Augmentations from Legendrian knots.

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Outline:

1. Chekanov-Eliashberg DGAs associated to Legendrian knots/tangles

2. Augmentations and cell decomposition.

3. Application: counting, dual boundary complexes.

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1. Chekanov-Eliashberg DGAs

Setup: Contact pair (V, Λ), V = contact manifold
Λ = Legendrian submanifold.

eg: $V = (\mathbb{R}^3_{x,y,z}, \alpha = dz - y\,dx)$, $\Lambda \hookrightarrow V$ Legendrian knot.

· "open string theory in $(\mathbb{R}_t \times V, \mathbb{R}_t \times \Lambda)$"

$\mathbb{R}_t \times V$: symplectization; $\mathbb{R}_t \times \Lambda$: Lagrangian cylinder

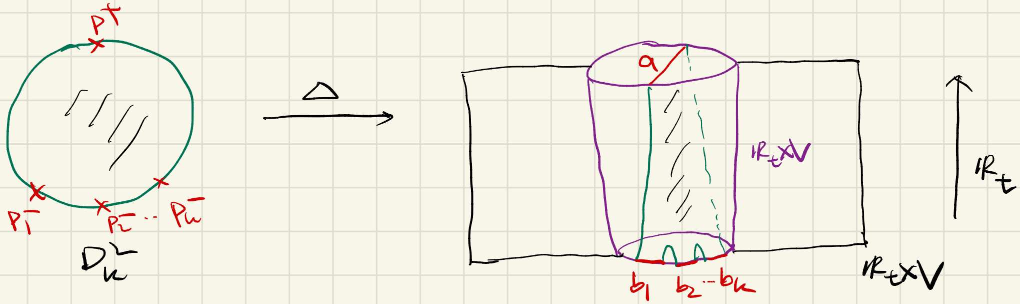
$\rightsquigarrow$ Relative contact homology (c SFT):

Contact pair $(V, \Lambda) \longmapsto$ Legendrian contact homology DGA $\mathcal{A}(V, \Lambda)$.

(LCH DGA or Chekanov-Eliashberg DGA)

· $\mathcal{A}(V, \Lambda)$: generators = Reeb chords of Λ.

• differential (deg −1) : counts certain holomorphic disks in the symplectization $\mathbb{R}_t \times V$, with boundary along the Lagrangian cylinder $\mathbb{R}_t \times \Lambda$, and meeting the Reeb chords at infinity at some punctures.

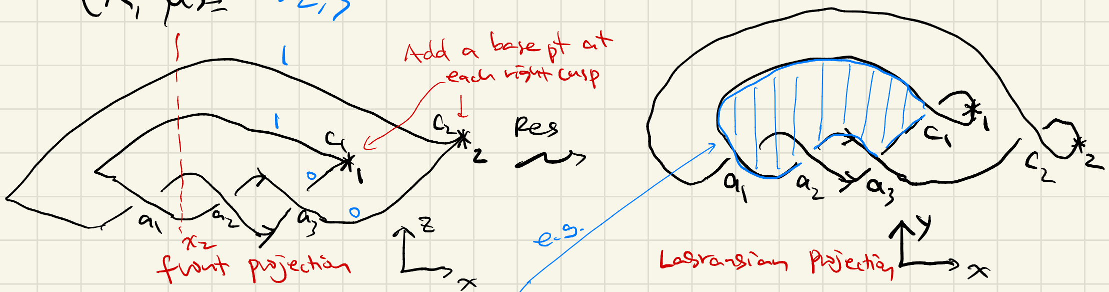


(holomorphic disks defining the LCH DGA differential)

$$\rightsquigarrow \quad \partial a = \sum \underset{\substack{\uparrow \\ \text{weights}}}{\varepsilon} \, b_1 b_2 \cdots b_k$$

Example:

$(\Lambda, \mu) = T_{2,3}$



$$\Rightarrow A(\Lambda) = \mathbb{Z}\langle t_1^{\pm 1}, t_2^{\pm 1}, a_1, a_2, a_3, c_1, c_2 \rangle, \quad |t_i^{\pm 1}| = 0, \; |a_i| = 0, \; |c_i| = 1.$$

$$\begin{cases} \partial c_1 = t_1^{-1} + a_1 + a_3 + a_1 a_2 a_3 \\ \partial c_2 = t_2^{-1} + a_2 + (1 + a_2 a_3)\, t_1 (1 + a_1 a_2) \end{cases} \qquad \partial t_i^{\pm 1} = 0, \; \partial a_i = 0.$$

Fact: the LCH DGA $\mathcal{A}(V,\Lambda)$ is a Legendrian isotopy invariant for $\Lambda \hookrightarrow V$, up to homotopy equivalence.

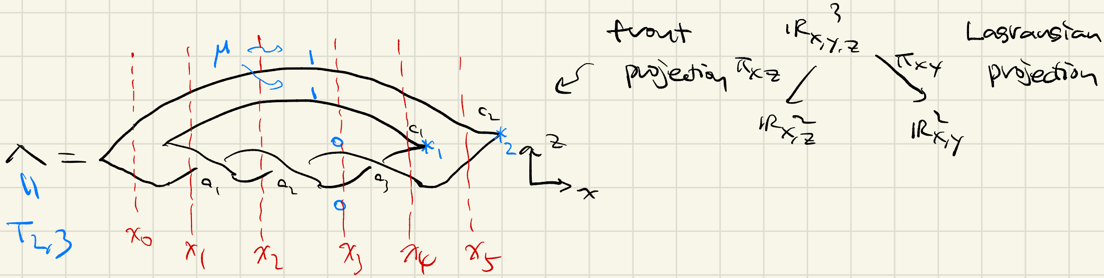
. Ekholm, Etnyre, Sullivan 05: when $V = \mathbb{R} \times P$ = contactification of exact symplectic mfld $\Rightarrow$ $\mathcal{A}(V,\Lambda)$ well-defined

our case:

$$V = (\mathbb{R}^3_{x,y,z}, \alpha = dz - y\,dx), \quad \Lambda \hookrightarrow V \text{ Legendrian knot.}$$

$\rightsquigarrow$ LCH DGA $\mathcal{A}(V,\Lambda) =$ C-E DGA $\mathcal{A}(\Lambda)$ admits a combinatorial description:

Illustration by example:



(the right-handed Legendrian trefoil knot $T_{2,3}$).

Additional data: (Conley-Zehnder index)

- μ = some $\mathbb{Z}/2r$-valued Maslov potential encoding the grading of the Reeb chords of Λ (r = rotation number of Λ).

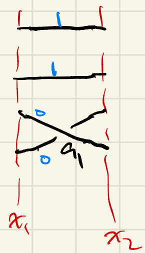
Assume for simplicity: $r = 0$.

- $*_1, \dots, *_B$ = some base points on Λ so that each right cusp is marked.

· For any open interval $(s_1, s_2) \subset \mathbb{R}_x$

$\rightsquigarrow$ Legendrian tangle $\Lambda(s_1,s_2) := \Lambda|_{\{s_1 < x < s_2\}}$.

<u>E.g.</u> $(s_1, s_2) = (x_1, x_2)$ (see above figure).

$\Rightarrow \Lambda(s_1, s_2) =$  $\hookrightarrow J^1(x_1, x_2)$

<u>prop.</u>:

$\exists$ an combinatorial assignment $\Lambda(s_1,s_2) \longmapsto \mathcal{A}(\Lambda(s_1,s_2))$

$\mathcal{A}(\Lambda(s_1,s_2)) \in$ $\mathbb{Z}/2r$-graded DGAs

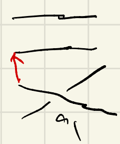
s.t.

1) As an algebra,

$$\mathcal{A}(\Lambda(s_1,s_2)) = \mathbb{Z}[t_{i_1}^{\pm 1}, \dots, t_{i_b}^{\pm 1}]\langle a_1, \dots, a_N \rangle$$

where · t_{i_j} corresponds to the base point $*_{i_j}$ on $\Lambda(s_1,s_2)$,

· a_i's correspond to the crossings, right cusps, and the pairs of left endpoints of $\Lambda(s_1,s_2)$.

(E.g.: $\mathcal{A}(\Lambda(x_1,x_2)) = \mathbb{Z}\langle a_1, a_{ij} : 1 \le i < j \le 4 \rangle$, e.g. $a_{23} =$ )

2) $\mathcal{A}(\Lambda(s_1,s_2))$ is an invariant of $\Lambda(s_1,s_2)$, up to homotopy equivalence

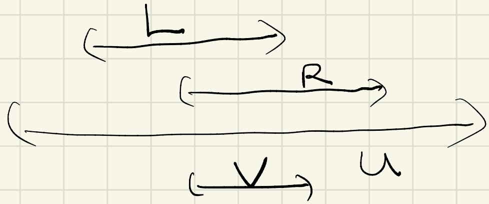
3) each inclusion $V \hookrightarrow U$ in $\mathbb{R}_x$ induces a co-restriction map (of $\mathbb{Z}/_{2r}$-graded DGAs):

$$\iota_{U,V} : A(\Lambda(V)) \to A(\Lambda(U))$$

4) the DGA's $A(\Lambda(s_1, s_2))$'s satisfy a co-sheaf/van-Kampen property: "gluing holo disks"

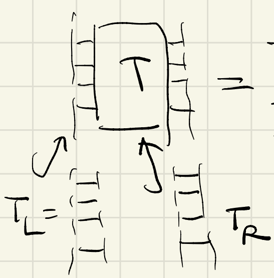
$L, R \hookrightarrow U$ (open intervals in $\mathbb{R}_x$)

with $V = L \cap R$



$$\Longrightarrow \quad \begin{tikzcd} A(\Lambda(V)) \arrow[r, "\iota_{R,V}"] \arrow[d, "\iota_{L,V}"'] \arrow[dr, phantom, "\ulcorner", very near end] & A(\Lambda(R)) \arrow[d, "\iota_{U,R}"] \\ A(\Lambda(L)) \arrow[r, "\iota_{U,L}"'] & A(\Lambda(U)) \end{tikzcd}$$

5). For each elementary Legendrian tangle



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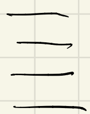
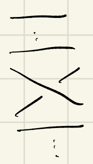
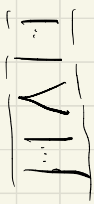


(parallel strands),



(a single crossing),



(a single cusp),

(a single (marked) right cusp)

$\rightsquigarrow$ have explicit combinatorial description of

$$\mathcal{A}(T_L) \to \mathcal{A}(T) \leftarrow \mathcal{A}(T_R).$$

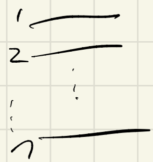
"counting of holo disks".

b) $\mathcal{A}(\Lambda(-\infty,\infty)) = \mathcal{A}(\Lambda)$ is the Chekanov-Eliashberg DGA associated to Λ.

(global co-section of a constructible co-sheaf of $\mathbb{Z}/2r$-graded DGAs).

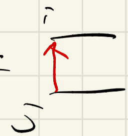
Example:

① $(T,\mu) =$ n parallel strands



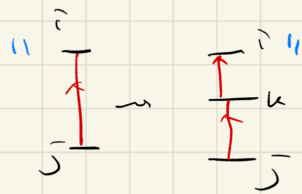
$$\Rightarrow \quad \mathcal{A}(T_L) \xrightarrow{id} \mathcal{A}(T) \xleftarrow{id} \mathcal{A}(T_R).$$

& $\mathcal{A}(T) = \mathbb{Z}\langle a_{ij} : 1 \le i < j \le n \rangle$, $\quad a_{ij} =$

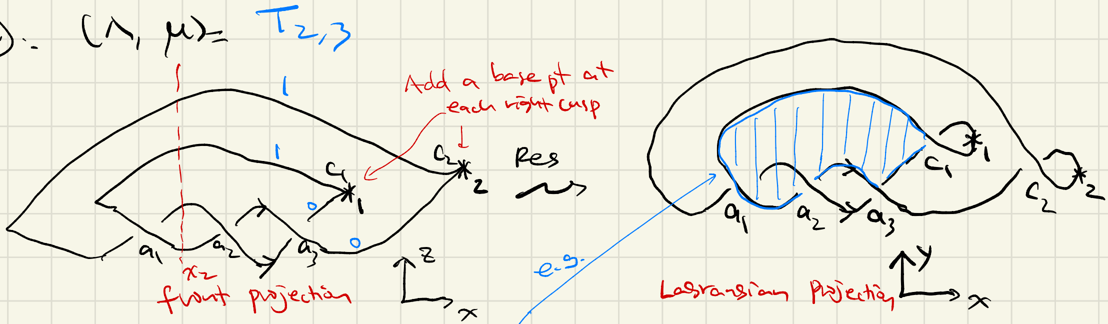


grading: $|a_{ij}| = \mu(i) - \mu(j) - 1$.

differential: $\partial a_{ij} = \sum_{i<k<j} (-1)^{|a_{ik}|+1} a_{ik} a_{kj}$,



② : $(\Lambda, \mu) = T_{2,3}$



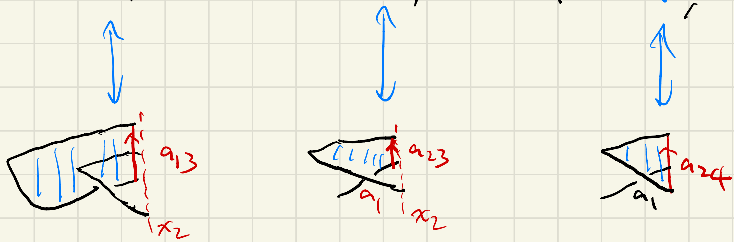
$$\Rightarrow \mathcal{A}(\Lambda) = \mathbb{Z}\langle t_1^{\pm1}, t_2^{\pm1}, a_1, a_2, a_3, c_1, c_2 \rangle, \ |t_i^{\pm1}| = 0, \ |a_i| = 0, \ |c_i| = 1.$$

$$\begin{cases} \partial c_1 = t_1^{-1} + a_1 + a_3 + a_1 a_2 a_3 \\ \partial c_2 = t_2^{-1} + a_2 + (1 + a_2 a_3)\, t_1 (1 + a_1 a_2) \end{cases} \qquad \partial t_i^{\pm1} = 0, \ \partial a_i = 0.$$

ι ↑

• $\mathcal{A}(\Lambda(x_2, x_2+\varepsilon)) = \mathbb{Z}\langle a_{ij} : 1 \le i < j \le 4 \rangle$, e.g. $a_{23} =$  etc.

$\iota(a_{13}) = 1, \ \iota(a_{23}) = a_1, \ \iota(a_{24}) = 1, \ \iota(a_{ij}) = 0$, else.



2. Augmentations

Given any Legendrian knot/link (Λ, μ),

$\forall$ open interval $(s_1, s_2) \rightsquigarrow$ Legendrian tangle

$$\begin{tikzcd}[column sep=small] & \Lambda(s_1,s_2) & \\ \Lambda(s_1, s_1+\varepsilon) \arrow[ur, hook] & & \Lambda(s_2-\varepsilon, s_2) \arrow[ul, hook'] \end{tikzcd}$$

$(0 < \varepsilon << 1)$

$\rightsquigarrow$ $\mathbb{Z}/2r$-graded DGAs:

$$\begin{tikzcd}[column sep=small] & \mathcal{A}(\Lambda(s_1,s_2)) & \\ \mathcal{A}(\Lambda(s_1, s_1+\varepsilon)) \arrow[ur] & & \mathcal{A}(\Lambda(s_2-\varepsilon, s_2)) \arrow[ul] \end{tikzcd}$$

Defn: the ($\mathbb{Z}/2r$-graded) augmentation variety

of a $\mathbb{Z}/2r$-graded DGA $\mathcal{A}$ is:

$$\mathrm{Aug}(\mathcal{A}; k) := \{ \varepsilon : \mathcal{A} \to (k, 0) \ \mathbb{Z}/2r\text{-graded DGA maps} \}$$

affine

fixed base field

trivial $\mathbb{Z}/2r$-graded DGA k with zero differential.

$\rightsquigarrow$

$$\begin{tikzcd}[column sep=small] & \mathrm{Aug}(\Lambda(s_1,s_2)) \arrow[dl] \arrow[dr] & \\ \mathrm{Aug}(\Lambda(s_1, s_1+\varepsilon)) & & \mathrm{Aug}(\Lambda(s_2-\varepsilon, s_2)) \end{tikzcd}$$

Example:

①: (T,μ) = n parallel strands.

· Fix $C(T) := \bigoplus_{i=1}^{n} k e_i$, $e_i \leftrightarrow \underset{i}{\text{———}}$ in (T,μ).

$$|e_i| = -\mu(i) \pmod{2r}$$

· Filtration $F^\bullet$ on $C(T)$:

$$F^i C(T) := \bigoplus_{i<j\leq n} k e_j, \quad 0 \leq i \leq n.$$

$\rightsquigarrow$ $(C(T), F^\bullet)$ is a $\mathbb{Z}/2r$-graded filtered free k-module.

· Define a map:

$$\begin{array}{ccc} \mathrm{Aug}(T,\mu) & \xrightarrow{d} & \mathrm{End}^{-1}((C(T),F^\bullet)) \\ \varepsilon & \longmapsto & d(\varepsilon) \end{array}$$

by $d(\varepsilon)\, e_i = \sum_{j>i} (-1)^{\mu(i)} \varepsilon(a_{ij})\, e_j$.

Fact: 1) $\mathrm{Aug}(T,\mu) \xrightarrow{\simeq} \{ d \in \mathrm{End}^{-1}(C(T),F^\bullet),\ d^2=0 \}$

$$\| $$

$\{ (C(T),d) : \mathbb{Z}/2r$-graded filtered complex of free k-modules$\}$

("Borel group")

2) $B_{2r}(T) := \mathrm{Aut}(C(T),F^\bullet) \curvearrowright \{ d \in \mathrm{End}^{-1}(C(T),F^\bullet) : d^2=0 \}$

via conjugation with finitely many orbits.

Defn: $\varepsilon \in \mathrm{Aug}(T,\mu)$ is acyclic, if:

$H^*(C(T), d(\varepsilon)) \cong 0$, i.e. the complex $(C(T), d(\varepsilon))$ is acyclic.

Notation: $\mathrm{Aug}^a(T,\mu) := \{\varepsilon \in \mathrm{Aug}(T,\mu) \text{ acyclic}\}$.

② : $(\Lambda, \mu) = T_{2,3}$

$$\Rightarrow \quad \mathrm{Aug}(T_{2,3};k) \cong \left\{ \varepsilon(t_1,a_1,a_2,a_3) \in k^* \times k^3 \;\middle|\; \varepsilon(t_1)^{-1} + \varepsilon(a_1) + \varepsilon(a_3) + \varepsilon(a_1)\varepsilon(a_2)\varepsilon(a_3) = 0 \right\}$$

$$\begin{tikzcd}
\varepsilon \arrow[d, mapsto] & \mathrm{Aug}(T_{2,3};k) \arrow[d, "\iota^*"'] \arrow[dr, dotted] & \\
\iota^*\varepsilon & \mathrm{Aug}(T_{2,3}(x_2,x_2+\varepsilon);k) & \mathrm{Aug}^a(T_{2,3}(x_2,x_2+\varepsilon);k) \arrow[l]
\end{tikzcd}$$

$$\mathrm{Aug}^a(T_{2,3}(x_2,x_2+\varepsilon);k) = \left\{ \det \varepsilon\begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix} \neq 0 \right\}.$$

$$\mathrm{Aug}(T_{2,3}(x_2,x_2+\varepsilon);k) = \left\{ \varepsilon(a_{13},a_{14},a_{23},a_{24}) \in k^4 \right\} \cong k^4$$

$(\iota^*\varepsilon)(a_{13}) = 1$, $(\iota^*\varepsilon)(a_{14}) = 0$

$(\iota^*\varepsilon)(a_{23}) = \varepsilon(a_1)$, $(\iota^*\varepsilon)(a_{24}) = 1$.

Can compute:

$$\underset{\substack{\circlearrowleft \\ \mathrm{Br}(T_{2,3}(x_2,x_2+\varepsilon)) \\ = 0}}{\mathrm{Aug}^a(T_{2,3}(x_2,x_2+\varepsilon);k)} = \overbrace{\underset{\substack{\cong \\ (k^*)^2 \times k^2}}{\{\varepsilon(a_{23}) \neq 0\}} \sqcup \underset{\substack{\cong \\ (k^*)^2 \times k}}{\{\varepsilon(a_{23}) = 0\}}}^{\mathrm{Br}(\dots)\text{-orbits}}$$

Cell decomposition for augmentation varieties.

- (Λ, μ) = Legendrian knot front projection.

- Let $x_0 < x_1 < \cdots < x_N$ be the generic x-coordinates separating the singularities (crossings, cusps) of $\pi_{xz}(\Lambda)$.

$\rightsquigarrow \quad \Lambda(x_i, x_i+\varepsilon) \hookrightarrow (\Lambda, \mu)$.

$$\rightsquigarrow \quad \begin{array}{c} \mathrm{Aug}^a(\Lambda(x_i, x_i+\varepsilon); k) \xleftarrow{\iota_i^*} \mathrm{Aug}((\Lambda,\mu); k) \\ \| \\ \bigsqcup \mathrm{Aug}^{\rho,a}(\Lambda(x_i, x_i+\varepsilon); k) \\ \uparrow \\ \text{finitely} \quad B_{2r}(\Lambda(x_i, x_i+\varepsilon); k)\text{-orbits} \end{array}$$

$$\rightsquigarrow \quad \mathrm{Aug}((\Lambda,\mu); k) = \bigsqcup_{\rho \in NR(\Lambda,\mu)} \mathrm{Aug}^\rho((\Lambda,\mu); k)$$

$$\| $$

$$\Rightarrow \text{Cell decomposition} \qquad \bigcap_{0 \le i \le N} (\iota_i^*)^{-1}\left(\mathrm{Aug}^{\rho_i, a}(\Lambda(x_i, x_i+\varepsilon); k)\right)$$

<u>Theorem</u>:

- $\exists$ a natural decomposition by locally closed subvarieties

$$\mathrm{Aug}((\Lambda,\mu); k) = \bigsqcup_{\rho \in NR(\Lambda,\mu)} \mathrm{Aug}^\rho((\Lambda,\mu); k)$$

with $\mathrm{Aug}^\rho((\Lambda,\mu); k) \cong (k^*)^{a(\rho)} \times k^{b(\rho)}$.

<u>Idea of proof</u>:

cosheaf property for C-E DGAs $\Rightarrow$ sheaf property for $\mathcal{A}(\Lambda(s_1, s_2))$ $\quad$ $\mathrm{Aug}(\Lambda(s_1,s_2); k)$.

3. Applications

① count augmentations:

augmentation number.

Theorem:

- $k = \mathbb{F}_q \Rightarrow q^{-\dim_{\mathbb{C}} Aug((\Lambda,\mu);\mathbb{C})} |Aug((\Lambda,\mu);\mathbb{F}_q)| \in \mathbb{Z}[q^{\pm 1}]$

is a Legendrian isotopy invariant, and:

$$q^{-\dim_{\mathbb{C}} Aug((\Lambda,\mu);\mathbb{C})} |Aug((\Lambda,\mu);\mathbb{F}_q)| = q^{-\frac{d+B}{2}} z^{B} R_{(\Lambda,\mu)}(z).$$

where $R_{(\Lambda,\mu)}(z) \in \mathbb{Z}[z^{\pm 1}]$ is the ruling polynomial

(a state-sum invariant of (Λ,μ)), $z = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$.

Rmk: N. Katz's theorem $\Rightarrow$ the E-polynomial of

$Aug((\Lambda,\mu);\mathbb{C})$ is:

$$E_c(Aug((\Lambda,\mu);\mathbb{C}); x, y) = |Aug((\Lambda,\mu);\mathbb{F}_q)|_{q=xy}.$$

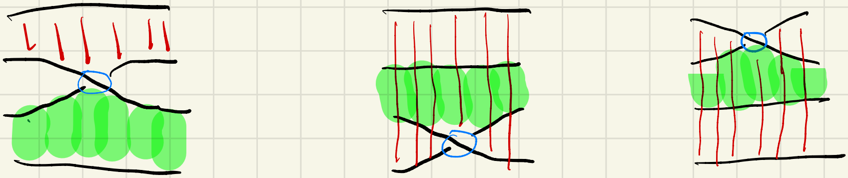
$\rightsquigarrow$ get formula for the E-polynomial.

Ruling polynomials:

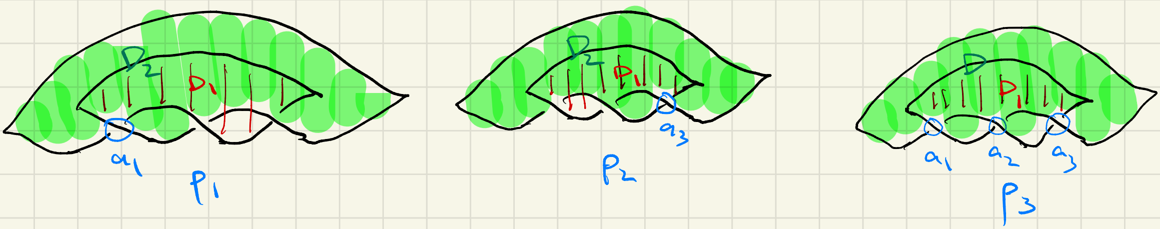
Defn: · A normal ruling ρ of (Λ,μ) is a decomposition of the front diagram $\pi_{xz}(\Lambda)$ into the boundaries of disks in $\mathbb{R}^2_{x,z}$ satisfying a normal condition.

. a switch of a normal ruling ρ is a crossing of deg 0 that appears as a corner in the boundary of some disk of ρ.

Normal condition: admissible disks near a switch.



E.g. $(\Lambda, \mu) = T_{2,3} \rightsquigarrow$ 3 normal rulings ρ_1, ρ_2, ρ_3.



Defn:

- $\rho \in NR(\Lambda, \mu)$ normal ruling $\rightsquigarrow$

$$\chi(\rho) := \#(\text{disks}) - \#(\text{switches})$$

- the ruling polynomial of (Λ, μ) is:

$$R_{(\Lambda,\mu)}(z) := \sum_{\rho \in NR(\Lambda,\mu)} z^{-\chi(\rho)} \in \mathbb{Z}[z^{\pm 1}].$$

$\rightsquigarrow$ Legendrian isotopy inv.

Example: $(\Lambda, \mu) = T_{2,3} \Rightarrow$

$$R_{T_{2,3}}(z) = z^{-\chi(\rho_1)} + z^{-\chi(\rho_2)} + z^{-\chi(\rho_3)} = 2z^{-1} + z$$

$$\Rightarrow d := \max \deg_z (R_{(\Lambda,\mu)}(z)) = 1.$$

On the other hand,

$$Aug(T_{2,3};k) = \{\varepsilon(t_1,a_1,a_2,a_3)\in k^*\times k^3 \mid \varepsilon(t_1)^{-1}+\varepsilon(a_1)+\varepsilon(a_3)+\varepsilon(a_1a_2a_3)=0\}$$

$$\underset{\text{cell decomposition}}{=} \bigsqcup_{\rho\in NR(T_{2,3})} Aug^{\rho}(T_{2,3};k)$$

$$\overset{\rho_3}{\leftrightarrow} \quad = \{\varepsilon(a_1)\neq 0,\ 1+\varepsilon(a_1a_2)\neq 0\} \cong (k^*)^3$$

$$\overset{\rho_1}{\leftrightarrow} \quad \sqcup \{\varepsilon(a_1)\neq 0,\ 1+\varepsilon(a_1a_2)=0\} \cong k^*\times k$$

$$\overset{\rho_2}{\leftrightarrow} \quad \sqcup \{\varepsilon(a_1)=0,\ 1+\varepsilon(a_1a_2)\neq 0\} \cong k^*\times k$$

$$\Rightarrow q^{-\dim}|Aug(T_{2,3};\mathbb{F}_q)| = q^{-3}\left(2(q-1)q+(q-1)^3\right)$$

$$= q^{-\frac{3}{2}}2z + q^{-\frac{3}{2}}z^3$$

$$= q^{-\frac{1+2}{2}}z^2 R_{T_{2,3}}(z) \quad (d=1,\ B=2) \quad ✓$$

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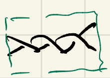
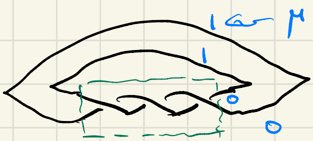
② dual boundary complexes of Betti moduli spaces.

β = n-strand positive braid so that $\Lambda = \beta^{>}$ is connected.

e.g. $n=2$, $\beta = \sigma_1^3 =$  $\Rightarrow$

$\Lambda = \beta^{>} =$ 

with standard Maslov potential μ

Theorem (S 21): $\Lambda = \beta^{>}$ connected $\Rightarrow$

1) $\exists$ natural action of $\mathbb{G}_m^n \curvearrowright Aug(\beta^{>};k)$

& $\operatorname{Spec} \mathcal{O}(Aug(\beta^{>};k))^{\mathbb{G}_m^n} \cong \mathcal{M}_1(\beta^{>})$ (say $k=\mathbb{C}$)

is an irregular Betti moduli space / wild character variety

on $\mathbb{CP}^1$ with one irregular singularity at ∞.

2) $\mathcal{M}_1(\beta^{>})$ is smooth, connected affine algebraic variety

3) the $\mathbb{G}_m^n$-action respects the cell decomposition

for $Aug(\beta^{>};k) \rightsquigarrow$ cell decomposition for

$$\mathcal{M}_1(\beta^{>}) = \bigsqcup_{p \in NR(\beta^{>})} \mathcal{M}_1^p(\beta^{>}), \quad \mathcal{M}_1^p(\beta^{>}) \cong (k^*)^{s(p)-n+1} \times k^{r(p)}$$

4) The homotopy type conjecture for $\mathcal{M}_1(\beta^{>})$ holds:

$\exists$ homotopy equivalence $\mathbb{D}\partial\mathcal{M}_1(\beta^{>}) \simeq S^{d-1}$,

where $d = \dim_{\mathbb{C}} \mathcal{M}_1(\beta)$. dual boundary complex.

E.g.: $\beta = \sigma_1^3$ (braid with three crossings) $\Rightarrow \Lambda = \beta^{>} = T_{2,3}$

$$\Rightarrow \mathbb{G}_m^2 \curvearrowright Aug(T_{2,3};\mathbb{C}) = \{\varepsilon(t_1,a_1,a_2,a_3) \in k^* \times k^3 \mid \varepsilon(t_1)^{-1} + \varepsilon(a_1) + \varepsilon(a_3) + \varepsilon(a_1a_2a_3) = 0\}$$

$\vec{\lambda} = (\lambda_1, \lambda_2)$

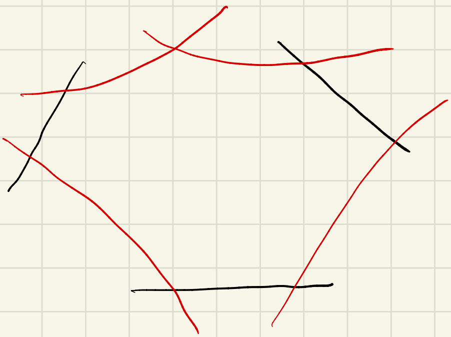
$$\vec{\lambda} \cdot \varepsilon(t_1, a_1, a_2, a_3) = (\lambda_2^{-1}\varepsilon(t_1)\lambda_1, \lambda_1^{-1}\varepsilon(a_1)\lambda_2, \lambda_2^{-1}\varepsilon(a_2)\lambda_1, \lambda_1^{-1}\varepsilon(a_3)\lambda_2).$$

$$\Rightarrow \mathcal{M}_1(\beta^3) \cong \operatorname{Spec} \mathcal{O}(\mathrm{Aug}(\beta^3;\mathbb{C}))^{\mathbb{G}_m^2}$$

$$\cong \{1+\varepsilon(a_1)+\varepsilon(a_3)+\varepsilon(a_1a_2a_3)=0\}, \quad \dim = 2$$

<u>Fact</u>: $\mathcal{M}_1(\beta^3)$ admits a log compactification $\overline{\mathcal{M}_1(\beta^3)}$

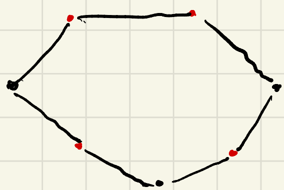
with boundary divisor $\partial\mathcal{M}_1(\beta^3) =$



$\Rightarrow$ dual intersection complex

$\mathbb{D}\partial\mathcal{M}_1(\beta^3)$

$\|$



$\simeq S^1$

$\square$