Nielsen Realization for sphere twist on 3-manifolds

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1) Nielsen Realisation

$$\begin{tikzcd}
\mathrm{Diff}^+(M) \arrow[r] & \pi_0(\mathrm{Diff}^+(M)) = MCG(M) \\
& \pi_0(\mathrm{Homeo}^+(M)) \arrow[u, phantom, "\shortparallel"] \\
& G \arrow[u, phantom, "\vee"] \arrow[uul, dashed]
\end{tikzcd}$$

$\dim M \leq 3$

For $\dim M = 2$, Nielsen 1934. cyclic finite subgp of $MCG(S_g)$ $g \geq 1$ is realizable.

Kerckhoff 1983. Any finite subgp of $MCG(S_g)$ is realizable.

Pf: using $MCG(S_g) \curvearrowright$ Teich space.

For infinite subgp of $MCG(S_g)$

1989 Morita: $g \geq 18$, $MCG(S_g)$ is not realizable in $Diff(S_g)$.

$$\begin{tikzcd} Diff(S_g) \arrow[r] & MCG(S_g) \arrow[dl, dashed] \\ H^*(Diff(S_g)) & H^*(MCG(S_g)) \arrow[l, "inj"] \end{tikzcd}$$

"homological flavor".

Thurston Thm: $H^*(Homeo(S_g)) = H^*(MCG(S_g))$

2007 Markovic: $MCG(S_g)$ $g \geq 6$ has no realization in $Homeo(S_g)$.

2018 C. $g \geq 2$ $MCG(S_g)$ has no realization in $Homeo(S_g)$.

(fixed point argument)

2020 C. Salter elementary pf of $MCG(S_g)$.

2019 C.–Markovic: Torelli group has no realization in $\boxed{Homeo_{area}(S_g)}$. $\to$ $Diff(S_g)$?

$$\begin{array}{c} S_g \to E^8 \\ \downarrow \\ M^6 \end{array}$$

Q: is not flat!

Q: Can you find an S_g-bundle with lower dim base which is not flat?

$$\boxed{\pi_1(M) \longrightarrow MCG(S_g)}$$

$$\rightsquigarrow$$

$$\begin{array}{c} S_g \to E \\ \downarrow \\ M \end{array}$$

$$\boxed{\pi_1(M) \longrightarrow \mathrm{Diff}(S_g)}$$

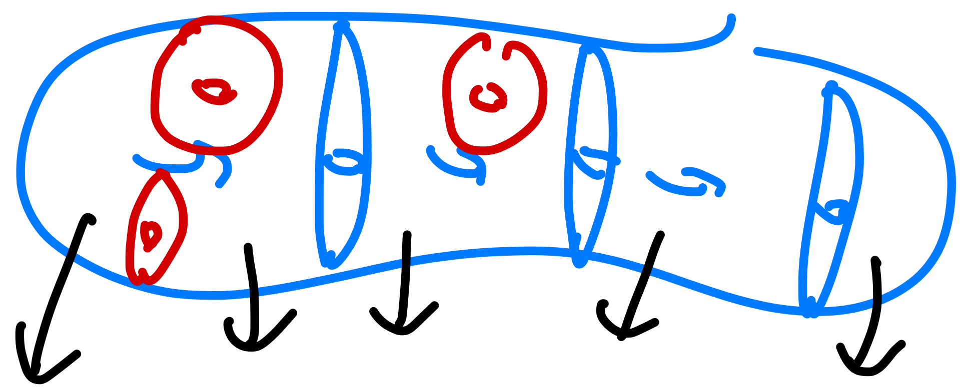
flat bundle

$$S_g \longrightarrow \underline{\tilde{M} \times \underline{\underline{S_g}} / \underline{\underline{\pi_1(M)}}} \longrightarrow M$$

2) 3-manifold MCG

① Prime decomposition (sphere)

② torus decomposition



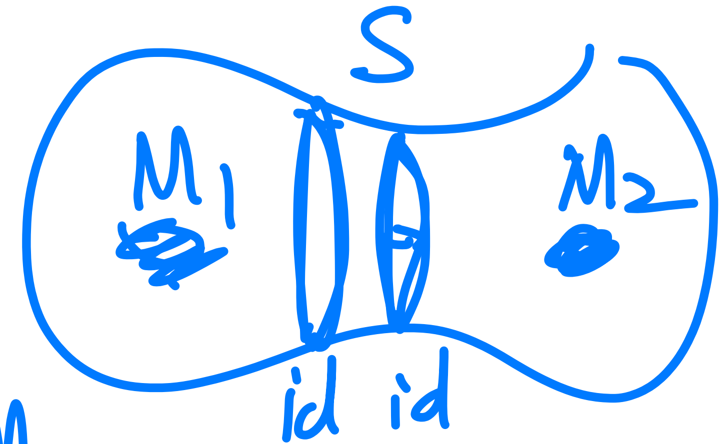
Geometric 3-manifolds.

(8 geometries Thurston)

Nielsen Realization for geometric 3-manifolds are mostly known.

Twist subgp

3) Twist subgroup



τ_S $\qquad\qquad S \subseteq M = M_1 \# M_2$

$\mathrm{supp}(\tau_S) \subseteq S^2 \times I$

$\tau_S(z,t) = (\rho(t)\, z,\ t) \Leftarrow$ sphere twist.

$\rho : I \longrightarrow \mathrm{Diff}(S^2)$

Twist subgp $= \langle \tau_S \mid \forall S \rangle \leq MCG(M^3)$

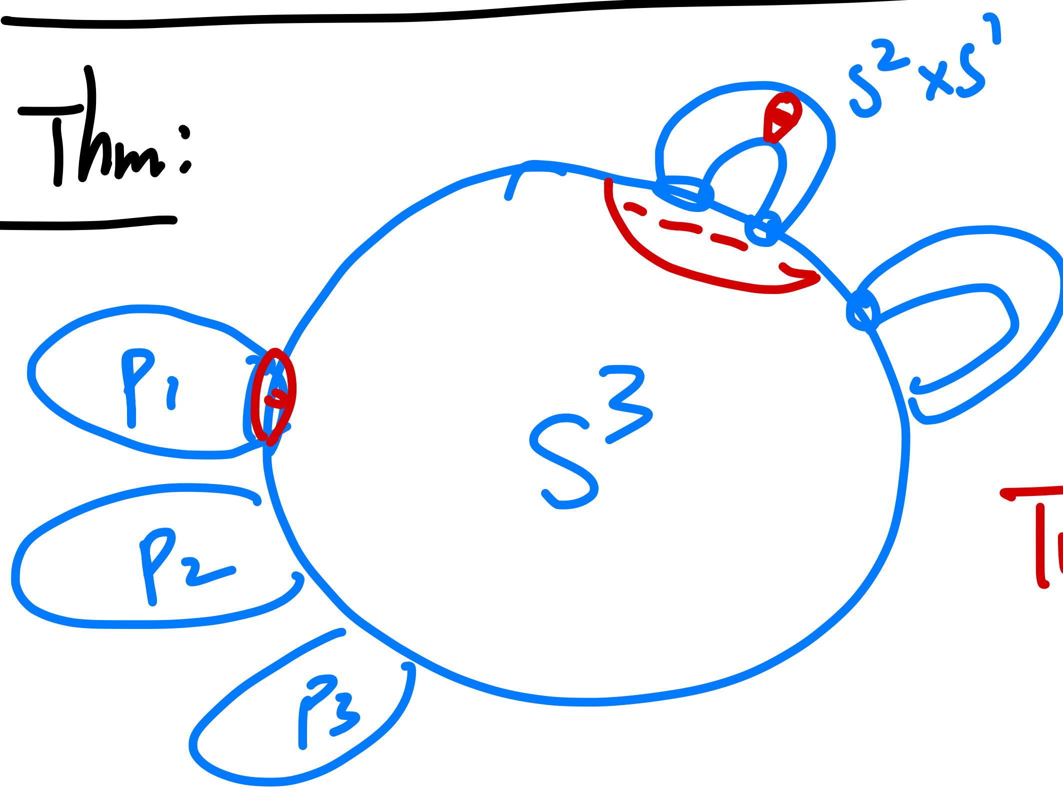
order$(\tau_S) \leq 2$ because $\pi_1(\mathrm{Diff}(S^2)) = \mathbb{Z}/2 = \pi_1(SO(3))$

4) **Thm:** $1 \neq G < \mathrm{Twist}(M^3)$

G is realizable $\iff$ $G =$ cyclic

$M =$ connected sum of lens spaces.

Old Thm:

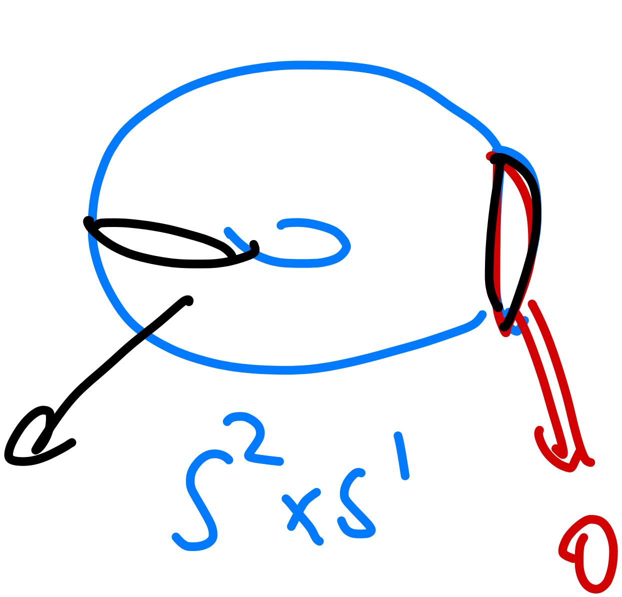


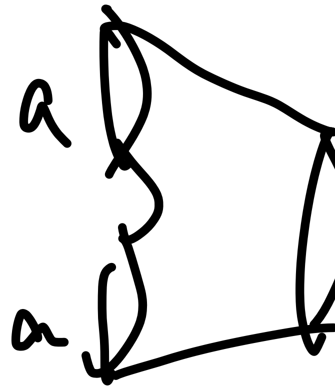
$$M = \#_K S^2 \times S^1 \# P_i$$

$\mathrm{Twist}(M) = (\mathbb{Z}/2)^K$ if $i=0$

$= (\mathbb{Z}/2)^{K+\ell}$ ℓ is non lens space.

$T_a + T_b = T_c$





$c = 2a = 0$

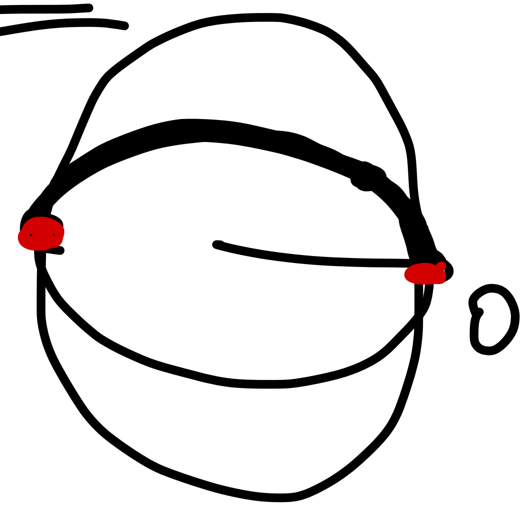
lens space $\Rightarrow$ $S^2 \times S^1$

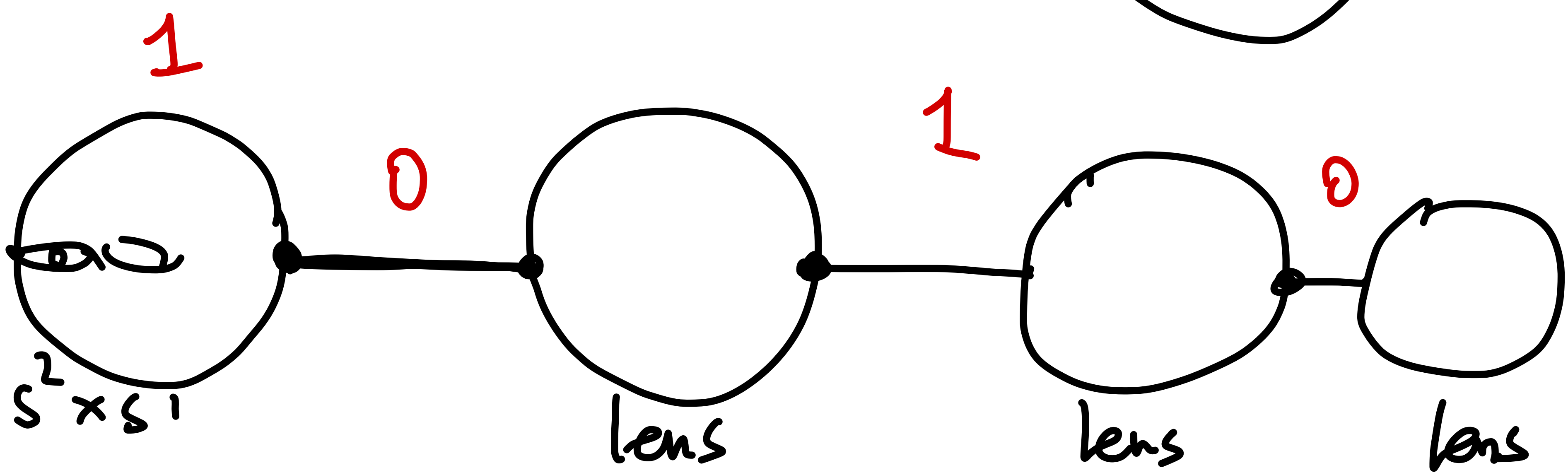
$$S^2 \times S^1 \longrightarrow S^2 \times S^1$$

R_0

$$(x, \theta) \longrightarrow \big(\underline{R_{\theta,\pi}}(x), \theta\big)$$

vector in black by deg π.





5) Obstruction $G < \mathrm{Twist}(M)$

Equivariant sphere Thm (Meeks – Yau)

G finite $\curvearrowright M^3$

$\Rightarrow \exists$ a collection of spheres disjoint $\mathcal{S}$

s.t. $\mathcal{S}$ is G-invariant, $M - \mathcal{S}$ is irreducible.

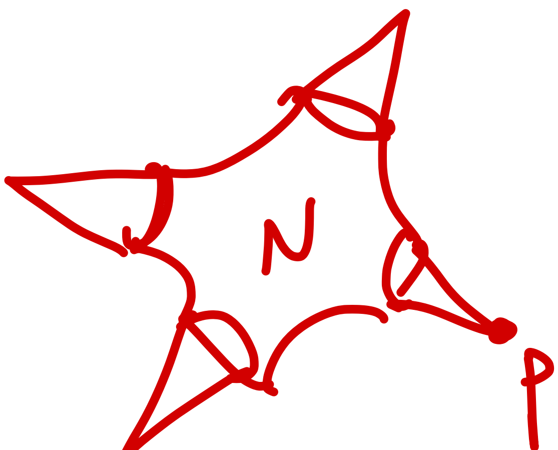
<u>Step 1:</u> $G \curvearrowright M$ but ~~trivially on $\pi_1(M)$~~ $\pi_2(M)$)

Lemma: G preserves every sphere in $\mathcal{S}$.

(homological argument) up to conjugation

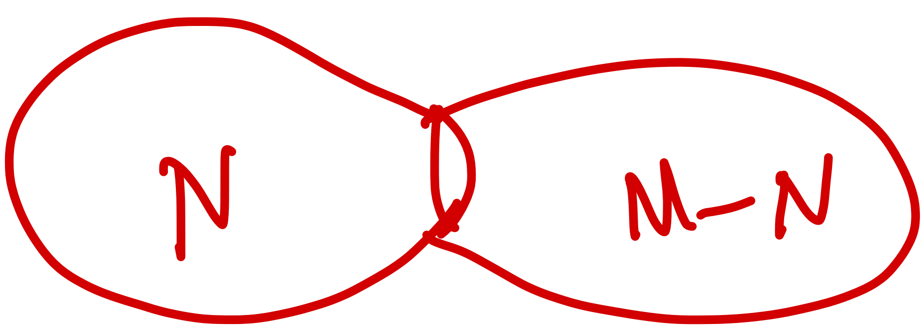
Step2: N is a component of $M - S$ after filling the sphere with balls.

Extend action of G on N with many fixed pts



<u>Lemma</u>: $G \circlearrowright \pi_1(N, p)$ is trivial.

Pf is group theory.



$\pi_1(N) * \pi_1(M-N) \circlearrowleft G$

Step 3: $\tilde{N}$ < contractible / S^3

irreducible

$N, p \curvearrowleft G$

trivial on $\pi_1(N, p)$

$\longrightarrow$

$\tilde{N}, \tilde{p} \curvearrowleft \tilde{G}$

lifting $\tilde{G}$ commutes with deck transformation $\pi_1(N) \curvearrowright \tilde{N}$

Fix$(\tilde{G})$ is a connected 2-manifold.

$\pi_1(N)$ free, proper $\Longrightarrow$ ~~$\pi_1(N)$ cyclic~~ $\Rightarrow N$ is lens spaces.