\usepackage{chalkboard}

Large-scale geometry of the saddle connection graph

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[rob-tang.github.io]

YMSC Topology Seminar

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tldr: the saddle connection graph is just like the Farey graph

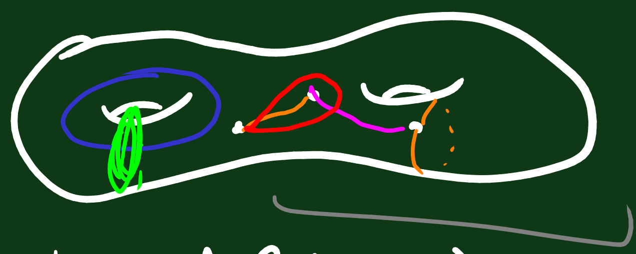
Plan:

1. Background
2. State main results
3. Nice partition via slopes
4. Nice paths

Combinatorial Complexes

(S, Z)

marked points



* Arc-and-curve graph $\mathcal{AC}(S, Z)$

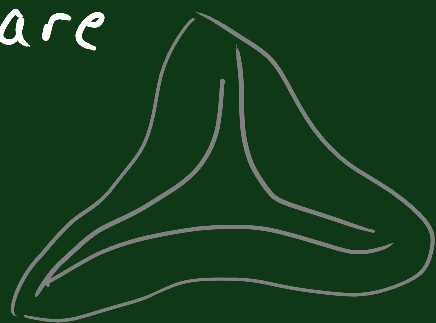
  vertices : arcs or curves / isotopy

  edges : disjoint rep's

* Arc graph $\mathcal{A}(S,Z) \hookrightarrow \mathcal{AC}(S,Z)$

* Curve graph $\mathcal{C}(S,Z) \hookrightarrow \mathcal{AC}(S,Z)$

<u>Facts</u> $\{\mathcal{A}, \mathcal{C}, \mathcal{AC}\}(S,Z)$ are



- connected
- locally infinite
- of infinite diameter
- Gromov hyperbolic

  [Masur-Minsky, Masur-Schleimer]

<u>Mapping class group</u>

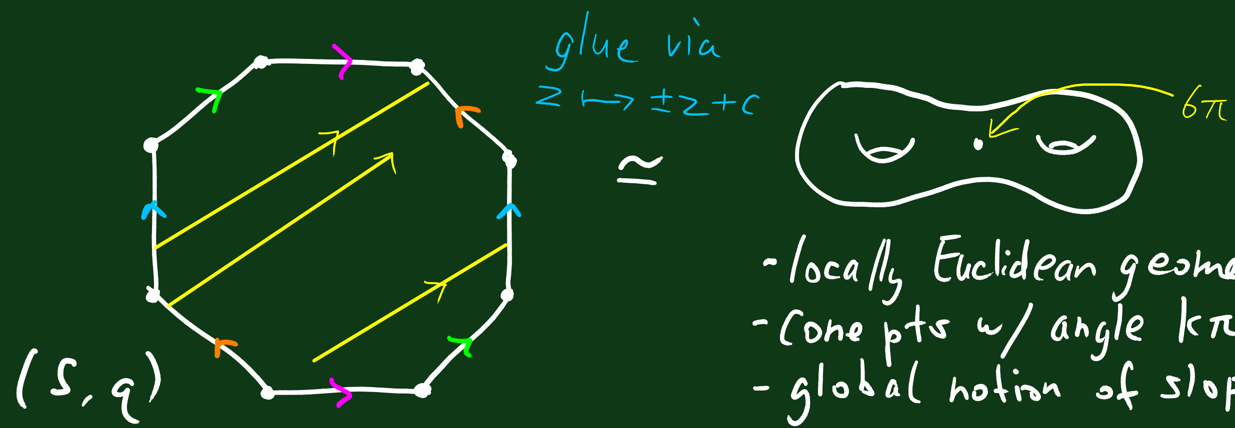
$$\mathrm{Mod}^{\pm}(S,Z) \xrightarrow{\ \simeq\ }{}^{*} \mathrm{Aut}\,\{\mathcal{A}, \mathcal{C}, \mathcal{AC}\}(S,Z)$$

$$\| $$

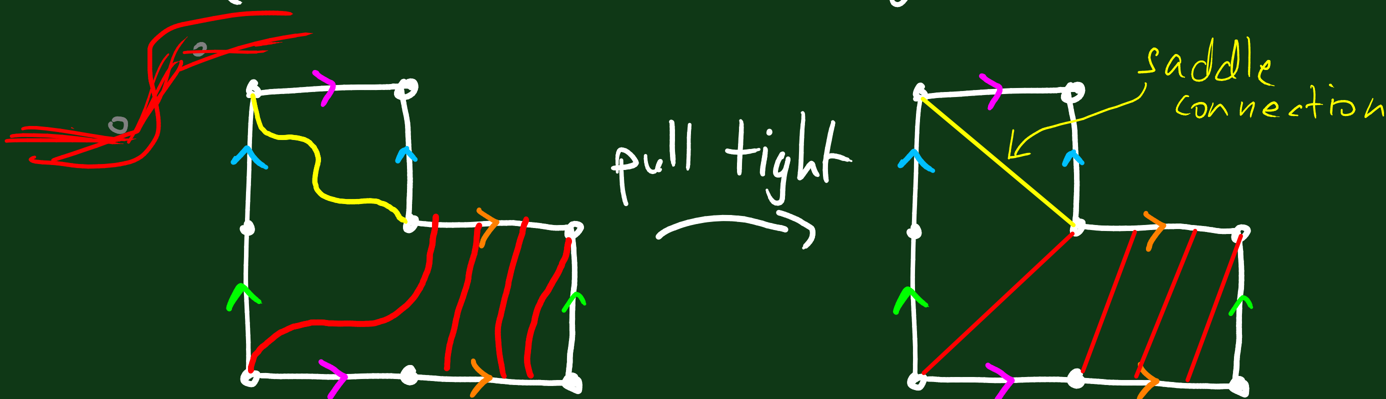
$$\mathrm{Homeo}^{\pm}(S,Z)/\text{isotopy}$$

[Ivanov, Irmak-McCarthy, ...]

(Half-) translation surfaces



- locally Euclidean geometry
- Cone pts w/ angle $k\pi$
- global notion of slope



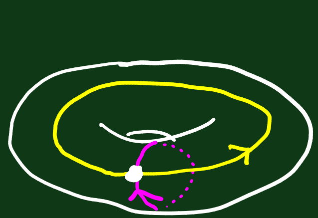
* Saddle connection graph $\mathcal{A}(S,q)$

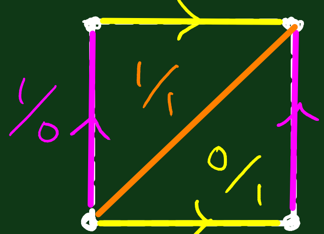
vertices : saddle conn's , edges : disjointness

[Minsky-Taylor] $\mathcal{A}(S,q) \hookrightarrow \mathcal{A}(S,Z)$ isometric emb.

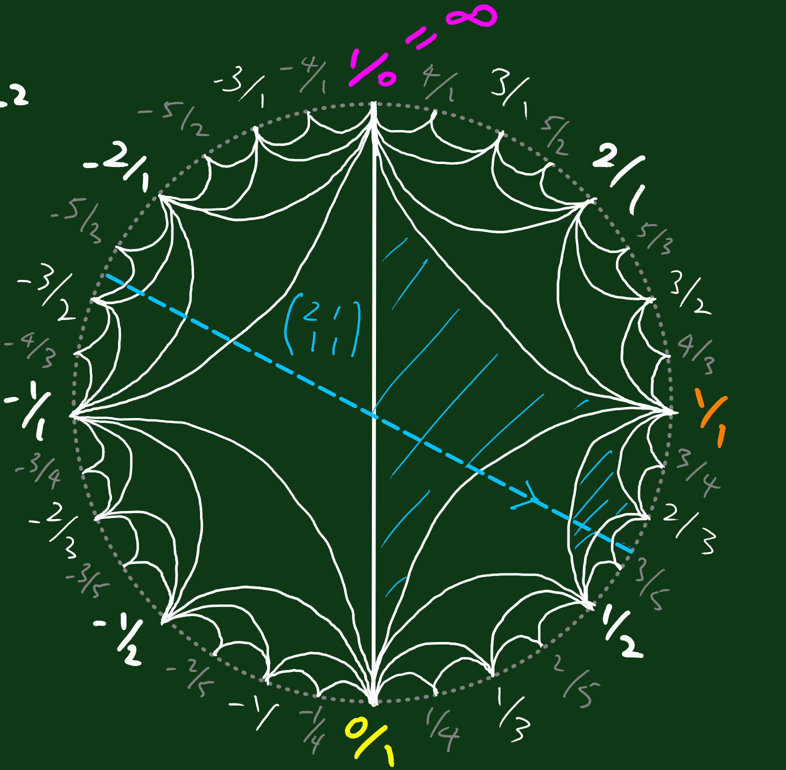
$\therefore$ connected & Gromov hyperbolic (Z = sing. pts)

Ex Once-marked torus $\mathbb{T}^2$



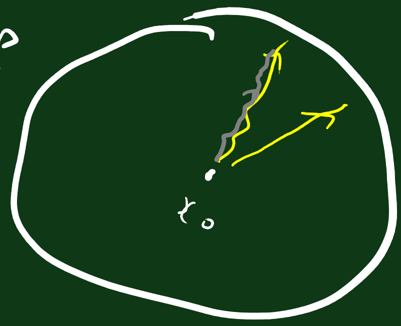


$\mathcal{A}(S,Z) = \mathcal{A}(S,q) =$ Farey graph



Gromov boundary

X hyp

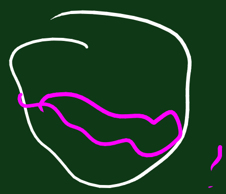


$\partial X \longleftrightarrow \{\text{geod rays}\} /$ fellow-travelling

[Klarreich] $\partial \mathcal{C}(S,2) \longleftrightarrow$ arational topological foliations

isom. emb.

$$\mathcal{A}(S,q) \hookrightarrow \mathcal{AC}(S,2) \simeq_{QI} \mathcal{C}(S,2)$$

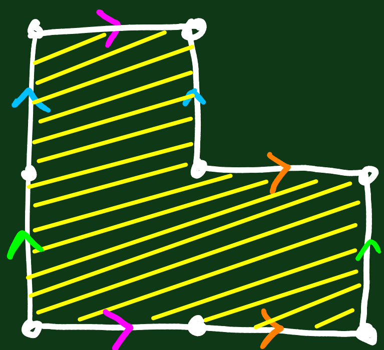


$$\partial \mathcal{A}(S,q) \hookrightarrow \partial \mathcal{AC}(S,2) \simeq \partial \mathcal{C}(S,2)$$

Q: Which foliations arise on $\partial \mathcal{A}(S,q)$?

[Disarlo-Pan-Randecker-T.]

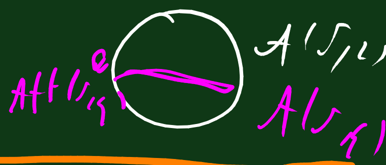
- $\partial_\infty \mathcal{A}(S,q) \longleftrightarrow$ straight foliations on (S,q) containing no saddle connections
- $\alpha_n \to \mathcal{F} \in \partial_\infty \mathcal{A}(S,q) \iff$ slopes $\theta_n \to \theta$ arational



Ex $\mathbb{T}^2$

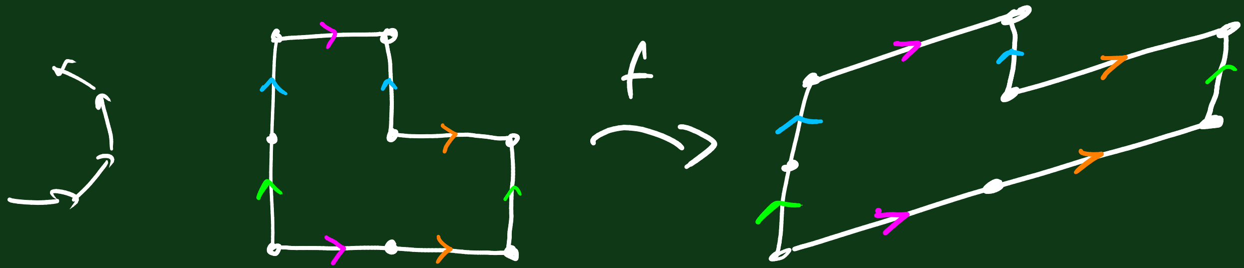
$\rightsquigarrow$ irrational slopes

Isomorphism vs. Quasi-isometry



[Disarlo-Randecker-T.] [Pan]

- Any (graph) isomorphism $\Phi : \mathcal{A}(S,q) \xrightarrow{\sim} \mathcal{A}(S',q')$ is induced by an affine diffeomorphism $f : (S,q) \to (S',q')$
- $\mathrm{Aff}(S,q) \xrightarrow{\sim} \mathrm{Aut}(\mathcal{A}(S,q))$



$F : (X,d) \to (X',d')$ is a (K,C)-quasi-isometry if

- $\dfrac{d(x,y) - C}{K} \leq d'(F(x),F(y)) \leq K\, d(x,y) + C \quad \forall x,y \in X$
- F is coarsely surjective.

[Disarlo-Pan-Randecker-T.]

$\forall (S,q)$, $\exists$ a surjective $(36,35)$-QI

$$\mathcal{A}(S,q) \to T_\infty$$

regular ∞-valent tree

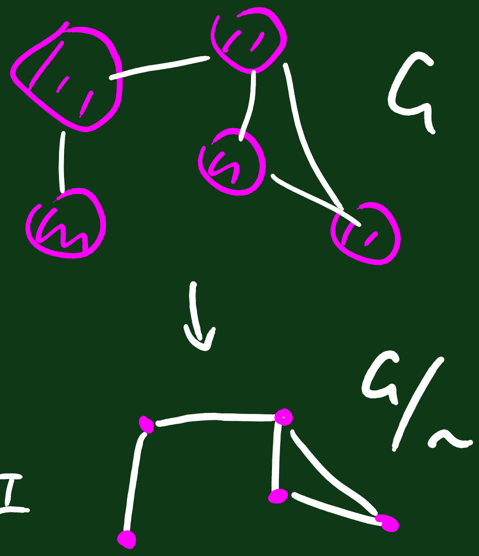
Only 1 QI-class!

[Rafi-Schleimer] $\mathcal{C}(S) \sim_{QI} \mathcal{C}(S') \iff \mathcal{C}(S) \cong \mathcal{C}(S')$

Q: What about the arc graph?

Quotient graphs

- G graph
- $\sim$ equiv. rel on $V(G)$



If each $\sim$-class has diam $\leq K$

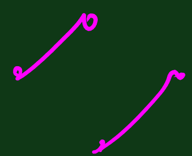
then $G \to G/\sim$ is a $(K+1, K)$-QI

Graph of slopes $\mathcal{G}(S,q)$

$\Theta : \mathcal{A}(S,q) \to \mathcal{G}(S,q)$

$(2,1)$-QI

identify parallel saddle conns.



Strategy:

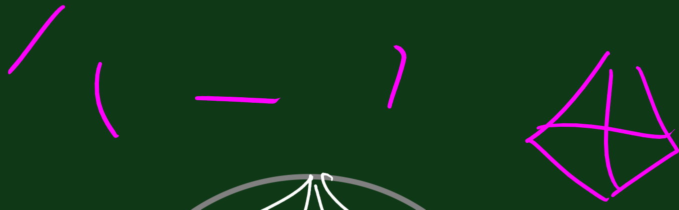
Find partition of $V(\mathcal{G}(S,q))$ s.t.

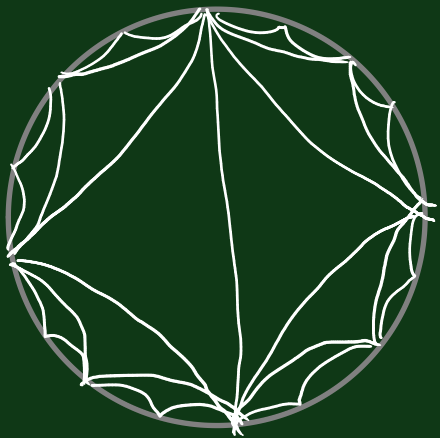
- each $\sim$-class has diam ≤ 17
- $\mathcal{G}(S,q)/\sim \;\cong\; T_\infty$

Guiding principle:

Pretend $\mathcal{G}(S,q)$ is the Farey graph

Picture on disc



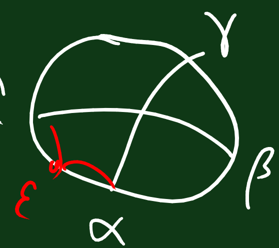


Farey: $\hat{\mathbb{Q}} \subset \mathbb{RP}^1$

$\mathcal{G}(S_{,q}) \subset \mathbb{RP}^1$

Linking slopes Lemma

$\alpha, \beta, \gamma, \delta \in \mathcal{A}(S_{,q})$ s.t.

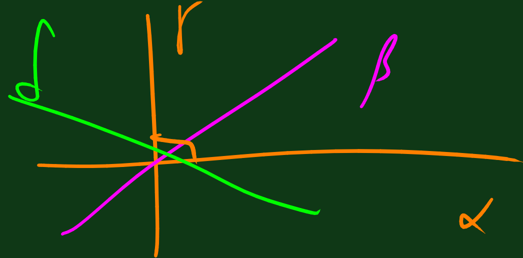


$$\Rightarrow d_{\mathcal{A}(S_{,q})}(\{\alpha,\gamma\}, \{\beta,\delta\}) \leq 2$$

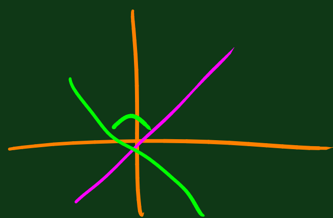
Apply affine transf

$\rightsquigarrow$ α vertical

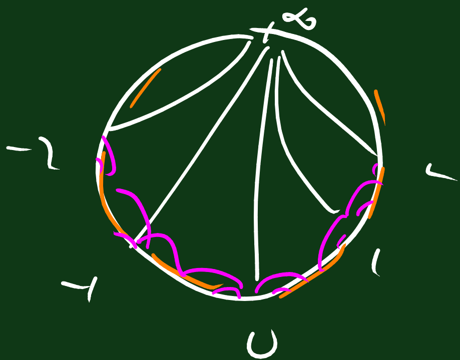
γ horizontal



Let ε be a shortest s.c.



α, γ disjoint

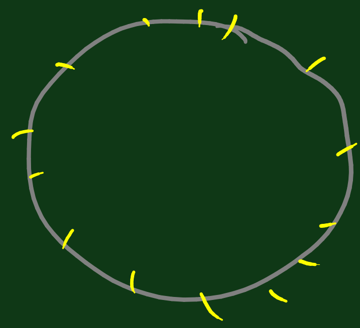


Balls in $\mathcal{G}(S,q)$

Fix $\theta_0 \in \mathcal{G}(S,q)$, set $B(k) = B(\theta_0, k)$

Prop $\forall k \geq 0$, $B(k)$ is closed in $\mathbb{RP}^1$

↳ $\mathbb{RP}^1 \setminus B(k)$ is a countably infinite union of open intervals.

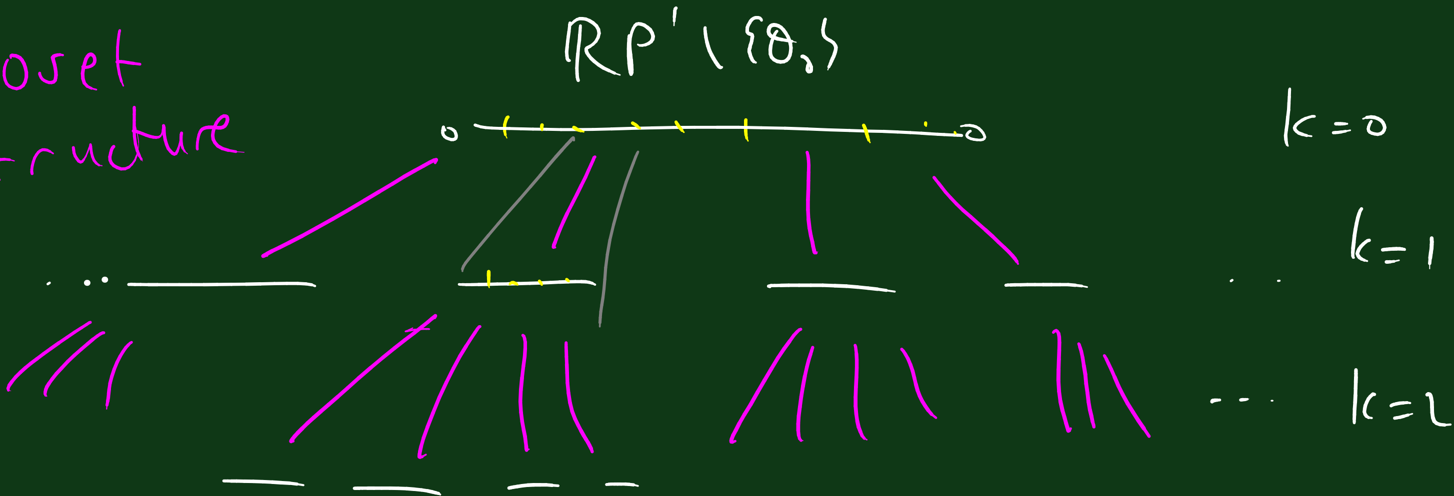


Set $\mathcal{I}_k = \{\text{max. intervals in } \mathbb{RP}^1 \setminus B(k)\}$

$$\mathcal{I} = \bigsqcup_{k \geq 0} \mathcal{I}_k$$

Lem Each $I \in \mathcal{I}_k$ contains ∞'ly many slopes from $B(k+1)$.

Poset structure



Hasse diagram of $\mathcal{I} \cong T_\infty$

A Partition

$I \subseteq I' \Rightarrow G(I) \subseteq G(I')$

Given an interval $I \subset \mathbb{RP}^1$, let

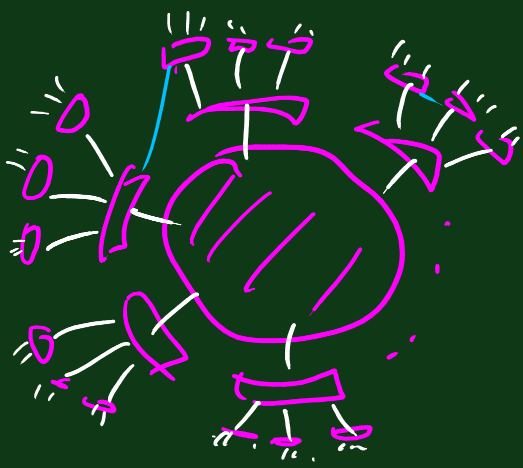
$$G(I) := \{\theta \in \mathcal{G}(S,q) \mid \theta \in I\} \subseteq \mathcal{G}(S,q)$$

Prop $G(I)$ spans a connected subgraph of $\mathcal{G}(S,q)$

For $I \in \mathcal{I}_k$, let

$$U(I) = \begin{cases} B(3) & , k = 0 \\ G(I) \cap S(k+3) & , k \geq 1 \end{cases}$$

slice



$I \subset I'$

$k+1 \quad k$

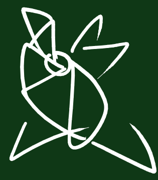
Prop $\forall I \in \mathcal{I}$, $\operatorname{diam}(U(I)) \leq 17$

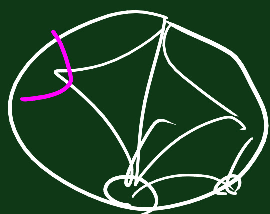
Prop $\mathcal{G}(S,q)/\sim \; \cong T_\infty$

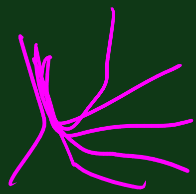
Thm $\mathcal{G}(S,q) \to T_\infty$ is a $(18,17)$-QI

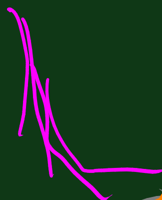
Boundary revisited

$\partial T_\infty \longleftrightarrow$ geod rays from fixed base point



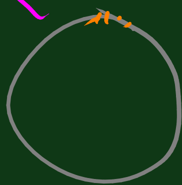






$x \in \partial T_\infty \longleftrightarrow$ nested sequence

$$I_0 \supset I_1 \supset I_2 \supset I_3 \supset \dots$$



Lem $\bigcap I_k = \{\theta\}$, $\theta \in \mathbb{RP}^1 \setminus \mathcal{G}(S,q)$

arational slope

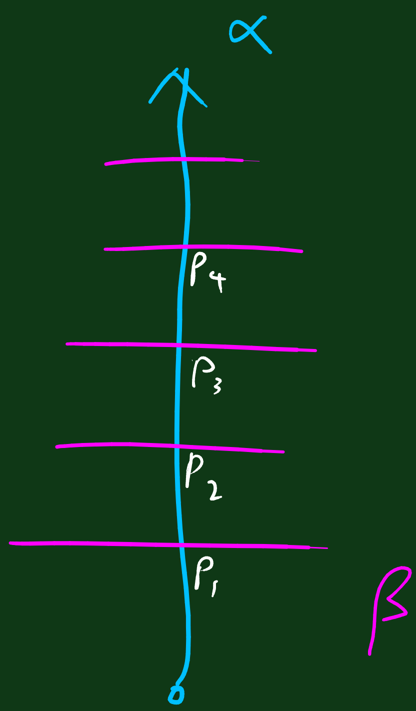
↳ $\partial \mathcal{G}(S,q) \longleftrightarrow \mathbb{RP}^1 \setminus \mathcal{G}(S,q)$

↳ $\partial \mathcal{A}(S,q) \longleftrightarrow$ straight arational foliations

Bicorn paths in $\mathcal{A}(S,z)$ (generalising unicorn paths)

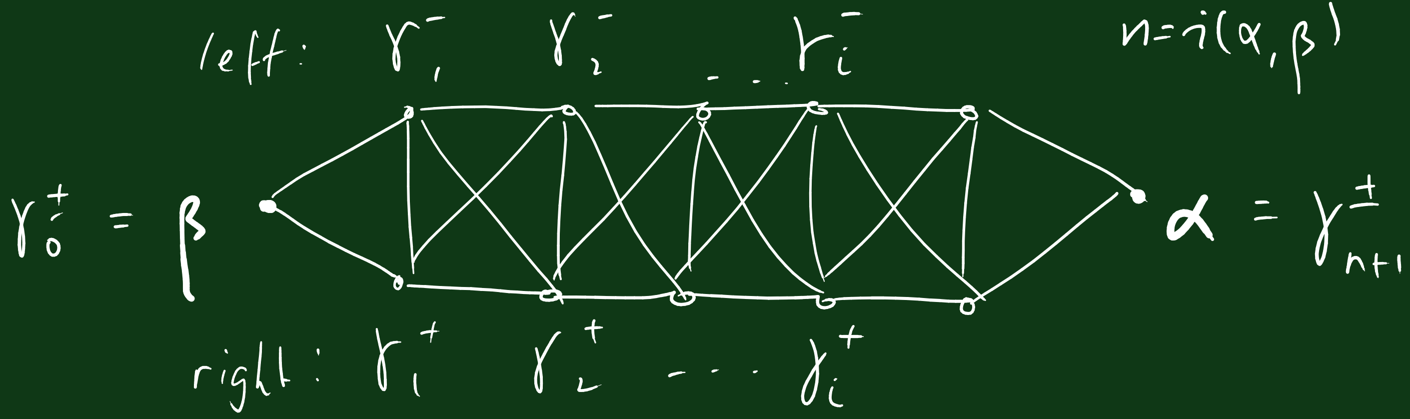
[Hensel – Przytycki – Webb]

$\alpha, \beta \in \mathcal{A}(S,z)$



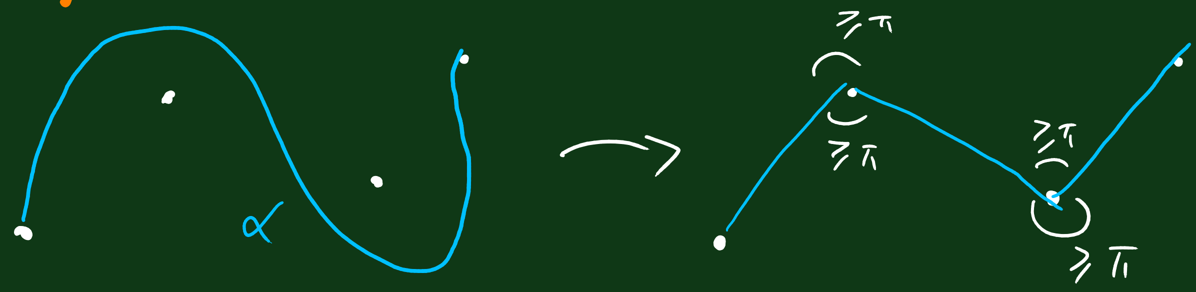
Ladder Lemma

$$i(\gamma_i^{\pm}, \gamma_i^{\mp}) = i(\gamma_i^{\pm}, \gamma_{i+1}^{\pm}) = i(\gamma_i^{\pm}, \gamma_{i+1}^{\mp}) = 0$$



Prop: Left/right ladder paths are a unif. Hausdorff dist. from any geodesic (using H-P-W)

Straight ladder paths

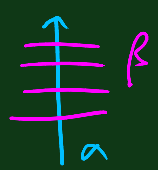


$\sigma_q(\alpha) :=$ set of saddle conn's appearing on geod. rep. of α

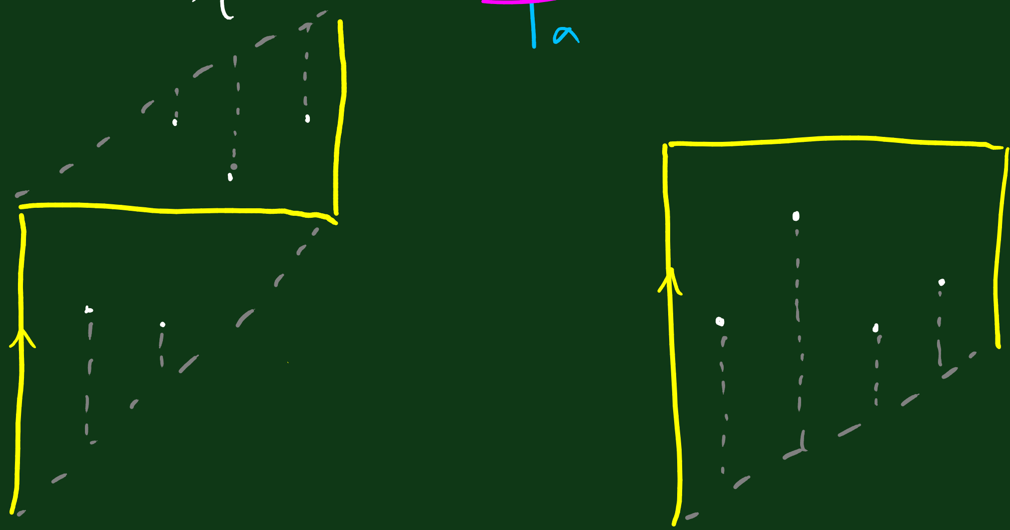
[Minsky–Taylor] $\alpha, \beta \in \mathcal{A}(S,2)$

- $i(a,a') = 0, \quad \forall a, a' \in \sigma_q(a)$
- $i(\alpha,\beta) = 0 \Rightarrow i(a,b) = 0, \; \forall a \in \sigma_q(\alpha), b \in \sigma_q(\beta)$

Let $\alpha, \beta \in \mathcal{A}(S,q)$



γ_i^+



$\delta_i^\pm :=$ first sadd. conn. appearing on geod rep of $\gamma_i^\pm$

Cor $i(\delta_i^\pm, \delta_i^\mp) = i(\delta_i^\pm, \delta_{i+1}^\pm) = i(\delta_i^\pm, \delta_{i+1}^\mp) = 0$

$\rightsquigarrow$ left / right straight ladder paths in $\mathcal{A}(S,q)$

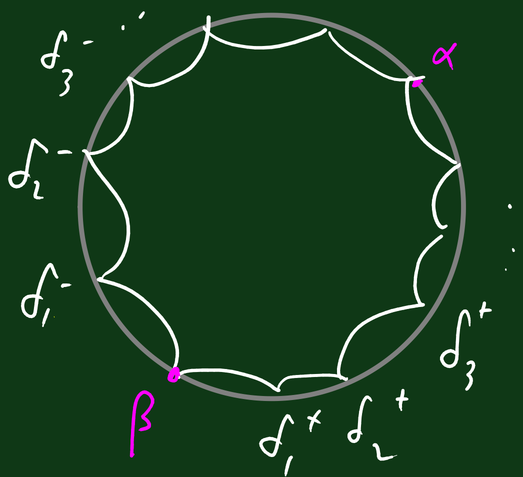
Nice properties

Prop: Monotone slopes

$\forall \alpha, \beta \in A(S,q)$ non-parallel, the slopes of

$$\beta, \delta_1^\pm, \delta_2^\pm, \ldots, \delta_n^\pm, \alpha$$

are monotone



Prop: Close to every path

$\forall \alpha, \beta \in A(S,q)$ and any path P from α to β in $A(S,q)$,

$$\{\beta, \delta_1^\pm, \delta_2^\pm, \ldots, \delta_n^\pm, \alpha\} \subseteq N_3(P)$$

$\leadsto$ $A(S,q)$ is QI to a tree by Manning's Bottleneck Criterion

The End!!