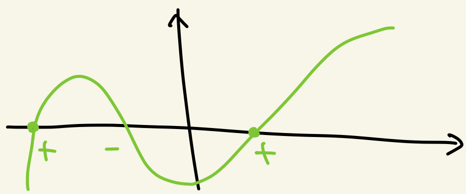
# Turning smooth 4-manifolds into maps between spheres

- Finit dimensional approximation

Q: Use topological method to count the number of solutions.
Assume 0 is a regular value

- $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) \to \pm\infty$ as $x \to \pm\infty$

Then $\# f^{-1}(0) := |\{x \in \mathbb{R} \text{ s.t. } f(x)=0 \ f'(x)>0\}|$ is always 1.
$\qquad - |\{x \in \mathbb{R} \text{ s.t. } f(x)=0 \ f'(x)<0\}|$




- $f: \mathbb{R}^n \to \mathbb{R}^n$ proper. $\quad f^+ : \mathbb{R}^{n+} \longrightarrow \mathbb{R}^{n+}$

. then $\# f^{-1}(0) = \deg(f^+)$

- $f: \mathbb{R}^m \to \mathbb{R}^n$ proper $\quad f^+ : (\mathbb{R}^m)^+ \longrightarrow (\mathbb{R}^n)^+$

$f^{-1}(0)$ is a framed submanifold of $\mathbb{R}^m$. It's framed cobordism class is determined by $[f^+] \in \pi_m(S^n)$.

Q: Can we use this idea to study the space of solutions of a P.D.E?

Our setting: $X$: smooth manifold $\quad E, F$: vector bundles over $X$ with same dimension.

Consider a differential operator $L+Q : \underbrace{C^\infty(X;E)}_{\text{Smooth sections of } E} \to C^\infty(X;F)$

where $L$ is a 1-st order elliptic operator (linear)

$Q$ is a 0-th order, nonlinear operator (e.g. $f \mapsto f^2$)

Key assumption: $(L+Q)^{-1}(0)$ is compact.

Thm (Schwarz, Bauer-Furuta) Such operator gives a well-defined element $\alpha \in \pi^{st}_{ind(L)}(S^0)$.

$\pi^{st}_{ind(L)}(S^0) := \lim_{m\to\infty} [S^{m+ind(L)}, S^m]_*$ Stable homotopy group of spheres.

(i.e. $T_* f^{-1}(0) \oplus \mathbb{R}^m$ has a trivializaton $m \gg 0$)

Remark: $f^{-1}(0)$ is a stably framed manifold. It's stably framed cobordism class is given by $\alpha$.

Sketch proof of theorem:

$k \geq 0$

$\forall s_1, s_2 \in C^\infty(X, E)$, define

$$\langle s_1, s_2 \rangle_{L^2_k} := \int_X \langle s_1, s_2 \rangle \, dvol + \int \langle \nabla s_1, \nabla s_2 \rangle \, dvol + \cdots + \int \langle \nabla^k s_1, \nabla^k s_2 \rangle \, dvol$$

$|s|_{L^2_k} := \langle s, s \rangle_{L^2_k}$ Sobolev norm.

Define: $L^2_k(X;E) :=$ completion of $C^\infty(X,E)$ w.r.t. $\|-\|_{L^2_k}$

Sobolev space. A Hilbert space.

Some properties: i) $L : L^2_{k+1}(X;E) \to L^2_k(X;F)$ is a Fredholm operator (i.e. finite dim ker, coker)

ii) $\exists C$ s.t. $\|f\|_{L^2_{k+1}} \leq C (\|Lf\|_{L^2_k} + \|f\|_{L^2})$

iii) For any sequence $\{f_i\}$ in $L^2_{k+1}(X;E)$ s.t. $\|f_i\|_{L^2_{k+1}}$ bounded

After passing to a subsequence, we may assume $\exists f_\infty \in L^2_{k+1}(X;E)$

- $f_i \xrightarrow{L^2_k} f_\infty$ $\quad (\|f_i - f_\infty\|_{L^2_k} \to 0)$
- $f_i \xrightarrow{\text{weakly}} L^2_{k+1}\ f_\infty$ $\quad (\forall g \in L^2_{k+1}\ \langle f_n, g\rangle_{L^2_{k+1}} \to \langle f_\infty, g\rangle_{L^2_{k+1}}$ as $n \to +\infty)$

vi) $2k > \dim(X)$. Then $Q$ extends to

$$Q : L^2_k(X;E) \to L^2_k(X;F).$$

Let $U = L^2_{k+1}(X;E)$ $\quad V = L^2_k(X;F)$. Then we have a continuous map.

$SW := L + Q : U \to V$ $\quad$ By key assumption $SW^{-1}(0) \subset B_R(U)$

$$B_R(U) = \{v \in U \mid \|v\| \le R\}$$

Take a sequence $V_1 \subset V_2 \subset \cdots \subset V$

s.t. iv) $\dim(V_i) < \infty$

v) $V_i + \operatorname{image}(L) = V \quad \forall i$

vi) $\forall x \in V \quad Pr_{V_n}(x) \to x$ as $n \to +\infty$

Let $U_n = L^{-1}(V_n)$ Then $\dim U_n - \dim V_n = \operatorname{index}(L)$

Define $SW_n := L + Pr_{V_n} \circ Q : U_n \to V_n$

Theorem: $\exists n_0 \gg 0$ small $\varepsilon_0 > 0$ s.t. $\forall n > n_0$

$$SW_n^{-1}(B_{\varepsilon_0}(V_n)) \cap S_{R+1}(U_n) = \phi$$

Corollary: $SW_n$ induces a map

$$SW_n^+ : B_{R+1}(U_n)/S_{R+1}(U_n) \longrightarrow V_n/(V_n - \mathring{B}_{\varepsilon_0}(V_n))$$

$$\begin{array}{ccc} \cong & & \cong \\ U_n^+ & & V_n^+ \end{array}$$

$[SW_n^+] \in \pi^{st}_{\mathrm{index}(L)}(S^0)$ only depends on $L + Q$.

proof of theorem (sketch): proof by contradiction

Assume $f_n \in S_{R+1}(U_n)$ with $\overset{\text{vii)}}{} SW_n(f_n) \xrightarrow{L^2_k} 0$.

By passing to subsequence we may assume $\exists f_\infty \in U$

$f_n \xrightarrow{L^2_k} f_\infty \qquad f_n \xrightarrow{\text{weak } L^2_{k+1}} f_\infty$

1) $Q(f_n) \xrightarrow{L^2_k} Q(f_\infty)$

$\Downarrow$ (vi)

2) $pr^Q_{V_n}(f_n) \xrightarrow{L^2_k} Q(f_\infty)$

3) $L(f_n) \xrightarrow{\text{weak}} {}_{L^2_k}\ L(f_\infty)$

$\Downarrow$

4) $(L + pr^Q_{V_n})(f_n) \xrightarrow{\text{weak}} {}_{L^2_k}\ (L+Q)(f_\infty)$

$\Downarrow$

5) $(L+Q)(f_\infty) = 0$

$\Downarrow$

6) $\|L(f_n) - L(f_\infty)\|_{L^2_k} \leq \|SW_n(f_n) - SW(f_\infty)\|_{L^2_k}$
$\quad + \|pr^Q_{V_n}(f_n) - Q(f_\infty)\|_{L^2_k} \xrightarrow{(5)+} 0$

$\Downarrow$

7) $f_n \xrightarrow{L^2_{k+1}} f_\infty$

$\Downarrow$

$\|f_\infty\|_{L^2_{k+1}} = \lim \|f_n\| = R+1$. contradiction $\square$.

• Bauer-Furuta invariant of smooth 4-manifolds

$X$: smooth 4-mfd $b_1(X)=0$

Consider the frame bundle $SO(4) \hookrightarrow Fr_X \to X$

A spin structure $\mathfrak{s}$ is a lift $Spin(4) \hookrightarrow P \to X$

Here $Spin(4) = SU(2) \times SU(2)$ is 2-fold cover of $SO(4)$.

($X$ has a spin str. $\Leftrightarrow$ $w_2(TX)=0$)

Given $\mathfrak{s}$, one can define the Seiberg-Witten equations

$$\left\{\begin{array}{llll} \boxed{d^+\alpha} & + & \boxed{\rho^{-1}(\phi\phi^*)_0} & = 0 \\ \boxed{\not{D}\phi} & + & \boxed{\rho(\alpha)\phi} & = 0 \\ \boxed{d^*\alpha} & & & = 0 \end{array}\right. \qquad \begin{array}{l} \alpha \in \Omega^1(X; i\mathbb{R}) \\ \phi \in C^\infty(X; E) \end{array}$$

$$\underbrace{\phantom{d^+\alpha}}_{L(\alpha,\phi)} \qquad \underbrace{\phantom{\rho(\alpha)\phi}}_{Q(\alpha,\phi)}$$

$E$: rank-2 complex vector bundle over $X$.

$SW = L + Q$ satisfies the Key assumption. So

We have invariant $BF(X, \mathfrak{s}) \in \pi^{st}_{\mathrm{ind}(L)}(S^0)$

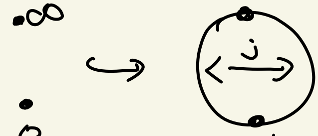
non-equivariant Bauer-Furuta invariant. $\quad \mathrm{ind}(L) = \dfrac{-\sigma(X)}{4} - b^+(X)$

E.g. $BF(S^4) = 1 \in \pi_0^{st}$ $\quad BF(S^2\times S^2) = 0 \in \pi^{st}_{-1}$

$BF(K3) = \eta \in \pi_1^{st}$ where $\eta: S^3 \to S^2$ is the Hopf map.

We actually have more, the S.W. eqs has a symmetry group of $Pin(2) = \{e^{i\theta}\} \cup \{j e^{i\theta}\} \subset \mathbb{H}$. ⇝ quaternion.

So $SW_n^+$ is really a $Pin(2)$-equivariant map and gives an element $BF^{Pin(2)} \in \pi_\star^{Pin(2)}(S^0)$

$BF^{Pin(2)}(S^2\times S^2) = [$  $]$ where $j$-acts on $S^1$ by reflection



Key observation: $BF^{Pin(2)}(S^2\times S^2)\neq 0.^{S^1}$

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Application: exotic phenomina and stabilization

Exotic phenomina in dim 4: Smooth category $\neq$ topological category

- $\exists X, X'$ s.t $X$ homeomorphic to $X'$ but not diffeomorphic (exotic smooth str.)
- $\exists f_0, f_1 : X \to X$ diffeomorphism s.t. $f_0$ is topologically isotopic to $f_1$ but not smoothly so (exotic diffeomorphism)
- $\exists i_0, i_1 : \Sigma \hookrightarrow X$ s.t. $i_0$ is topologically isotopic to $i_1$ but not smoothly so (exotic surfaces)

Stabilization: Taking connected sum with $S^2\times S^2$.

Theorem (Wall, Perron, Quinn) Exotic pheonomina on simply connected 4-mfds all dissapears after sufficiently many stabilizatoin.

Q: Is one stabilization enough? A: No

Thm (L.) $\exists$ exotic diffeomorphism $f_0, f_1 : K3\# K3 \circlearrowleft$ that remains exotic after one stabilization.

Thm (L.-Mukherjee) $\exists$ exotic surfaces $\bigsqcup_{22} D^2 \hookrightarrow K3 \setminus \mathring{D}^4$ that remains exotic after one stabilization.

proved using $BF^{Pin(2)}$.