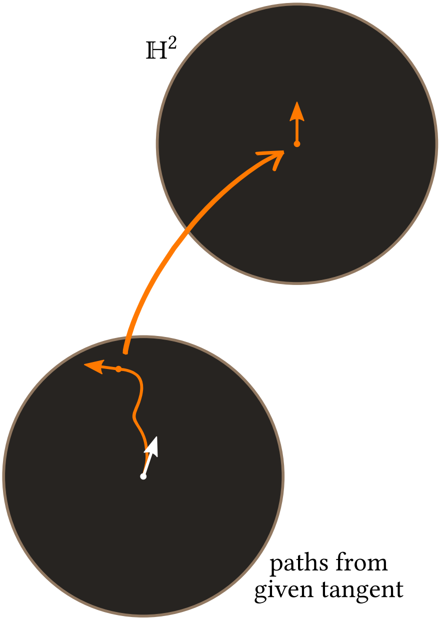
Hyperbolic charts
via the bundle of paths

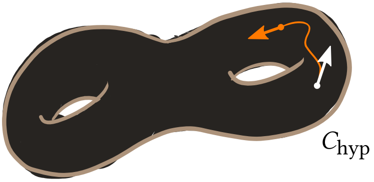
For each path, one local chart sends ending tangent to base point in $U\mathbb{H}^2$.

Thus, M parameterizes local charts.

Say a section of M is flat if the ending tangent stays still.

The local system of flat sections of M is the local system of charts.





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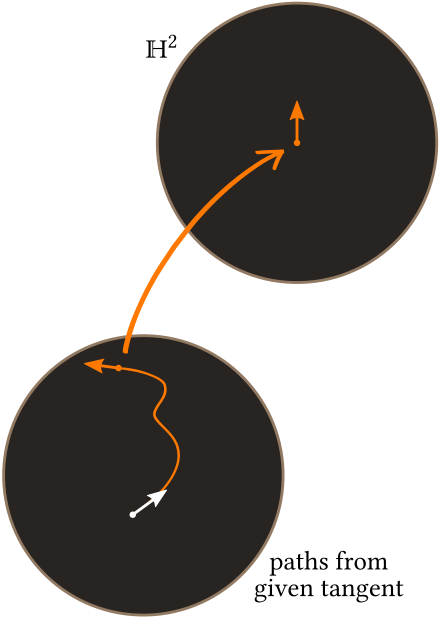
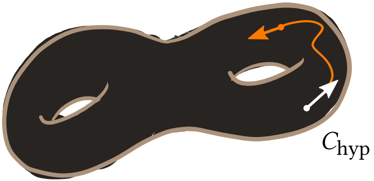




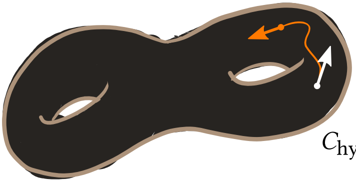
Spin charts
via the bundle of spin paths

Take smooth paths up to:

- Homotopies fixing starting and ending unit tangent vectors.
- Removing **double** loops.

Projection to starting tangent gives bundle $E \to UC_{\text{hyp}}$.

It parameterizes spin charts.



C_{hyp}

paths from given tangent

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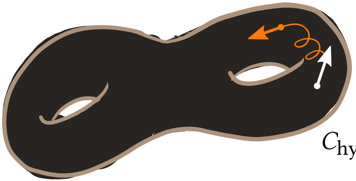

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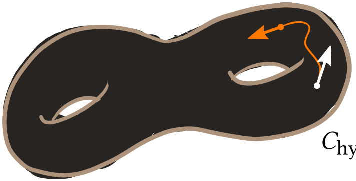

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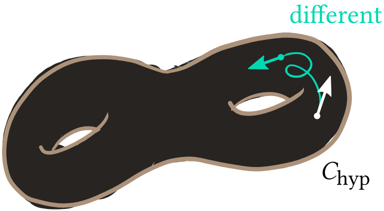

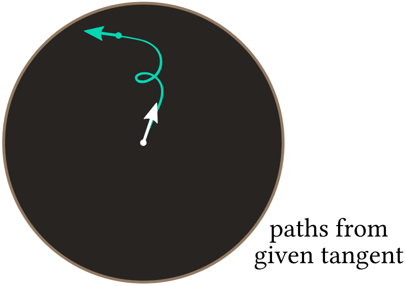

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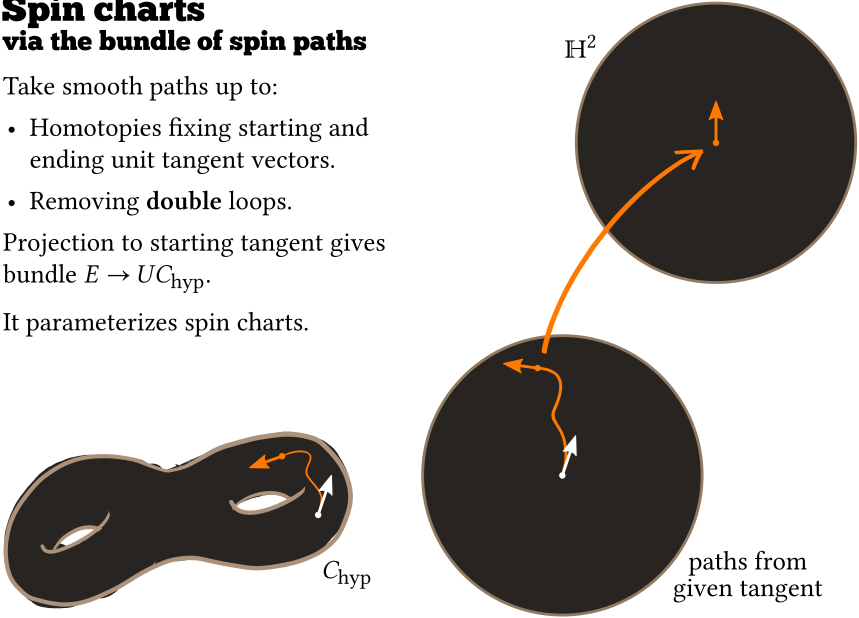

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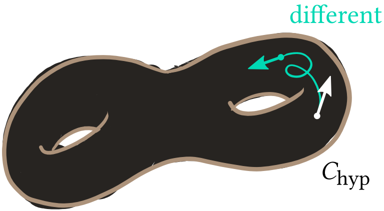
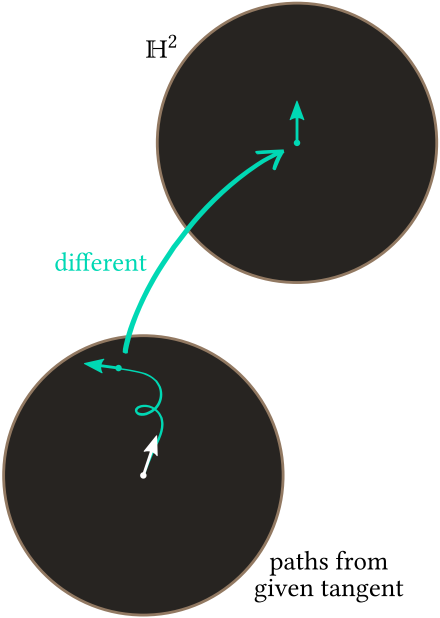
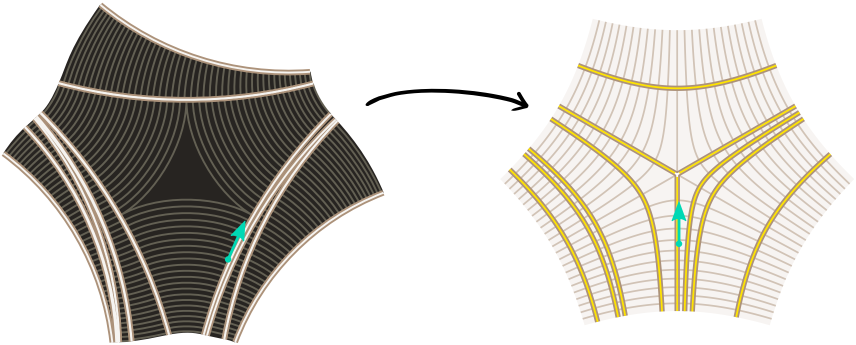




Abelianization
in action



In the local system of spin charts, pushed forward to C_{flat}, parallel transport across a singular leaf looks like this.

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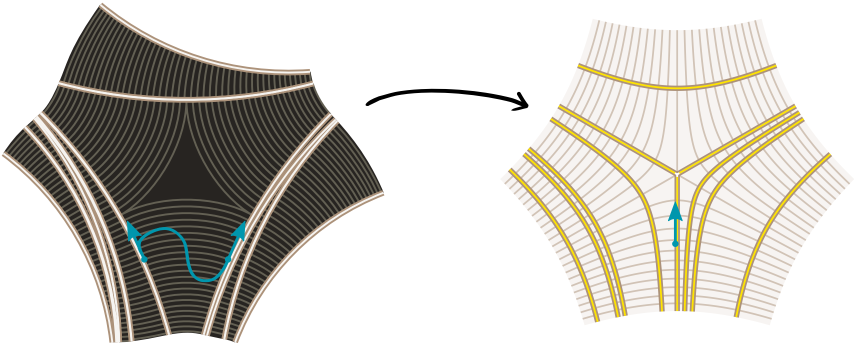

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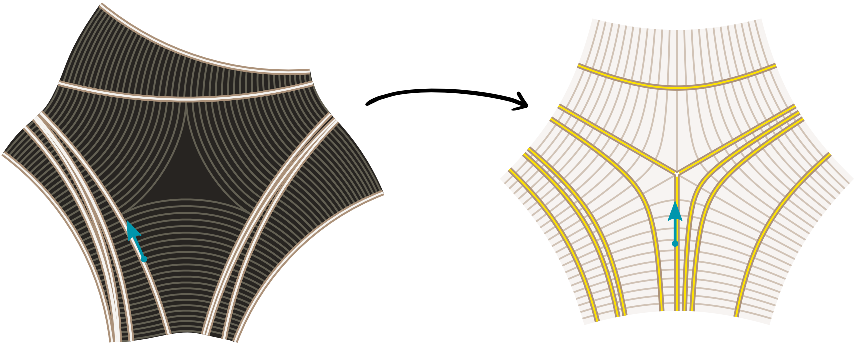
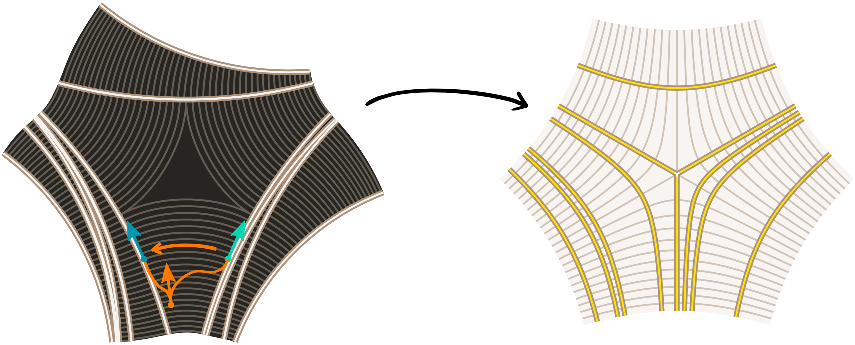


In the abelianized local system, parallel transport takes us here instead.

Abelianization
in action



To abelianize, we cut along the singular leaf, apply a special "slithering automorphism" of E, and reglue.

The slithering automorphism acts on the endings of paths by an isometry of the local universal cover.