2022. 4. 21.

YMSC Topology Seminar

# Wilson lines & $\mathcal{A}=\mathcal{U}$ problem for moduli spaces of $G$-local systems

joint work w/ Hironori Oya & Linhui Shen

1) Introduce "Wilson lines" on $\mathcal{A}^\times_{G,\Sigma}$

2) Applications → cluster algebra
→ (skein theory)

## Wilson loops


## Wilson lines


$$\begin{array}{ccc} \rho_{[\gamma]} : \mathcal{L}oc_{G,\Sigma} & \longrightarrow & [G/\mathrm{Ad}G] \\ & & \downarrow \mathrm{tr}_V \\ & & \mathbb{C} \end{array}$$

$$\begin{array}{ccc} g_{[c]} : \mathcal{A}^\times_{G,\Sigma} & \longrightarrow & G \\ & & \downarrow c^V_{f,v} \\ & & \mathbb{C} \end{array}$$

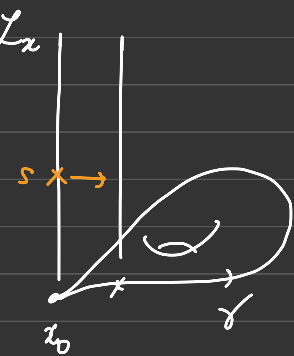
# §1 Introduction

## 1) Local systems & monodromy (Wilson loops)

$\Sigma$ : a closed surface, $G$ : an (alg.) Lie group

- A <u>$G$-local system</u> on $\Sigma$ is a principal $G$-bundle $\mathcal{L}$ equipped w/ a flat connection.

  ↪ parallel transport




- Associated to $\mathcal{L}$ is its monodromy hom.

$$\rho : \pi_1(\Sigma, x_0) \longrightarrow G.$$

Its conjugacy class determines the isom. class of $\mathcal{L}$.

⇝ Moduli space (stack) of $G$-local systems

$$\mathcal{L}oc_{G,\Sigma} := \left[ \mathrm{Hom}(\pi_1(\Sigma), G) / G \right]$$

- alg. str. of $\mathrm{Loc}_{G,\Sigma}$ is complicated (e.g. non-rational)
- "quantization" of $\mathrm{Loc}_{G,\Sigma}$? (cf. skein theory)
- good coordinate systems?

$$\rightsquigarrow \begin{cases} \Sigma \rightsquigarrow \text{marked surface} \\ \mathrm{Loc}_{G,\Sigma} \rightsquigarrow \text{"decorated" moduli space} \end{cases}$$

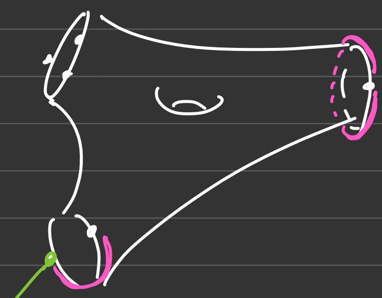
2) Fock–Goncharov's moduli space $\mathcal{A}_{G,\Sigma}$

A marked surface $(\Sigma, \mathbb{M})$ is a compact ori. surface $\Sigma$ equipped w/ a fin. set $\mathbb{M} \subset \Sigma$ of "marked pts".

<u>Today</u> Assume $\mathbb{M} \subset \partial\Sigma$

$\mathbb{B} := \{$ conn. comp's of $\partial\Sigma \setminus \mathbb{M} \}$

(boundary intervals)



$G$ : a $\underbrace{\text{semisimple}}_{}$ alg. group $/\mathbb{C}$ (e.g. $G = SL_N$)

simply-connected

Choose : $G \supset B^{\pm} \supset H$ $U^{\pm} := [B^{\pm}, B^{\pm}]$

opposite Borels — Cartan — unipotent

An element of $\mathcal{A}_G := G/U^+$ is called a <u>decorated flag</u>.

$\ni g.[U^+]$

e.g. $\mathcal{A}_{SL_N} = \left\{ \begin{array}{l} 0 \subset F_1 \subset \cdots \subset F_N = \mathbb{C}^N, \ \dim F_i = i \\ f_i \in \Lambda^i F_i \setminus \{0\}, \ i = 1, \cdots, N-1 \end{array} \right\}$ ← flags ← decorations

## <u>Moduli space $\mathcal{A}_{G,\Sigma}$</u>

$\mathcal{L}$ : a $G$-loc. sys. on $\Sigma$ $\rightsquigarrow$ $\mathcal{L}_{\mathcal{A}} := \mathcal{L} \times_G \mathcal{A}_G$

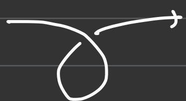
A <u>decoration</u> of $\mathcal{L}$ is a flat section $\alpha$ of $\mathcal{L}_{\mathcal{A}}$ defined near $\mathbb{M}$.

$\mathcal{A}_{G,\Sigma}$ "$:=$" moduli space of pairs $(\mathcal{L}, \alpha)$.

Slight modification : twistings (for positivity)

- $T'\Sigma := T\Sigma \setminus \{0\}$ : punctured tangent bundle.

  We actually consider twisted $G$-local systems $\mathcal{L}$, which are defined on $T'\Sigma$ s.t. $\rho(\text{fiber}) = s_G$

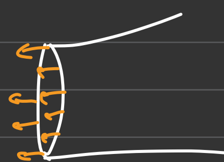
 $= s_G \cdot \longrightarrow$

a special central element

e.g. $s_{SL_N} = (-1)^{N-1}$

- Lift $\partial\Sigma \hookrightarrow T'\Sigma$ by <u>outward</u> tangent vectors.

  $$\begin{array}{ccc} \partial\Sigma & \hookrightarrow & T'\Sigma \\ \rotatebox{90}{$\in$} & & \\ x & \longmapsto & (x, v_{\mathrm{out}}(x)) \end{array}$$



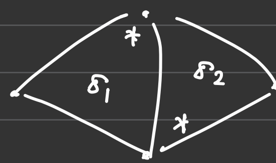
<u>Def</u> ([FG'06])

$\mathcal{A}_{G,\Sigma}$ := moduli space of decorated twisted $G$-local systems $(\mathcal{L}, \alpha)$.

[FG'06], [Le'16], [GS'19]

3) Cluster structure on $\mathcal{A}_{G,\Sigma}$ (quick review)

We have cluster coordinate systems




$$(A_i^{\Delta})_{i \in I} : \mathcal{A}_{G,\Sigma} \longrightarrow \mathbb{C}^I$$

associated w/ "decorated" triangulations of $\Sigma$.

$\left\{ \begin{array}{l} \text{a reduced word of } w_0 \in W(G) \\ \text{a choice of vertex} \end{array} \right.$ for each triangle

They give birational equiv's $\mathcal{A}_{G,\Sigma} \xrightarrow{\sim} (\mathbb{C}^\times)^I$,

and transition maps are cluster $K_2$-transformations

In particular, we get:

$$\underbrace{\mathscr{A}_{\mathfrak{g},\Sigma}}_{\text{cluster alg.}} \subset \underbrace{\mathscr{U}_{\mathfrak{g},\Sigma}}_{\text{upper cluster alg}} \subset \mathcal{K}(\mathcal{A}_{G,\Sigma})$$

inside the field of rational functions.

General problem : $\mathcal{A} \overset{?}{=} \mathcal{U}$

explicit generators

categorified

geometric

(function ring of the "cluster variety")

Main Theorem (I.- Oya- Shen)

Assume : • $\Sigma$ has $\geq 2$ marked points

• $G$ admits a non-triv. "minuscule" rep.

(i.e. not of type $E_8, F_4, G_2$)

Then

an open subspace

$$\boxed{\mathcal{A}_{\mathfrak{g},\Sigma} = \mathcal{U}_{\mathfrak{g},\Sigma} = \mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma})}$$

• Muller '16 : $\mathcal{A} = \mathcal{U}$ for "locally acyclic" cases

In particular for $G = SL_2$.

• Shen - Weng '21 : $\mathcal{A} = \mathcal{U}$ for double Bott-Samelson cells.

In particular for $\Sigma =$ disks.

## (11) Strategy


1) Prove $\mathcal{U}_{\mathfrak{g},\Sigma} \cong \mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma})$ by a "covering" argument.

$$\mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma}) = \bigcap_{E \in e(\Delta)} \mathcal{O}(\mathcal{A}^{\Delta;E}_{G,\Sigma}) = \bigcap_{E} \mathcal{U}^{\Delta;E}_{\mathfrak{g},\Sigma} = \mathcal{U}_{\mathfrak{g},\Sigma}$$

3-gon, 4-gon cases

upper bound thm

2) Prove that $\mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma})$ is generated by matrix coefficients of Wilson lines.

3) Among those, pick up a generating set which are cluster monomials.

Then $$\mathcal{A}_{\mathfrak{g},\Sigma} \subset \mathcal{U}_{\mathfrak{g},\Sigma} = \mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma}) \subset \mathcal{A}_{\mathfrak{g},\Sigma},$$

and we are done.

# §2. Wilson lines on $\mathcal{A}^{\times}_{G,\Sigma}$

**Fundamental groupoid**   Fix $x_E \in E$, $E \in \mathbb{B}$

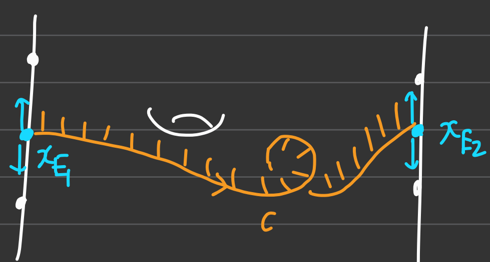
Let $\Pi_1(T'\Sigma, \mathbb{B}^{\pm})$ be the groupoid,

where obj. $E^{\pm} := (x_E, \pm \underbrace{v_{ori}}_{\text{positive v.f. along } \partial\Sigma}(x_E)) \in \partial(T'\Sigma)$

morph. $[c] : E_1^{\varepsilon_1} \longrightarrow E_2^{\varepsilon_2}$.

homotopy classes of paths in $T'\Sigma$

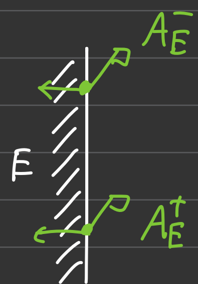
("framed arc classes")




$[c] : E_1^- \longrightarrow E_2^+$

## Wilson lines

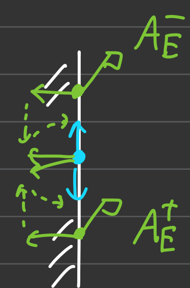
Let $\mathcal{A}^{\times}_{G,\Sigma} \subset \mathcal{A}_{G,\Sigma}$ be the open subspace consisting of $(\mathcal{L}, \alpha)$ s.t. the pair $(A_E^-, A_E^+)$ of decorated flags associated to each $E \in \mathbb{B}$ is generic.




$\rightsquigarrow$ "pinnings" [GS'19]

$$\begin{cases} p_{E^-} := (A_{E^-}, \underbrace{B_{E^+}}) & \text{at } E^- \\ p_{E^+} := (A_{E^+}, \underbrace{B_{E^-}}) & \text{at } E^+ \end{cases}$$

underlying flag




<u>Remark</u> The space of generic pairs $(A_1, B_2) \in \mathcal{A}_G \times \mathcal{B}_G$ is a principal $G$-space

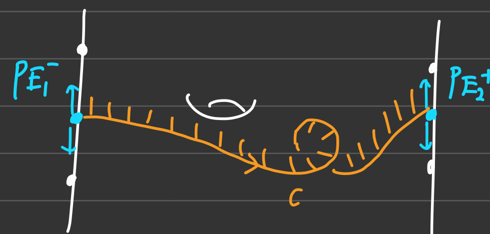
$\Rightarrow$ pinnings determines <u>local trivializations</u> of $\mathcal{L}$.

- $p_{std} := ([U^+], B^-)$

**Def** The Wilson line $g = g_{[c]}([\mathcal{L}, \alpha])$

along a framed arc class $[c] : E_1^{\varepsilon_1} \longrightarrow E_2^{\varepsilon_2}$

is the unique element s.t.




$$p_{E_2^{\varepsilon_2}} = g \cdot p_{std}^{*} = g\, \underbrace{\overline{w_0}}_{\text{a lift } \in N_G(H) \text{ of } w_0 \in W(G)}{}^{-1} \cdot p_{std}$$

in the loc. triv. given by $p_{E_1^{\varepsilon_1}}$ (extended along $[c]$)

**Remark**

- It defines a morphism $g_{[c]} : \mathcal{A}^{\times}_{G,\Sigma} \longrightarrow G$.
- "Twisted" Wilson lines $g^{tw}_{[c]} := g_{[c]}\, \overline{w_0}^{-1}$

are multiplicative : $\boxed{g^{tw}_{[c_1]*[c_2]} = g^{tw}_{[c_1]} \cdot g^{tw}_{[c_2]}}$

**Theorem** For any unpunctured marked surface $\Sigma$, Wilson lines give a closed embedding

$$\begin{array}{ccc} \mathcal{A}^{\times}_{G,\Sigma} & \hookrightarrow & \mathrm{Hom}(\Pi_1(T'\Sigma, \mathbb{B}^{\pm}), G) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [\mathcal{L}, \alpha] & \longmapsto & g_{\bullet}^{tw}([\mathcal{L}, \alpha]) \end{array}$$

← affine variety

**Cor** $\mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma})$ is generated by the matrix coefficients of (twisted) Wilson lines.

**Recall**: Peter–Weyl isom

$$\mathcal{O}(G) \cong \bigoplus_{\lambda : \text{irrep.}} (V_\lambda \otimes V_\lambda^*)^G$$

$$\langle f, g.v \rangle_V \longleftarrow\!\!\shortmid (v, f)$$

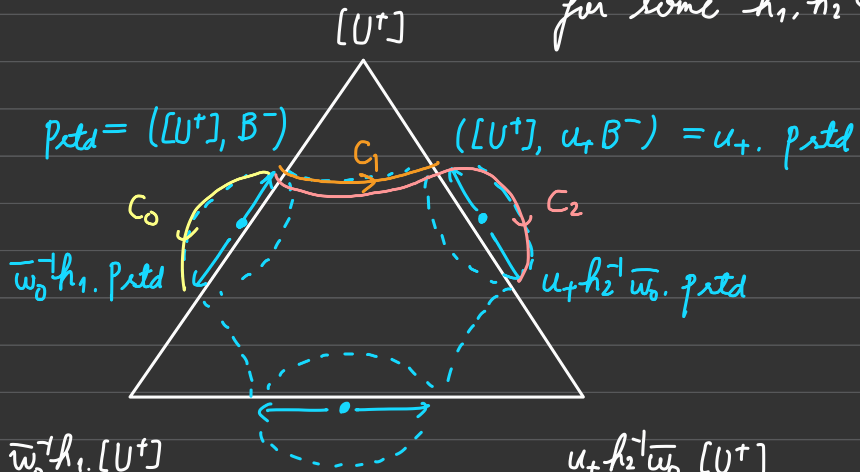
## Example ($\Sigma = T$ : triangle)

$\mathcal{A}_{G,T} \cong G \backslash \mathcal{A}_G^{\times 3}$ : config. space of 3 dec. flags

A generic config. has a representative

$$(A_1, A_2, A_3) = \left( [U^+],\ \overline{w}_0^{-1} h_1 . [U^+],\ u_+ h_2^{-1} \overline{w}_0 . [U^+] \right)$$

for some $h_1, h_2 \in H$, $u_+ \in U_*^+$




$\Rightarrow \quad g_{[C_0]} = \overline{w}_0^{-1} h_1 \overline{w}_0 = w_0(h_1) \in H$ (boundary)

$g_{[C_1]} = u_+ \overline{w}_0 \in U_*^+ \overline{w}_0$ (corner)

$g_{[C_2]} = u_+ h_2^{-1} \overline{w}_0^2 = u_+ h_2^{-1} s_G \in B^+$ ("simple")

## §3. Wilson lines & cluster variables

Special kind of matrix coefficients of

Wilson lines are cluster variables (up to frozen).

1) Generalized minors. [BFZ'07]

$w, w' \in W(G)$, $\lambda$ : dominant weight

$\rightsquigarrow \Delta_{w\lambda, w'\lambda}(g) := \langle \overline{w}.f_{\lambda^*}, g\overline{w'}.v_\lambda \rangle_{V_\lambda}$

<u>Lem</u> $\mathcal{O}(G)$ is generated by generalized minors, except for $E_8$, $F_4$, $G_2$.

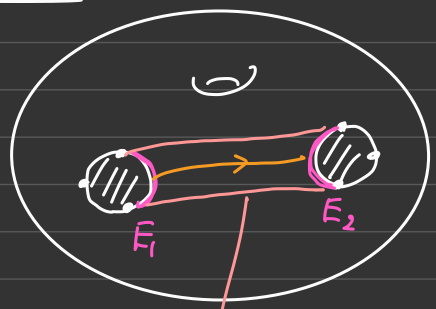
2) Simple Wilson lines

$[c] : E_1^- \longrightarrow E_2^-$ w/ • $E_1 \neq E_2$

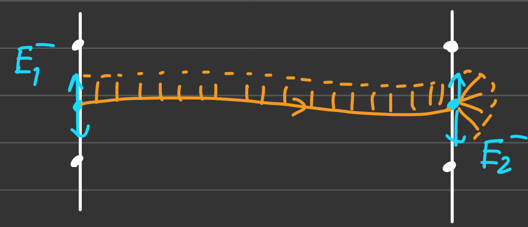
• "standard" framing

$\Rightarrow g_{[c]}$ is called a simple Wilson line.

$E_1 \neq E_2$

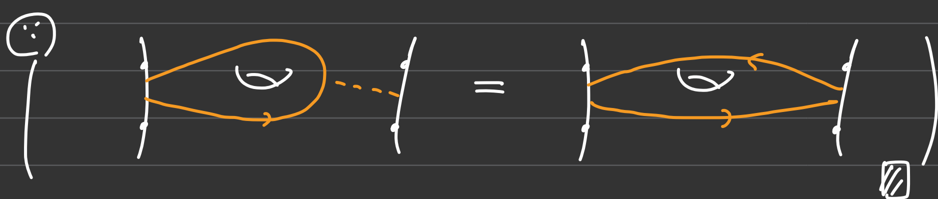



std. framing




cf. "good lift" of Costantino – Lê '19

Lem If $\Sigma$ has $\geq 2$ marked points, then $\pi_1(T'\Sigma, \mathbb{B}^\Sigma)$ is generated by simple classes.



Combining two lemmas :

Cor If $G \neq E_8, F_4, G_2$ & $|\mathbb{M}| \geq 2$, then $\mathcal{O}(\mathcal{A}^\times_{G,\Sigma})$ is gen'ed by generalized minors of simple Wilson lines.

<u>Proposition</u> For a simple class $[c] : E_1^- \longrightarrow E_2^-$,

GS cluster var.

$$\Delta_{w\varpi_s, w'\varpi_s}(g_{[c]}) = \frac{A_{w\varpi_s, w'\varpi_s}}{\text{frozen var's on } E_1 \,\&\, E_2}$$

☺ $\left( \begin{array}{l} \text{BFZ variables} \propto \text{GS variables} \\ H^2 \times G^{w_0, w_0} \cong \mathcal{A}^{\times}_{G, [\,\dots\,]} \end{array} \right)$ $\blacksquare$

Then we get $\mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma}) \subseteq \mathcal{A}_{g,\Sigma}$

(under the assumption above). //

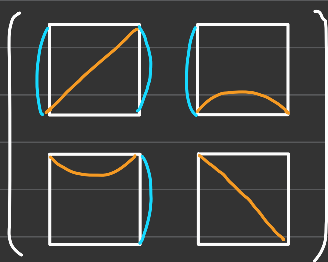
# §4. Examples & skein realizations

## $G = SL_2$

$\mathfrak{sl}_2$-skein alg. at $q=1$

Muller'16: $\mathcal{A}_{\mathfrak{sl}_2,\Sigma} \cong \underbrace{\mathcal{S}^1_{\mathfrak{sl}_2,\Sigma}[\partial^{-1}]}$ "boundary localized"

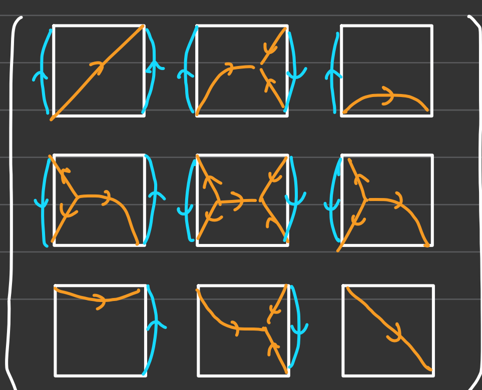
cluster var. $\leftrightarrow$ arc

$g_{[c]} =$  $\in SL_2\left(\mathcal{S}^1_{\mathfrak{sl}_2,\Sigma}[\partial^{-1}]\right)$

— : inverse

## $G = SL_3$

I.-Yuasa'21: $\mathcal{A}_{\mathfrak{sl}_3,\Sigma} \supseteq \mathcal{S}^1_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}]$

$g_{[c]} =$ 

# Final remarks

- This also shows $\mathcal{O}(\mathcal{A}^{\times}_{SL_3,\Sigma}) \subseteq \mathcal{S}^1_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}]$,

  and hence $\mathcal{A}_{\mathfrak{sl}_3,\Sigma} \cong \mathcal{S}^1_{\mathfrak{sl}_3,\Sigma}[\partial^{-1}]$.

- Similar story goes for $G = Sp_4$ (Type $C_2$)

  [I.-Yuasa] in prep.

- A quantum version of the formula

  " Wilson lines $\leftrightarrow$ cluster variables "

  establishes

$$\begin{pmatrix} \text{Reduced stated} \\ \mathfrak{g}\text{-skein alg} \end{pmatrix} \cong \begin{pmatrix} \text{clasped} \\ \mathfrak{g}\text{-skein alg} \end{pmatrix}$$

Bonahon-Wong, Lê, Higgins, H.K. Kim, ⋯      Muller, I.-Yuasa, ⋯

Thank You !