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Homological stability for the ribbon Higman–Thompson groups

Xiaolei Wu

J.w. Rachel Skipper

Shanghai Center for Mathematical Sciences

12.04.2022

History of Thompson groups

- [1965] The groups F, T, and V were first -defined by Richard Thompson. T and V are the first known examples of finitely presented infinite simple groups.

- [1974] Higman introduced what we call nowdays the Higman–Thompson groups.

- [70s] Fred–Heller, independtly Dydak, rediscovered F as the universal group encoding a free homotopy idempotent.

History of braided Thompson groups

- [2006] Brin and Dehornoy independently defined braided V.

- [2017] Thumann introduced ribbon Thompson groups.

- [$\approx$ 2020] Braided and ribbon version of Higman–Thompson groups was first studied by Aroca–Cumplido and Skipper-W.

History of braided Thompson groups

- [2004] Funar–Kapoudjian introduced the asymptotic mapping class group on an infinite type surface.

$\mathrm{Map}(S_g \setminus \mathcal{C})$

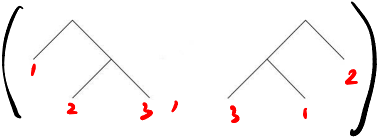
- [2006] Brin and Dehornoy independently defined braided V.

- [2017] Thumann introduced the ribbon Thompson groups.

- [$\approx$ 2020] Braided and ribbon version of Higman–Thompson groups was first studied by Aroca–Cumplido and Skipper-W.

Define Thompson's group V using paired tree diagram

- An element in V is a paired tree diagram $[T_1, \sigma, T_2]$ where T_1 and T_2 are two finite rooted binary trees with the same number of leaves, and σ is a bijection from the leaves of T_1 to T_2.

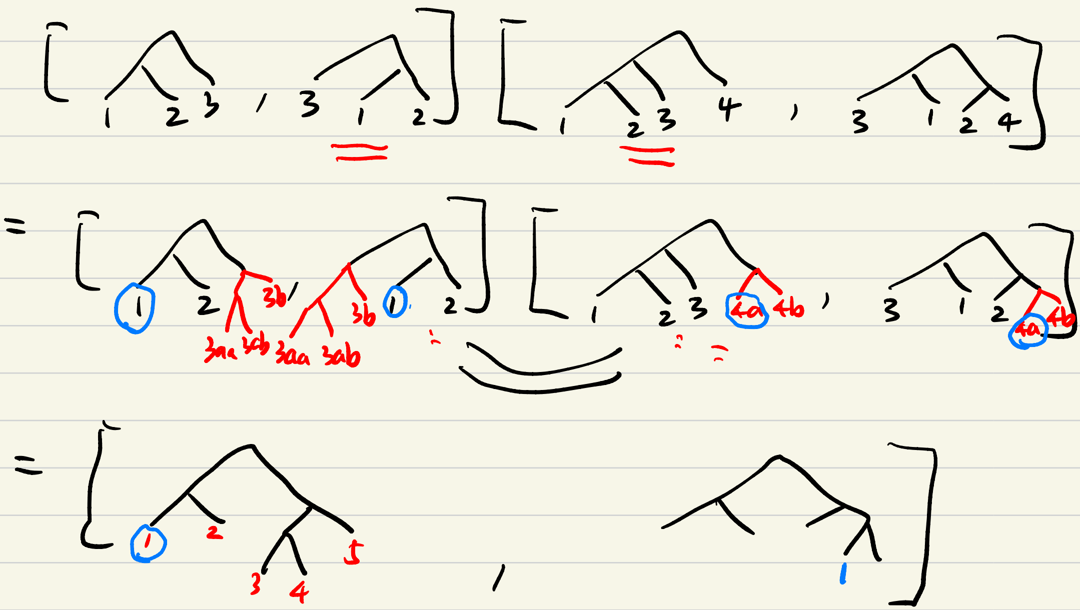


- Equivalence relation:

- Multiplication:

$[T_1, \sigma, T_2][T_2, \sigma', T_3] = [T_1, \sigma\sigma', T_3]$

$[T_1, \sigma, T_2)[T_3, \sigma', T_4)$



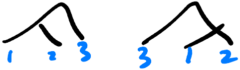
1 2 3 , 3 1 2
1 2 3 4 , 3 1 2 4
=
1 2 3b 3aa 3ab 3aa 3ab 3b 1 2
1 2 3 4a 4b , 3 1 2 4a 4b
=
1 2 3 4 5 , 1

F and T as subgroups of V

$V = \{ [T_1, \sigma, T_2] \mid \sigma \text{ bijection} \}$

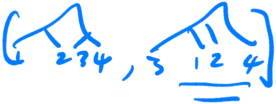
$\rotatebox{90}{$\subseteq$}$

$T = \{ \ \cdots \ \mid \sigma \ \text{cyclic permutation} \}$



$\rotatebox{90}{$\subseteq$}$

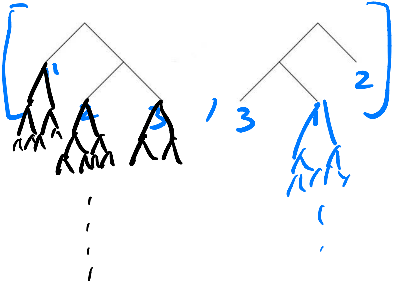
$F = \{ \ \cdots \ \mid \sigma \ \text{"id"} \}$



$$[T_1, \sigma, T_2]^{-1} = [T_2, \sigma^{-1}, T_1]$$

$$id = [\ \cdot\ ,\ \cdot\ ]$$

Thompson's group V as a subgroup of homeomorphisms of the Cantor set



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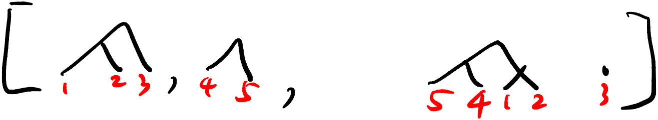
Higman–Thompson groups

$$V_1 = \{ [T_1, \sigma, T_2] \mid \sigma \text{ bijection} \}$$

$$V_r = \left\{ \left[\bigcup_{i=1}^{r} T_{1i}, \sigma, \bigcup_{i=1}^{r} T_{2i} \right] \,\middle|\, \sigma \text{ bijection} \right\}$$



$\in V_2$

Why V?

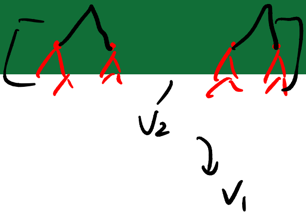
$\{[T, \sigma, T] / \sigma\}$

- V contains all finite groups.
- [Brown 1987] V is of type F_∞.
- V is dense in $\mathrm{Homeo}(\mathcal{C})$.
- [Szymik–Wahl 2019] V is acyclic.

G is of type F_∞ if it acts freely on a contractible CW-cpx X, s.t. X/G has finite in each dim.

$\tilde{H}_i(V,\mathbb{Z}) = \{0\}$ for any i.

How to show V is acyclic



$$2 \subseteq \frac{2}{2}$$

Step 1: Homological Stability

$$V_1 \underset{\subsetneq}{\leq} V_2 \leq V_3 \leq \cdots V_n$$

$$[T_1, \sigma, T_2] \to [T_1 \cup *, \bar{\sigma}, T_2 \cup *]$$

Thm: for any i and n, $(n \gg i)$

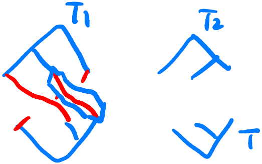
$$i_*: H_i(V_n) \to H_i(V_{n+1})$$

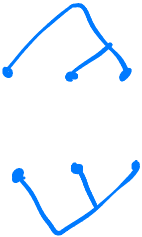
i_* is isom

Step 2: Stable homology.

Braided V [Brin 2006, Dehornoy 2006]

- An element in bV is a braided paired tree diagram $[T_1, b, T_2]$ where T_1 and T_2 are two finite rooted trees with the same number of leaves, b is a braid.





- Equivalence relation:

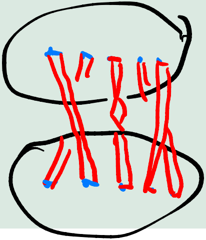
- multiplication:

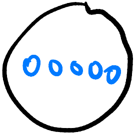
We define braided Higman–Thompson groups bV_r similarly.

Ribbon braid groups

Definition

An element in the ribbon braid group RB_n is





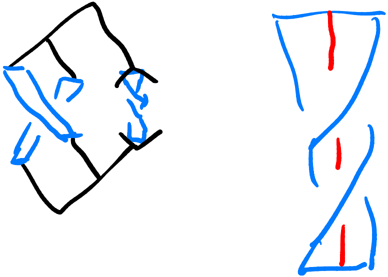
Fact

- B_n *can be identified with the mapping class group of a disk with* n *punctures.*
- PRB_n *can be identified with the mapping class group of a* n*-holed disk.*

Ribbon Higman–Thompson groups

- An element in RV_r is a braided paired tree diagram $[F_1, \mathfrak{r}, F_2]$ where F_1 and F_2 are two finite rooted forests with the same number of leaves, $\mathfrak{r}$ is a ribbon braid.

- Equivalence relation:



Question

Question

Is bV also acyclic? How about RV?

Remark

It is known that $H_1(bV) = 0$.

How one might prove bV is acyclic?

① $bV_1 \le bV_2 \le bV_3 \le \cdots$

②

Our theorem

Theorem (Skipper–W)

The inclusion maps

$$\iota_* : H_i(RV_r, \mathbb{Z}) \to H_i(RV_{r+1}, \mathbb{Z})$$

induce isomorphisms on homology in all dimensions $i \geq 0$, for all $r \geq 1$.

$$G_1 \leq G_2 \leq G_n \leq \cdots$$
$$\parallel \qquad \parallel$$
$$\mathrm{Aut}(X) \leq \mathrm{Aut}(X^{\oplus 2}) \leq \cdots \mathrm{Aut}(X^{\oplus n})$$

A quick review on Homogeneous category

Definition

A monoidal category $(\mathcal{C}, \oplus, 0)$ *is called homogeneous*

- 0 *is initial in* $\mathcal{C}$*;*
- $\mathrm{Hom}(A, B)$ *is a transitive* $\mathrm{Aut}(B)$*-set under postcomposition;*
- *The map* $\mathrm{Aut}(A) \to \mathrm{Aut}(A \oplus B)$ *taking* f *to* $f \oplus \mathrm{id}_B$ *is injective with image*

$$\mathsf{Fix}(B) := \{\phi \in \mathrm{Aut}(A \oplus B) \mid \phi \circ (\imath_A \oplus \mathrm{id}_B) = \imath_A \oplus \mathrm{id}_B\}$$

where $\imath_A \colon 0 \to A$ *is the unique map.*

The space $S_n(X)$

$Aut(X) \hookrightarrow Aut(X^2) \hookrightarrow \cdots \quad Aut(X^n)$

$G_1 \quad \leq \quad G_2 \quad \leq \cdots \quad G_n$

Let X be a object of the homogeneous category $(\mathcal{C}, \oplus, 0)$, then $\underline{S_n(X)}$ denote the simplicial complex

- Vertices: morphisms $f\colon X \to X^{\oplus n}$;
- p-simplices: $(p+1)$-sets $\{f_0, \ldots, f_p\}$ such that there exists a morphism $f\colon X^{\oplus p+1} \to X^{\oplus n}$ with $f \circ i_j = f_j$ for some order on the set, where

$$i_j = \imath_{X^{\oplus j}} \oplus \mathrm{id}_X \oplus \imath_{X^{\oplus p-j}}\colon X = 0 \oplus X \oplus 0 \longrightarrow X^{\oplus p+1}.$$

How to prove homological stability?

Theorem (Randal-Williams–Wahl)

Let $(\mathcal{C}, \oplus, 0)$ be a homogeneous category such that the space $S_n(X)$ is $\frac{n-2}{k}$ for some $k \geq 2$, then

$$H_i(\mathrm{Aut}(X^{\oplus n})) \longrightarrow H_i(\mathrm{Aut}(X^{\oplus n+1}))$$

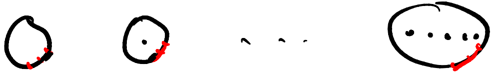
induced by the natural inclusion map is an isomorphism if $n \geq ki + 1$.

Key fact: B_n can be identified with the mapping class group of a disk with n punctures.

$B_1 \leq B_2 \leq B_3 \leq \cdots B_n$

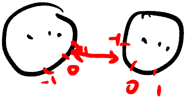
The homogeneous category consists of

- Objects:



$0 \qquad X \qquad \cdots \qquad X^n$

- monoid operation:



$X^n \oplus X^m = X^{n+m}.$

- morphisms:

$$\mathrm{Hom}(X^n, X^m) = \text{isotopy Class of based} = \begin{cases} \phi & n>m \\ \text{Homeo} & n=m \\ \text{embe} & n<m \end{cases}$$

An example: braid groups II

The complex $S_n(X)$:

- vertices: $\mathrm{Hom}(X, X^n)$  

- p-simplices: $p+1$ vertices with disjoint boundary loops outside the base pt.

Arc complex: Vertices: isotopy class of arcs connecting base pt and a marked pt.

Key difficulty

Build a geometric model for the Ribbon Higman–Thompson groups.

Asymptotic mapping class group I: rigid structure

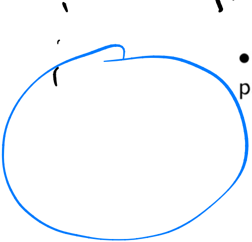
Map(D|e)

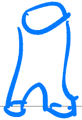
- Reconstruction of the surface:

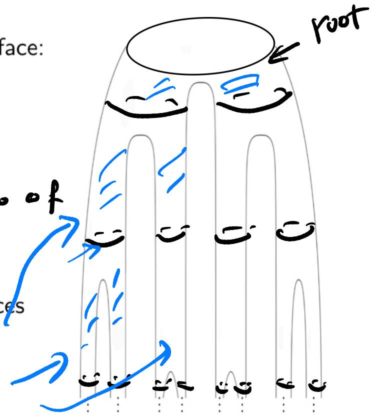
- Suited subsurface:

first connected union of large ad parts.

- Suited loops and parameterization of the pieces







Asymptotic mapping class group II: paired surface diagram

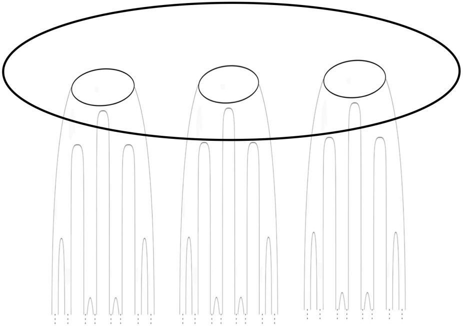
An element in $\mathcal{B}V = \mathcal{B}V_1$ is a paired surface diagram

$$[\Sigma_1, \phi, \Sigma_2]$$

such that

- Σ_1 and Σ_2 are suited subsurface with the same number of suited loops.
- ϕ is a homeomorphism from Σ_1 to Σ_2 which coincides with the parametrization of the suited loops.
- Equivalence relation: ~ isotopy
- Composition: ~ adding parts.

$\mathcal{B}_r$ for any r

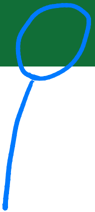


Isomorphism results

Theorem (Skipper-W)

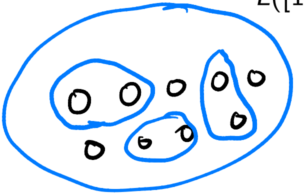
The groups $\mathcal{B}V_r$ and RV_r are isomorphic in a canonical way.

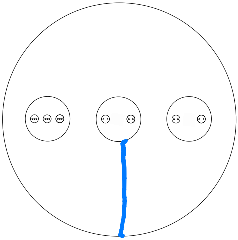
Key gadget: Lollipop



- Let $A = [0, 2]/1 \sim 2$.

- $L : (A, 0) \to (D \setminus \mathcal{C}, 0)$ is a lollipop if
  - L is an embedding.
  - $L\,|_{[1,2]}$ is isotopic to a suited loop in $D \setminus \mathcal{C}$,
  - $L\,|_{[0,1]}$ is an arc connecting the base point 0 to $L([1, 2])$.





The Lollipop complex

- vertices: isotopy classes of lollipops;

- $L_0, L_1, \cdots, L_p$, form a p-simplex if they are pairwise disjoint outside the base point 0 and there exists at least one suited loop which does not lie inside the disks bounded by the L_is.

The Lollipop complex is contractible

Theorem (Skipper-W 2021)

The Lollipop complex is contractible.

v''

$v' \in Lk(v)$

v

$Lk(v) \cap Lk(v'')$

Connectivity of the lollipop complex