FILTRATION OF COHOMOLOGY VIA SYMMETRIC SEMI SIMPLICIAL SPACES

Goal / Motto : Compare 'indecomposables in a topological sense' with 'indecomposables' in cohomology (derived). OR

Motivating Examples / Context :

1. $\mathrm{Mor}_n(\mathbb{P}^1, \mathbb{P}^r) = \{[f_0 : \dots f_r] : f_i$ homogeneous degn in 2 variables, no common zeroes$\}$

$\cap$

$\widetilde{\mathrm{Mor}}_n(\mathbb{P}^1, \mathbb{P}^r) = \{[f_0 : \dots f_r] : f_i$ degn hom. 2 var.$\}$

- Clear notion of (in)decomposables in a topological sense
- More precisely $\coprod_n \widetilde{\mathrm{Mor}}_n(\mathbb{P}^1, \mathbb{P}^r)$ is a (graded associative) module over $\mathrm{Sym}\,\mathbb{P}^1 \left(\coprod_n \mathrm{Sym}^n \mathbb{P}^1\right)$.

topological monoid

FACE MAPS

$$\mathbb{P}^1 \times \widetilde{\mathrm{Mor}}_n(\mathbb{P}^1, \mathbb{P}^r) \longrightarrow \widetilde{\mathrm{Mor}}_{n+1}(\mathbb{P}^1, \mathbb{P}^r)$$

$$[a:b],\ [f_0 : \cdots f_r] \longmapsto [(ay-bx)f_0 : (ay-bx)f_1 : \cdots (ay-bx)f_r]$$

dictates the module structure of the monoid $\mathrm{Sym}\,\mathbb{P}^1$.

$$\coprod_n \widetilde{\mathrm{Mor}}_n(\mathbb{P}^1, \mathbb{P}^r) - \left(\coprod_n \widetilde{\mathrm{Mor}}_n(\mathbb{P}^1, \mathbb{P}^r) \times \mathbb{P}^1\right)$$

Often $H^*(\mathrm{Mor}_n(\mathbb{P}^1, \mathbb{P}^r))$? (FW'13, Segal'70s)
we want to compute cohom. of the space of topological indecomposables.

In our framework 'ΔS category' makes a unified framework for relating top. indecomposables with cohom. indecomp.

$H^*(\mathrm{Mor}_n(\mathbb{P}^1, \mathbb{P}^r))$ in terms of H^* of $\coprod_n \widetilde{\mathrm{Mor}}_n(\mathbb{P}^1, \mathbb{P}^r)$ & $(\mathbb{P}^1)^p \times \widetilde{\mathrm{Mor}}_{n-p}(\mathbb{P}^1, \mathbb{P}^r)$ 'decomposables'.

H^* sing / $\mathbb{Q}$ or étale / $\mathbb{Q}_\ell$ (side remark: everything translates algebraically)

2. $\mathbb{P}^1$ can by replaced by any smooth proj. curve (aka any Riemann surface) of higher genus.

3. unordered Config space $\mathrm{UConf}_n X \subset \mathrm{Sym}^n X$ $H^*(\mathrm{UConf}_n X)$ (plenty exciting work!)

4. $\mathcal{L}$ line bundle on X (smooth proj.)

$H^*(U_{\mathcal{L}})$, $U_{\mathcal{L}} = \{$ smooth global section of $\mathcal{L}\}$

$X = \mathbb{P}^n$ (Tommasi '06)

X (Aumonier '21) homotopy theoretic methods

I. Banerjee

1. Motto
2. Motivating examples
3. ΔS - what is it anyway?
4. Toy computation

ΔS – why and what is it?

<u>why</u> Decomposables (e.g. $\coprod_n \widetilde{\mathrm{Mor}}_n(\mathbb{P}^1, \mathbb{P}^r)$ decomposables are $(\mathbb{P}^1)^p \times \coprod_n \widetilde{\mathrm{Mor}}_n(\mathbb{P}^1, \mathbb{P}^r)$ - for $p \geq 1$)

'Δ' involvement of (semi) simplicial spaces is unavoidable (think of an upgraded nerve thm.)

$[n] = \{0 < 1 < 2 < \dots n\}$

$\underline{\Delta S}$ (more generally ΔG) ΔG is called a crossed simplicial category if

$\{G_n\}_{n \geq 0}$ grps equipped with

a small cat. ΔG s.t (i) obj $[n]$ (ie that of Δ)

(Fiederowicz Loday '69
Krasaukas '68
independently)

(ii) $\Delta G \supset \Delta$ sub cat

(iii) $\mathrm{Aut}_{\Delta G}([n]) = G_n^{op}$

(iv)
$$\begin{tikzcd}
{[n]} \arrow[r] \arrow[dr, "\phi \in \Delta"'] & {[m]} \\
& {[m]} \arrow[u, "g \in G_m^{op}"']
\end{tikzcd}$$

Special case : $G_n = S_{n+1}$ $\forall n$

Upshot: Instead of taking Δ spaces we take ΔS spaces.

Toy computation:

$$T_p := (\mathbb{P}^1)^{p+1} \times \widetilde{\mathrm{Mor}}_{n-(p+1)}(\mathbb{P}^1, \mathbb{P}^r)$$

$H^*(\mathrm{Mor}_n(\mathbb{P}^1, \mathbb{P}^r))$, we focus on $H^*(\mathbb{P}^1)^{\otimes p} \otimes H^* \widetilde{\mathrm{Mor}}_{n-p}(\mathbb{P}^1, \mathbb{P}^r)$

Cor (B'21)

$$E_1^{p,q} = H_c^q(T_p) \otimes \mathrm{sgn}_{S_{p+1}} \Rightarrow H_c^{p+q}(\mathrm{Mor}_n(\mathbb{P}^1, \mathbb{P}^r))$$

$$\left(H^*(\mathbb{P}^1)^{\otimes p+1} \otimes \mathrm{sgn}_{S_{p+1}}\right) \otimes H^*(\widetilde{\mathrm{Mor}}_{n-(p+1)}(\mathbb{P}^1, \mathbb{P}^r))$$

$\Lambda H^{\mathrm{even}}(\mathbb{P}^1) \otimes \mathrm{Sym}\, H^{\mathrm{odd}}(\mathbb{P}^1)$

gives all Betti nos. wts.

$$H^*(\mathrm{Mor}_n(\mathbb{P}^1, \mathbb{P}^1)) \simeq H^*(PGL_2) \qquad n \geq 2$$

THANK

YOU !! :)

Thm B.2.1

$$H^i(\mathcal{U}_2) \cong \begin{cases} \mathrm{Sym}^{p-2} H^1(X)(-(p-1)) \oplus \mathrm{Sym}^p H^1(X)(-p) & i = 2p \\ \mathrm{Sym}^{p-1} H^1(X)(-(p-1)) \oplus \mathrm{Sym}^p H^1(X)(-(p+1)) & i = 2p+1 \end{cases}$$

$\mathbb{Q}$ or $\mathbb{Q}_\ell$

$i \leq \frac{m}{2}$ where the degree of ampleness of $\mathcal{L}$.

Aumonier homotopy theory, Ishan Banerjee

$\mathbb{Z}$ $\mathbb{Z}$