Nonexistence of symplectic structures on certain family of 4-manifolds

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YMSC Topology Seminar

I. Motivation and Backgrounds

Existence and uniqueness of symplectic structures

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- One goal today: Use Seiberg-Witten theory to study (3).

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- So Question (2) is a refined version of the uniqueness problem of isotopies.

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- So it is **not** feasible to study $i_{r,*} : \pi_r(\mathrm{Symp}(X, \omega)) \to \pi_r(\mathrm{Diff}(X))$ by finding the homotopy type of $\mathrm{Symp}(X, \omega)$ and $\mathrm{Diff}(X)$.

II. Non-symplectic loop of diffeomorphisms

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- Seidel's proof uses deep results in quantum cohomology ring $QH(X)$ and the Floer homomlogy $HF(\phi)$ for $\phi \in \mathrm{Symp}(X,\omega)$. It applies to large family of symplectic manifolds. (e.g. complete intersections in $\mathbb{CP}^{n+2}$ other than $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$)

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For many complex surfaces (X, ω) *(including hypersurfaces in* $\mathbb{CP}^3$ *with degree* $\neq 1$ *or* 4*), the map* $i_{1,*} : \pi_1(\mathrm{Symp}(X, \omega)) \to \pi_1(\mathrm{Diff}(X))$ *is not surjective.*

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- The case of self-intersection -1 confirms a conjectured by McDuff.
- New ingredient: a new gluing formula for the family Seiberg-Witten invariant.

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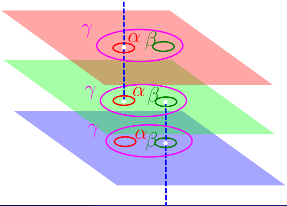

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- These loops are are contractible in the space of non-degenerate 2-forms.
- The condition $b_2^+(X) \neq 3$ is related to the wall-crossing phenomena for family Seiberg-Witten invariants.

III. Non-symplectic families of smooth 4-manifolds

Loops in Diff(X) $\longleftrightarrow$ Families of X over S^2

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- $[\gamma] \in \mathrm{Image}(\pi_1(\mathrm{Symp}(X, \omega)) \xrightarrow{i_{1,*}} \pi_1(\mathrm{Diff}(X))) \iff$ There is a fiberwise symplectic structure $\{\omega_b\}_{b \in S^2} \subset \mathcal{S}_\omega(X)$ on $\{X_b\}_{b \in S^2}$.

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- To show $i_{1,*}$ is not surjective, it suffices to establish non-symplectic families over S^2.

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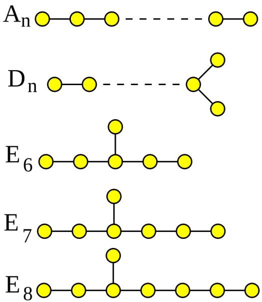

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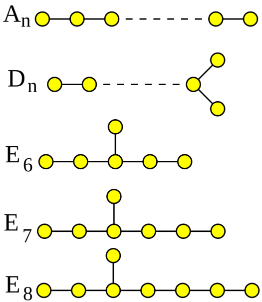

- Kronheimer proved that $\tilde{\Sigma}$ is an ALE space (i.e. it admits a hyper-Kähler metric that approaches the Euclidian metric at infinity.)

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Let $\{\omega_b\}$ be a family of minimal symplectic structure on an ADE family $X \hookrightarrow E \to S^2$. Then at least one of the following two situation happens:

- *For any $a \in H_2(\widetilde{\Sigma}; \mathbb{Z})$ with $a \cdot a = -2$, the function $S^2 \to \mathbb{R}$ defined by $b \mapsto \langle [\omega_b], a \rangle$ takes both positive and negative values. Or,*
- *$b^+(X) = 3$, $c_1(K)$ is torsion, and $\{\omega_b\}$ is a winding family.*

(Here K is the canonical bundle.) In particular, the ADE family never admits minimal fiberwise symplectic structures in a constant cohomology class.

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- McDuff conjectured that a similar result holds in all dimensions.

IV. A gluing formula for family Seiberg-Witten invariants.

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- For generic choice of $(\{g_b\}, \{\mu_b\})$, $\mathcal{M}$ is a compact, smooth manifold of dimension $d(\mathfrak{s}_E)$.
- We focus on the particular case $d(\mathfrak{s}_E) = 0$. Then we define **the family Seiberg-Witten invariant**

$$\mathrm{FSW}_\xi(E, \mathfrak{s}_E) := \#\mathcal{M} \in \mathbb{Z}.$$

Here $\xi \in [B, S^{b^+(X)-1}]$ is the "chamber" for our choice of $(\{g_b\}, \{\mu_b\})$.

The family Seiberg-Witten invariants

- Consider a smooth fiber bundle $X \hookrightarrow E \xrightarrow{p} B$.
- Pick a family Spin^c-structure $\mathfrak{s}_E$. Choose metrics $\{g_b\}_{b\in B}$ and perturbations $\{\mu_b\}_{b\in B}$ on each fiber X_b. Here $\mu_b \in \Omega^2(X_b)$.
- We study the Seiberg-Witten equations on the fiber X_b

$$F^+_{A^t_b} = \rho^{-1}(\phi_b\phi_b^*)_0 + i\mu_b, \quad \not{D}^+_{A_b}\phi_b = 0.$$

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Assuming $b_2^+(X) > 1$, then we have

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$\mathrm{FSW}_{\xi_\omega}(E, \mathfrak{s}_E) = 0$ *if for any $b \in B$ we have* $[\omega_b] \cdot c_1(\mathfrak{s}_E|_{X_b}) < -[\omega_b] \cdot c_1(K)$

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$$\widetilde{W}_1 := E_1 \backslash (B \times D^4) : S^3 \to Y$$

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- Given (family) Spin^c-structures $\mathfrak{s}_{E_1}$ on E_1 and $\mathfrak{s}_{X_2}$ on X_2. We get cobordism-induced maps

$$\widehat{HM}_*(\widetilde{W}_1, \mathfrak{s}_{E_1}) : \widehat{HM}_*(S^3) \to \widehat{HM}_*(Y)$$

$$\overrightarrow{HM}^*(W_2, \mathfrak{s}_{X_2}) : \widetilde{HM}^*(S^3) \to \widehat{HM}^*(Y)$$

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Theorem (Kronheimer-Mrowka, L.)

One has $\sum_{\mathfrak{s}} \mathrm{FSW}(E,\mathfrak{s}) = \langle \widehat{HM}_*(\widetilde{W}_1, \mathfrak{s}_{E_1})(\hat{1})\ ,\ \overrightarrow{HM}^*(W_2, \mathfrak{s}_{X_2})(\check{1})\rangle$*. Here* $\hat{1}$ *and* $\check{1}$ *are canonical generators of the monopole Floer (co)homology of* S^3*. The sum is taken over all family* Spin^c*-structures* $\mathfrak{s}$ *on* E *that satisfies* $\mathfrak{s}|_{E_1} = \mathfrak{s}_{E_1}$, $\mathfrak{s}|_{E_2} = p^*(\mathfrak{s}_{X_2})$, $d(\mathfrak{s}_E) = 0$.

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Corollary

Assume Y *is an* L*-space and* $b_2^+(X_2) > 1$. *Given a family* Spin^c*-stricture* $\mathfrak{s}_E$ *on* E *and a* Spin^c*-structure* $\mathfrak{s}_X$ *on* X *that satisfy the conditions* $\mathfrak{s}_E|_{E_2} = p^*(\mathfrak{s}_X|_{X_2})$ *and* $d(\mathfrak{s}_E) = d(\mathfrak{s}_X) = 0$. *Then we have*

$$\mathrm{FSW}(E, \mathfrak{s}_E) = \langle c_{\frac{\dim(B)}{2}}(-\mathrm{Ind}(\not{D}(E, \mathfrak{s}_E))), [B]\rangle \cdot \mathrm{SW}(X, \mathfrak{s}_X)$$

if $b_2^+(X_1) = 0$. *We have* $\mathrm{FSW}(E, \mathfrak{s}_E) = 0$ *if* $b_2^+(X_1) \neq 0$.

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- We use this formula to compute FSW of ADE families and blow up families and use Taubes' result to show that they have no symplectic structures.

Thank you for listening.