Extended Graph 4-Manifolds, and Einstein Metrics

Luca F. Di Cerbo

Geometry and Topology Seminar,
Yau Mathematical Sciences Center, Tsinghua University,
November 4, 2021

Discussion will mention results from

Discussion will mention results from

Extended Graph 4-Manifolds, and Einstein Metrics

Discussion will mention results from

Extended Graph 4-Manifolds, and Einstein Metrics

e-print arXiv:2106.13279v3 [math.DG]

Discussion will mention results from

Extended Graph 4-Manifolds, and Einstein Metrics

e-print arXiv:2106.13279v3 [math.DG]

to appear in Annales Mathématiques du Quebéc, 2022

The Goal(s) of this Lecture

- Give an overview of the study of Einstein Metrics on Closed Manifolds especially in Dimension Four;

The Goal(s) of this Lecture

- Give an overview of the study of Einstein Metrics on Closed Manifolds especially in Dimension Four;

- Prove a Non-Existence Theorem for Einstein metrics on Extended Graph 4-Manifolds;

The Goal(s) of this Lecture

- Give an overview of the study of Einstein Metrics on Closed Manifolds especially in Dimension Four;
- Prove a Non-Existence Theorem for Einstein metrics on Extended Graph 4-Manifolds;
- Ideally, inspire you to find this Stuff Interesting;

The Goal(s) of this Lecture

- Give an overview of the study of Einstein Metrics on Closed Manifolds especially in Dimension Four;
- Prove a Non-Existence Theorem for Einstein metrics on Extended Graph 4-Manifolds;
- Ideally, inspire you to find this Stuff Interesting;
- Finally, I want to discuss some related open problems for Higher Graph 4-Manifolds and possible generalizations to Higher Dimensions.

Introduction: Metrics and Distance

The objects of this talk are Riemannian Manifolds equipped with special Riemannian metrics known as Einstein Metrics.

Introduction: Metrics and Distance

The objects of this talk are Riemannian Manifolds equipped with special Riemannian metrics known as Einstein Metrics.

Definition

Let M^n be a smooth orientable n-manifold (e.g., a smooth embedded surface in $\mathbb{R}^3$). A Riemannian metric g on M is choice of a positive definite inner product on each tangent space T_pM varying smoothly with $p \in M$.

Introduction: Metrics and Distance

The objects of this talk are Riemannian Manifolds equipped with special Riemannian metrics known as Einstein Metrics.

Definition

Let M^n be a smooth orientable n-manifold (e.g., a smooth embedded surface in $\mathbb{R}^3$). A Riemannian metric g on M is choice of a positive definite inner product on each tangent space T_pM varying smoothly with $p \in M$.

Given a smooth path in a Riemannian manifold (M, g)

$$\alpha : [a, b] \to M,$$

we define its length by setting

$$L(\alpha) = \int_a^b g(\alpha'(t), \alpha'(t))^{1/2} dt, \quad \alpha'(t) \in T_{\alpha(t)}M.$$

We then define the distance between $p_1, p_2 \in M$, denoted by $d(p_1, p_2)$, to be the infimun for L over all smooth paths joining p_1 and p_2.

Volumes, Ricci Curvature, and Einstein Metrics

The **expansion** of the **Volume Element** in Geodesic Normal Coordinates and is given by:

$$d\mu_g = \left(1 - \frac{1}{6} Ric_{ij} x^i x^j + O(|x|^3)\right) dx^1 \wedge ... \wedge dx^n$$

Volumes, Ricci Curvature, and Einstein Metrics

The **expansion** of the **Volume Element** in Geodesic Normal Coordinates and is given by:

$$d\mu_g = \left(1 - \frac{1}{6} Ric_{ij} x^i x^j + O(|x|^3)\right) dx^1 \wedge \ldots \wedge dx^n$$

This defines a symmetric two tensor (a symmetric bilinear form on each tangent space $T_p M$)

$$Ric_g = Ric_{ij} dx^i \otimes dx^j$$

know as the **Ricci Tensor**.

Volumes, Ricci Curvature, and Einstein Metrics

The **expansion** of the **Volume Element** in Geodesic Normal Coordinates and is given by:

$$d\mu_g = \left(1 - \frac{1}{6}Ric_{ij}x^i x^j + O(|x|^3)\right)dx^1 \wedge \ldots \wedge dx^n$$

This defines a symmetric two tensor (a symmetric bilinear form on each tangent space T_pM)

$$Ric_g = Ric_{ij}dx^i \otimes dx^j$$

know as the **Ricci Tensor**.

Volumes, Ricci Curvature, and Einstein Metrics

The **expansion** of the **Volume Element** in Geodesic Normal Coordinates and is given by:

$$d\mu_g = \left(1 - \frac{1}{6} Ric_{ij} x^i x^j + O(|x|^3)\right) dx^1 \wedge \ldots \wedge dx^n$$

This defines a symmetric two tensor (a symmetric bilinear form on each tangent space $T_p M$)

$$Ric_g = Ric_{ij} dx^i \otimes dx^j$$

know as the **Ricci Tensor**.

Definition

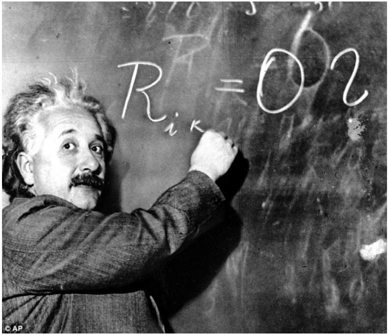
A Riemannian metric is said to be **Einstein** *if its Ricci Tensor satisfies*

$$Ric_g = \lambda g,$$

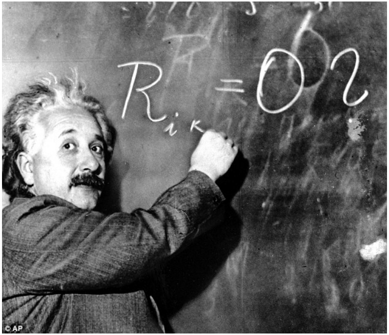
where the constant $\lambda \in \mathbb{R}$ is known as the cosmological or Einstein constant.

Here is a famous picture of **Einstein** looking for such metrics:

Here is a famous picture of **Einstein** looking for such metrics:

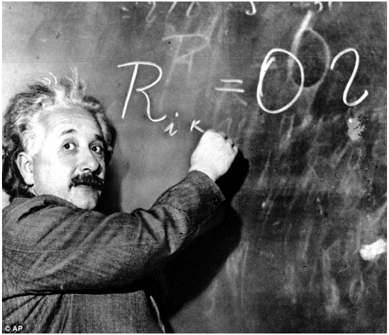


Here is a famous picture of **Einstein** looking for such metrics:



Apparently, here he is interested in the case $\lambda = 0$!

Here is a famous picture of **Einstein** looking for such metrics:



Apparently, here he is interested in the case $\lambda = 0$!
And of course, he was exploring the **Lorentzian** case...

Classical Examples of Einstein Metrics

Einstein metrics always exist in dimension $n = 2$. Indeed, we have

$$Ric_g = Kg$$

where K is the **Gauss** curvature function, and by the **uniformization theorem** we can always find metrics with constant **Gauss** curvature!

Classical Examples of Einstein Metrics

Einstein metrics always exist in dimension $n = 2$. Indeed, we have

$$Ric_g = Kg$$

where K is the **Gauss** curvature function, and by the **uniformization theorem** we can always find metrics with constant **Gauss** curvature!

More precisely, if $\mathbf{G} \geq 0$ is the genus of a **orientable closed surface** we then have

Classical Examples of Einstein Metrics

Einstein metrics always exist in dimension $n = 2$. Indeed, we have

$$Ric_g = Kg$$

where K is the **Gauss** curvature function, and by the **uniformization theorem** we can always find metrics with constant **Gauss** curvature!

More precisely, if $\mathbf{G} \geq 0$ is the genus of a **orientable closed surface** we then have

$$K = \begin{cases} 1 & \text{spherical} & \iff & \mathbf{G} = 0, \\ 0 & \text{flat} & \iff & \mathbf{G} = 1, \\ -1 & \text{hyperbolic} & \iff & \mathbf{G} \geq 2. \end{cases}$$

Classical Examples of Einstein Metrics

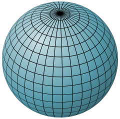
Einstein metrics always exist in dimension $n = 2$. Indeed, we have

$$Ric_g = Kg$$

where K is the **Gauss** curvature function, and by the **uniformization theorem** we can always find metrics with constant **Gauss** curvature!

More precisely, if $\mathbf{G} \geq 0$ is the genus of a **orientable closed surface** we then have

$$K = \begin{cases} 1 & \text{spherical} & \iff & \mathbf{G} = 0, \\ 0 & \text{flat} & \iff & \mathbf{G} = 1, \\ -1 & \text{hyperbolic} & \iff & \mathbf{G} \geq 2. \end{cases}$$



Similarly, Riemannian manifolds (M^n, g) with constant sectional curvature (real space forms) are example of Einstein manifolds

Similarly, Riemannian manifolds (M^n, g) with constant sectional curvature (real space forms) are example of Einstein manifolds

$$Sec = \begin{cases} 1 & \text{Universal cover is } \mathbb{S}^n, \quad \Longrightarrow \quad Ric_g = (n-1)g; \\ 0 & \text{Universal cover is } \mathbb{R}^n, \quad \Longrightarrow \quad Ric_g = 0; \\ -1 & \text{Universal cover is } \mathbb{H}^n, \quad \Longrightarrow \quad Ric_g = -(n-1)g. \end{cases}$$

Similarly, Riemannian manifolds (M^n, g) with constant sectional curvature (real space forms) are example of Einstein manifolds

$$Sec = \begin{cases} 1 & \text{Universal cover is } \mathbb{S}^n, \quad \Longrightarrow \quad Ric_g = (n-1)g; \\ 0 & \text{Universal cover is } \mathbb{R}^n, \quad \Longrightarrow \quad Ric_g = 0; \\ -1 & \text{Universal cover is } \mathbb{H}^n, \quad \Longrightarrow \quad Ric_g = -(n-1)g. \end{cases}$$

Constant curvature examples are then plentiful and very interesting from a **global** point of view especially in the hyperbolic case, e.g., deep connections with group theory, lattices in $\mathrm{PO}(n, 1)$, and so on.

Similarly, Riemannian manifolds (M^n, g) with constant sectional curvature (real space forms) are example of Einstein manifolds

$$Sec = \begin{cases} 1 & \text{Universal cover is } \mathbb{S}^n, \quad \Longrightarrow \quad Ric_g = (n-1)g; \\ 0 & \text{Universal cover is } \mathbb{R}^n, \quad \Longrightarrow \quad Ric_g = 0; \\ -1 & \text{Universal cover is } \mathbb{H}^n, \quad \Longrightarrow \quad Ric_g = -(n-1)g. \end{cases}$$

Constant curvature examples are then plentiful and very interesting from a **global** point of view especially in the hyperbolic case, e.g., deep connections with group theory, lattices in $\mathrm{PO}(n, 1)$, and so on.

With that said, these examples of Einstein manifolds are all locally symmetric and as such a bit **boring** from a local geometry point of view! Indeed, around each point they look exactly the same...

Do we have examples of Einstein metrics which are NOT locally symmetric?

Do we have examples of Einstein metrics which are NOT locally symmetric?

I will try to answer this question with a picture:

Do we have examples of Einstein metrics which are NOT locally symmetric?

I will try to answer this question with a picture:

Do we have examples of Einstein metrics which are NOT locally symmetric?

I will try to answer this question with a picture:

S.-T. Yau (1949 –), **Fields Medal** in 1982.

Theorem (Yau)

- *A compact Kähler manifold M admits a* **Kähler-Einstein metric** *with $\lambda = 0$ (Ricci-flat) if and only if $c_1(M) = 0 \quad (c_1(M) \in H^2_{dR}(M))$;*
- *A compact Kähler manifold M admits a* **Kähler-Einstein metric** *with $\lambda < 0$ (negative Ricci curvature) if and only if $c_1(M) < 0$.*

Theorem (Yau)

- *A compact Kähler manifold* M *admits a* **Kähler-Einstein metric** *with* $\lambda = 0$ *(Ricci-flat) if and only if* $c_1(M) = 0 \quad (c_1(M) \in H^2_{dR}(M))$;
- *A compact Kähler manifold* M *admits a* **Kähler-Einstein metric** *with* $\lambda < 0$ *(negative Ricci curvature) if and only if* $c_1(M) < 0$.

Theorem (Yau)

- *A compact Kähler manifold M admits a* **Kähler-Einstein metric** *with $\lambda = 0$ (Ricci-flat) if and only if $c_1(M) = 0 \quad (c_1(M) \in H^2_{dR}(M))$;*
- *A compact Kähler manifold M admits a* **Kähler-Einstein metric** *with $\lambda < 0$ (negative Ricci curvature) if and only if $c_1(M) < 0$.*

Recall that

Theorem (Yau)

- *A compact Kähler manifold M admits a* **Kähler-Einstein metric** *with $\lambda = 0$ (Ricci-flat) if and only if $c_1(M) = 0 \quad (c_1(M) \in H^2_{dR}(M))$;*
- *A compact Kähler manifold M admits a* **Kähler-Einstein metric** *with $\lambda < 0$ (negative Ricci curvature) if and only if $c_1(M) < 0$.*

Recall that

A Kähler manifold is an even dimensional real manifold which can be covered by holomorphic charts, equipped with a metric ω which can be locally written as $\quad \omega = \sqrt{-1}\partial\bar{\partial}\phi, \quad \phi : U \to \mathbb{R}$.

Theorem (Yau)

- *A compact Kähler manifold M admits a* **Kähler-Einstein metric** *with $\lambda = 0$ (Ricci-flat) if and only if $c_1(M) = 0 \quad (c_1(M) \in H^2_{dR}(M))$;*
- *A compact Kähler manifold M admits a* **Kähler-Einstein metric** *with $\lambda < 0$ (negative Ricci curvature) if and only if $c_1(M) < 0$.*

Recall that

A Kähler manifold is an even dimensional real manifold which can be covered by holomorphic charts, equipped with a metric ω which can be locally written as $\omega = \sqrt{-1}\partial\bar{\partial}\phi, \quad \phi : U \to \mathbb{R}.$

Reference (Cited 958 times in Mathscinet! 11/04/2021)

Theorem (Yau)

- *A compact Kähler manifold M admits a* **Kähler-Einstein metric** *with $\lambda = 0$ (Ricci-flat) if and only if $c_1(M) = 0 \quad (c_1(M) \in H^2_{dR}(M))$;*
- *A compact Kähler manifold M admits a* **Kähler-Einstein metric** *with $\lambda < 0$ (negative Ricci curvature) if and only if $c_1(M) < 0$.*

Recall that

A Kähler manifold is an even dimensional real manifold which can be covered by holomorphic charts, equipped with a metric ω which can be locally written as $\quad \omega = \sqrt{-1}\partial\bar{\partial}\phi, \quad \phi : U \to \mathbb{R}$.

Reference (Cited 958 times in Mathscinet! 11/04/2021)

S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampére equation. I.*, **Comm. Pure Appl. Math. 31** (1978), no. 3, 339–411.

Examples in Dimension n=4

- **K3 Surfaces**. Examples of such surfaces can be concretely constructed by looking at degree four hypersurfaces in $\mathbb{P}^3_{\mathbb{C}}$. For example

$$M := \{z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0, [z] \in \mathbb{P}^3_{\mathbb{C}}\} \quad \Rightarrow \quad c_1(M) = 0.$$

By Yau they support Ricci-flat Kähler metrics;

Examples in Dimension n=4

- **K3 Surfaces**. Examples of such surfaces can be concretely constructed by looking at degree four hypersurfaces in $\mathbb{P}^3_{\mathbb{C}}$. For example

$$M := \{z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0, [z] \in \mathbb{P}^3_{\mathbb{C}}\} \quad \Rightarrow \quad c_1(M) = 0.$$

  By Yau they support Ricci-flat Kähler metrics;

- **Degree $d > 4$ Hypersurfaces in Complex Projective 3-Space**. For example

$$M := \{z_0^d + z_1^d + z_2^d + z_3^d = 0, d > 4, [z] \in \mathbb{P}^3_{\mathbb{C}}\} \quad \Rightarrow \quad c_1(M) < 0.$$

  By Yau they support Kähler-Einstein metrics with $\lambda < 0$;

Examples in Dimension n=4

- **K3 Surfaces**. Examples of such surfaces can be concretely constructed by looking at degree four hypersurfaces in $\mathbb{P}^3_{\mathbb{C}}$. For example

$$M := \{z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0, [z] \in \mathbb{P}^3_{\mathbb{C}}\} \quad \Rightarrow \quad c_1(M) = 0.$$

  By Yau they support Ricci-flat Kähler metrics;

- **Degree $d > 4$ Hypersurfaces in Complex Projective 3-Space**. For example

$$M := \{z_0^d + z_1^d + z_2^d + z_3^d = 0, d > 4, [z] \in \mathbb{P}^3_{\mathbb{C}}\} \quad \Rightarrow \quad c_1(M) < 0.$$

  By Yau they support Kähler-Einstein metrics with $\lambda < 0$;

- Anderson's Dehn Filling Examples ($n \geq 4$, $\lambda < 0$);

Examples in Dimension n=4

- **K3 Surfaces**. Examples of such surfaces can be concretely constructed by looking at degree four hypersurfaces in $\mathbb{P}^3_{\mathbb{C}}$. For example

$$M := \{z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0, [z] \in \mathbb{P}^3_{\mathbb{C}}\} \quad \Rightarrow \quad c_1(M) = 0.$$

  By Yau they support Ricci-flat Kähler metrics;

- **Degree $d > 4$ Hypersurfaces in Complex Projective 3-Space**. For example

$$M := \{z_0^d + z_1^d + z_2^d + z_3^d = 0, d > 4, [z] \in \mathbb{P}^3_{\mathbb{C}}\} \quad \Rightarrow \quad c_1(M) < 0.$$

  By Yau they support Kähler-Einstein metrics with $\lambda < 0$;

- Anderson's Dehn Filling Examples ($n \geq 4$, $\lambda < 0$);

- Fine-Premoselli's Branched Covering Examples ($n = 4$, $\lambda < 0$, $Sec < 0$);

Examples in Dimension n=4

- **K3 Surfaces**. Examples of such surfaces can be concretely constructed by looking at degree four hypersurfaces in $\mathbb{P}^3_{\mathbb{C}}$. For example

$$M := \{z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0, [z] \in \mathbb{P}^3_{\mathbb{C}}\} \quad \Rightarrow \quad c_1(M) = 0.$$

  By Yau they support Ricci-flat Kähler metrics;

- **Degree $d > 4$ Hypersurfaces in Complex Projective 3-Space**. For example

$$M := \{z_0^d + z_1^d + z_2^d + z_3^d = 0, d > 4, [z] \in \mathbb{P}^3_{\mathbb{C}}\} \quad \Rightarrow \quad c_1(M) < 0.$$

  By Yau they support Kähler-Einstein metrics with $\lambda < 0$;

- Anderson's Dehn Filling Examples ($n \geq 4$, $\lambda < 0$);

- Fine-Premoselli's Branched Covering Examples ($n = 4$, $\lambda < 0$, $Sec < 0$);

Examples in Dimension n=4

- **K3 Surfaces**. Examples of such surfaces can be concretely constructed by looking at degree four hypersurfaces in $\mathbb{P}^3_{\mathbb{C}}$. For example

$$M := \{z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0, [z] \in \mathbb{P}^3_{\mathbb{C}}\} \quad \Rightarrow \quad c_1(M) = 0.$$

  By Yau they support Ricci-flat Kähler metrics;

- **Degree $d > 4$ Hypersurfaces in Complex Projective 3-Space**. For example

$$M := \{z_0^d + z_1^d + z_2^d + z_3^d = 0, d > 4, [z] \in \mathbb{P}^3_{\mathbb{C}}\} \quad \Rightarrow \quad c_1(M) < 0.$$

  By Yau they support Kähler-Einstein metrics with $\lambda < 0$;

- Anderson's Dehn Filling Examples ($n \geq 4$, $\lambda < 0$);

- Fine-Premoselli's Branched Covering Examples ($n = 4$, $\lambda < 0$, $Sec < 0$);

- **Hopefully** many more to come in my life time!

Here are the relevant papers for the most recent classes of examples:

Here are the relevant papers for the most recent classes of examples:

M. T. Anderson, *Dehn filling and Einstein metrics in higher dimensions*, **J. Differential Geom. 73** (2006), no. 2, 219-261.

Here are the relevant papers for the most recent classes of examples:

M. T. Anderson, *Dehn filling and Einstein metrics in higher dimensions*, **J. Differential Geom. 73** (2006), no. 2, 219-261.

R. Bamler, *Construction of Einstein metrics by generalized Dehn filling*, **J. Eur. Math. Soc. (JEMS) 14** (2012), no. 3, 887-909.

Here are the relevant papers for the most recent classes of examples:

M. T. Anderson, *Dehn filling and Einstein metrics in higher dimensions*, **J. Differential Geom. 73** (2006), no. 2, 219-261.

R. Bamler, *Construction of Einstein metrics by generalized Dehn filling*, **J. Eur. Math. Soc. (JEMS) 14** (2012), no. 3, 887-909.

J. Fine, B. Premoselli, *Examples of compact Einstein four-manifolds with negative curvature*, **J. Amer. Math. Soc. 33** (2020), no. 4, 991-1038.

Obstructions to the Existence of Einstein Metrics

Theorem (Hitchin, 1974)

Let (M, g) *be a closed orientable Einstein* 4*-manifold with signature* σ *and Euler characteristic* χ. *Then*

$$\chi(M) \geq \frac{3}{2}|\sigma(M)|$$

Furthermore, if equality occurs $\pm M$ *is either flat or its universal covering is a* **K3** *surface.*

Proof.

$\square$

Obstructions to the Existence of Einstein Metrics

Theorem (Hitchin, 1974)

Let (M, g) be a closed orientable Einstein 4-manifold with signature σ and Euler characteristic χ. Then

$$\chi(M) \geq \frac{3}{2}|\sigma(M)|$$

Furthermore, if equality occurs $\pm M$ is either flat or its universal covering is a **K3** *surface.*

Proof.

By **Chern**

$\square$

Obstructions to the Existence of Einstein Metrics

Theorem (Hitchin, 1974)

Let (M, g) be a closed orientable Einstein 4-manifold with signature σ and Euler characteristic χ. Then

$$\chi(M) \geq \frac{3}{2}|\sigma(M)|$$

Furthermore, if equality occurs $\pm M$ is either flat or its universal covering is a **K3** *surface.*

Proof.

By **Chern** $\Rightarrow$

∎

Obstructions to the Existence of Einstein Metrics

Theorem (Hitchin, 1974)

Let (M, g) be a closed orientable Einstein 4-manifold with signature σ and Euler characteristic χ. Then

$$\chi(M) \geq \frac{3}{2}|\sigma(M)|$$

Furthermore, if equality occurs $\pm M$ is either flat or its universal covering is a **K3** *surface.*

Proof.

By **Chern** $\Rightarrow \chi(M) = \frac{1}{8\pi^2}\int_M |W^+|^2 + |W^-|^2 + \frac{s^2}{24} - \frac{|\mathring{Ric}|^2}{2} d\mu_g$

$\square$

Obstructions to the Existence of Einstein Metrics

Theorem (Hitchin, 1974)

Let (M, g) be a closed orientable Einstein 4-manifold with signature σ and Euler characteristic χ. Then

$$\chi(M) \geq \frac{3}{2}|\sigma(M)|$$

Furthermore, if equality occurs $\pm M$ is either flat or its universal covering is a **K3** *surface.*

Proof.

By **Chern** $\Rightarrow \chi(M) = \frac{1}{8\pi^2}\int_M |W^+|^2 + |W^-|^2 + \frac{s^2}{24} - \frac{|\mathring{Ric}|^2}{2} d\mu_g$

By the **Einstein** condition $Ric_g = \lambda g$

□

Obstructions to the Existence of Einstein Metrics

Theorem (Hitchin, 1974)

Let (M, g) be a closed orientable Einstein 4-manifold with signature σ and Euler characteristic χ. Then

$$\chi(M) \geq \frac{3}{2}|\sigma(M)|$$

Furthermore, if equality occurs $\pm M$ is either flat or its universal covering is a **K3** *surface.*

Proof.

By **Chern** $\Rightarrow \chi(M) = \frac{1}{8\pi^2}\int_M |W^+|^2 + |W^-|^2 + \frac{s^2}{24} - \frac{|\mathring{Ric}|^2}{2} d\mu_g$

By the **Einstein** condition $Ric_g = \lambda g \quad \Rightarrow$

$\square$

Obstructions to the Existence of Einstein Metrics

Theorem (Hitchin, 1974)

Let (M, g) be a closed orientable Einstein 4-manifold with signature σ and Euler characteristic χ. Then

$$\chi(M) \geq \frac{3}{2}|\sigma(M)|$$

Furthermore, if equality occurs $\pm M$ is either flat or its universal covering is a **K3** *surface.*

Proof.

By **Chern** $\Rightarrow \chi(M) = \frac{1}{8\pi^2}\int_M |W^+|^2 + |W^-|^2 + \frac{s^2}{24} - \frac{|\mathring{Ric}|^2}{2} d\mu_g$

By the **Einstein** condition $Ric_g = \lambda g \quad \Rightarrow \quad \mathring{Ric} = 0$

□

Obstructions to the Existence of Einstein Metrics

Theorem (Hitchin, 1974)

Let (M, g) be a closed orientable Einstein 4-manifold with signature σ and Euler characteristic χ. Then

$$\chi(M) \geq \frac{3}{2}|\sigma(M)|$$

Furthermore, if equality occurs $\pm M$ is either flat or its universal covering is a **K3** *surface.*

Proof.

By **Chern** $\Rightarrow \chi(M) = \frac{1}{8\pi^2}\int_M |W^+|^2 + |W^-|^2 + \frac{s^2}{24} - \frac{|\mathring{Ric}|^2}{2} d\mu_g$

By the **Einstein** condition $Ric_g = \lambda g \Rightarrow \mathring{Ric} = 0$

By **Hirzebruch**

$\square$

Obstructions to the Existence of Einstein Metrics

Theorem (Hitchin, 1974)

Let (M, g) be a closed orientable Einstein 4-manifold with signature σ and Euler characteristic χ. Then

$$\chi(M) \geq \frac{3}{2}|\sigma(M)|$$

Furthermore, if equality occurs $\pm M$ is either flat or its universal covering is a **K3** *surface.*

Proof.

By **Chern** $\Rightarrow \chi(M) = \frac{1}{8\pi^2}\int_M |W^+|^2 + |W^-|^2 + \frac{s^2}{24} - \frac{|\mathring{Ric}|^2}{2} d\mu_g$

By the **Einstein** condition $Ric_g = \lambda g \Rightarrow \mathring{Ric} = 0$

By **Hirzebruch** $\Rightarrow$

$\square$

Obstructions to the Existence of Einstein Metrics

Theorem (Hitchin, 1974)

Let (M, g) be a closed orientable Einstein 4-manifold with signature σ and Euler characteristic χ. Then

$$\chi(M) \geq \frac{3}{2}|\sigma(M)|$$

Furthermore, if equality occurs $\pm M$ is either flat or its universal covering is a **K3** *surface.*

Proof.

By **Chern** $\Rightarrow \chi(M) = \frac{1}{8\pi^2}\int_M |W^+|^2 + |W^-|^2 + \frac{s^2}{24} - \frac{|\mathring{Ric}|^2}{2} d\mu_g$

By the **Einstein** condition $Ric_g = \lambda g \quad \Rightarrow \quad \mathring{Ric} = 0$

By **Hirzebruch** $\Rightarrow \sigma(M) = \frac{1}{12\pi^2}\int_M |W^+|^2 - |W^-|^2 d\mu_g$

$\square$

Obstructions to the Existence of Einstein Metrics

Theorem (Hitchin, 1974)

Let (M, g) be a closed orientable Einstein 4-manifold with signature σ and Euler characteristic χ. Then

$$\chi(M) \geq \frac{3}{2}|\sigma(M)|$$

Furthermore, if equality occurs $\pm M$ is either flat or its universal covering is a **K3** *surface.*

Proof.

By **Chern** $\Rightarrow \chi(M) = \frac{1}{8\pi^2}\int_M |W^+|^2 + |W^-|^2 + \frac{s^2}{24} - \frac{|\mathring{Ric}|^2}{2} d\mu_g$

By the **Einstein** condition $Ric_g = \lambda g \quad \Rightarrow \quad \mathring{Ric} = 0$

By **Hirzebruch** $\Rightarrow \sigma(M) = \frac{1}{12\pi^2}\int_M |W^+|^2 - |W^-|^2 d\mu_g$

Combining these identities we obtain the desired inequality! $\square$

Many 4-Manifolds do NOT support Einstein Metrics!

Many 4-Manifolds do NOT support Einstein Metrics!

- $S^1 \times S^3$ (M. Berger). Notice that

$$\chi(S^1 \times S^3) = \sigma(S^1 \times S^3) = 0,$$

but $S^1 \times S^3$ is **not** flat! Indeed, its universal cover is $\mathbb{R} \times S^3$.

Many 4-Manifolds do NOT support Einstein Metrics!

- $S^1 \times S^3$ (M. Berger). Notice that
$$\chi(S^1 \times S^3) = \sigma(S^1 \times S^3) = 0,$$
but $S^1 \times S^3$ is **not** flat! Indeed, its universal cover is $\mathbb{R} \times S^3$.

- **Non-Abelian 4-(Infra)Nilmanifolds** N^4. Again, it is easy to realize that
$$\chi(N^4) = \sigma(N^4) = 0,$$
but $\pi_1(N^4)$ grows polynomially of degree $d \geq 5$ (cannot be flat!).

Many 4-Manifolds do NOT support Einstein Metrics!

- $S^1 \times S^3$ (M. Berger). Notice that

$$\chi(S^1 \times S^3) = \sigma(S^1 \times S^3) = 0,$$

but $S^1 \times S^3$ is **not** flat! Indeed, its universal cover is $\mathbb{R} \times S^3$.

- **Non-Abelian 4-(Infra)Nilmanifolds** N^4. Again, it is easy to realize that

$$\chi(N^4) = \sigma(N^4) = 0,$$

but $\pi_1(N^4)$ grows polynomially of degree $d \geq 5$ (cannot be flat!).

- **Connected sum of two 4-dimensional tori :** $T^4 \# T^4$ (M. Berger). Notice that $\chi(T^4) = 0$, so that

$$\chi(T^4 \# T^4) = -2 < 0 \quad \Rightarrow \quad \textbf{No Einstein Metrics!}$$

Many 4-Manifolds do NOT support Einstein Metrics!

- $S^1 \times S^3$ (M. Berger). Notice that

$$\chi(S^1 \times S^3) = \sigma(S^1 \times S^3) = 0,$$

but $S^1 \times S^3$ is **not** flat! Indeed, its universal cover is $\mathbb{R} \times S^3$.

- **Non-Abelian 4-(Infra)Nilmanifolds** N^4. Again, it is easy to realize that

$$\chi(N^4) = \sigma(N^4) = 0,$$

but $\pi_1(N^4)$ grows polynomially of degree $d \geq 5$ (cannot be flat!).

- **Connected sum of two 4-dimensional tori :** $T^4 \# T^4$ (M. Berger). Notice that $\chi(T^4) = 0$, so that

$$\chi(T^4 \# T^4) = -2 < 0 \quad \Rightarrow \quad \textbf{No } \text{Einstein Metrics!}$$

- Many more...

Main Result, Conclusions, and Open Problems

The **Main Result I** want to present is

The **Main Result I** want to present is

A Non-Existence Theorem for Einstein Metrics

The **Main Result I** want to present is

A Non-Existence Theorem for Einstein Metrics

On a Large Class of 4-Manifolds

The **Main Result I** want to present is

A Non-Existence Theorem for Einstein Metrics

On a Large Class of 4-Manifolds

That satisfies the **Hitchin-Thorpe** Inequality

The **Main Result I** want to present is

A Non-Existence Theorem for Einstein Metrics

On a Large Class of 4-Manifolds

That satisfies the **Hitchin-Thorpe** Inequality

And

The **Main Result I** want to present is

A Non-Existence Theorem for Einstein Metrics

On a Large Class of 4-Manifolds

That satisfies the **Hitchin-Thorpe** Inequality

And

Generalizes the Class of **Geometric Aspherical** 3-Manifolds

The **Main Result I** want to present is

A Non-Existence Theorem for Einstein Metrics

On a Large Class of 4**-Manifolds**

That satisfies the **Hitchin-Thorpe** Inequality

And

Generalizes the Class of **Geometric Aspherical** 3-Manifolds

To

Main Result, Conclusions, and Open Problems

The **Main Result I** want to present is

A Non-Existence Theorem for Einstein Metrics

On a Large Class of 4-Manifolds

That satisfies the **Hitchin-Thorpe** Inequality

And

Generalizes the Class of **Geometric Aspherical** 3-Manifolds

To

Dimension $n = 4$

The **Main Result I** want to present is

A Non-Existence Theorem for Einstein Metrics

On a Large Class of 4-Manifolds

That satisfies the **Hitchin-Thorpe** Inequality

And

Generalizes the Class of **Geometric Aspherical** 3-Manifolds

To

Dimension $n = 4$

Known As

The **Main Result I** want to present is

A Non-Existence Theorem for Einstein Metrics

On a Large Class of 4-Manifolds

That satisfies the **Hitchin-Thorpe** Inequality

And

Generalizes the Class of **Geometric Aspherical** 3-Manifolds

To

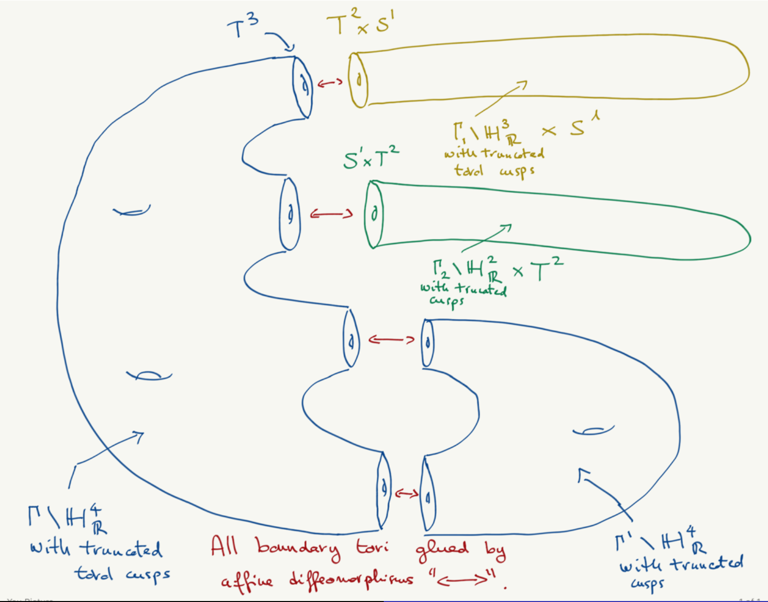
Dimension $n = 4$

Known As

Extended Graph 4-Manifolds

Definition: Extended Graph 4-Manifolds

Definition: Extended Graph 4-Manifolds



- Extended Graph n-manifolds were introduced by Frigerio-Lafont-Sisto: R. Frigerio, J.-F. Lafont, A. Sisto, *Rigidity of High Dimensional Graph Manifolds*, **Astérisque Volume 372**, 2015.

- Extended Graph n-manifolds were introduced by Frigerio-Lafont-Sisto: R. Frigerio, J.-F. Lafont, A. Sisto, *Rigidity of High Dimensional Graph Manifolds*, **Astérisque Volume 372**, 2015.

- Extended Graph n-manifolds were introduced by Frigerio-Lafont-Sisto: R. Frigerio, J.-F. Lafont, A. Sisto, *Rigidity of High Dimensional Graph Manifolds*, **Astérisque Volume 372**, 2015.

Extended Graph n-manifolds are manufactured out of finite volume real-hyperbolic $\Gamma\backslash\mathbb{H}^n_{\mathbb{R}}$ with torus cusps (pure pieces) and

- Extended Graph n-manifolds were introduced by Frigerio-Lafont-Sisto: R. Frigerio, J.-F. Lafont, A. Sisto, *Rigidity of High Dimensional Graph Manifolds*, **Astérisque Volume 372**, 2015.

Extended Graph n-manifolds are manufactured out of finite volume real-hyperbolic $\Gamma\backslash\mathbb{H}^n_{\mathbb{R}}$ with torus cusps (pure pieces) and product pieces modeled on $\Gamma'\backslash\mathbb{H}^{n-k}_{\mathbb{R}} \times T^k$, where T^k is a k-dimensional torus

- Extended Graph n-manifolds were introduced by Frigerio-Lafont-Sisto: R. Frigerio, J.-F. Lafont, A. Sisto, *Rigidity of High Dimensional Graph Manifolds*, **Astérisque Volume 372**, 2015.

Extended Graph n-manifolds are manufactured out of finite volume real-hyperbolic $\Gamma\backslash\mathbb{H}^n_{\mathbb{R}}$ with torus cusps (pure pieces) and

product pieces modeled on $\Gamma'\backslash\mathbb{H}^{n-k}_{\mathbb{R}} \times T^k$, where T^k is a k-dimensional torus

glued together along their tori boundaries via affine diffeomorphisms.

- Extended Graph n-manifolds were introduced by Frigerio-Lafont-Sisto: R. Frigerio, J.-F. Lafont, A. Sisto, *Rigidity of High Dimensional Graph Manifolds*, **Astérisque Volume 372**, 2015.

Extended Graph n-manifolds are manufactured out of finite volume real-hyperbolic $\Gamma\backslash\mathbb{H}^n_{\mathbb{R}}$ with torus cusps (pure pieces) and

product pieces modeled on $\Gamma'\backslash\mathbb{H}^{n-k}_{\mathbb{R}} \times T^k$, where T^k is a k-dimensional torus

glued together along their tori boundaries via affine diffeomorphisms.

Notice we always have **more than** one piece!

Non-Existence Theorem

Non-Existence Theorem

Theorem (DC, 2021)

Closed Extended graph 4-manifolds do not support Einstein metrics.

Remarks and Comments

Non-Existence Theorem

Theorem (DC, 2021)

Closed Extended graph 4-manifolds do not support Einstein metrics.

Remarks and Comments

- Notice that Closed Extended graph 3-manifolds do **NOT** support Einstein metrics

Non-Existence Theorem

Theorem (DC, 2021)

Closed Extended graph 4-manifolds do not support Einstein metrics.

Remarks and Comments

- Notice that Closed Extended graph 3-manifolds do **NOT** support Einstein metrics
- Indeed, $Ric_g = \lambda g$ implies constant sectional curvature in dimension $n = 3$!

Non-Existence Theorem

Theorem (DC, 2021)

Closed Extended graph 4-manifolds do not support Einstein metrics.

Remarks and Comments

- Notice that Closed Extended graph 3-manifolds do **NOT** support Einstein metrics
- Indeed, $Ric_g = \lambda g$ implies constant sectional curvature in dimension $n = 3$!
- This theorem then shows that graph-like manifolds carry over their aversion to Einstein metrics from dimension three to four.

Question

Question

Do extended graph n-manifolds with $n \geq 5$ support Einstein metrics?

Remarks and Comments

Question

Question

Do extended graph n-manifolds with $n \geq 5$ support Einstein metrics?

Remarks and Comments

- The study of Einstein metrics on manifolds of dimension $n \geq 5$ remains rather obscured when compared to dimension $n = 4$.

Question

Do extended graph n-manifolds with $n \geq 5$ support Einstein metrics?

Remarks and Comments

- The study of Einstein metrics on manifolds of dimension $n \geq 5$ remains rather obscured when compared to dimension $n = 4$.
- In fact, **no uniqueness or non-existence results** are currently known in higher dimensions!

Question

Question

Do extended graph n-manifolds with $n \geq 5$ support Einstein metrics?

Remarks and Comments

- The study of Einstein metrics on manifolds of dimension $n \geq 5$ remains rather obscured when compared to dimension $n = 4$.
- In fact, **no uniqueness or non-existence results** are currently known in higher dimensions!
- Maybe someone in the audience, perhaps a student, will take on the challenge!

Sketch of the Proof

Sketch of the Proof

The Proof can be roughly divided into Three Lemmas

Sketch of the Proof

The Proof can be roughly divided into Three Lemmas

Lemma (Improved Hithin-Thorpe Inequality due to LeBrun, 1999)

Let (M, g) be a closed orientable Einstein 4-manifold with signature σ, Euler characteristic χ, and $\lambda < 0$. Then

$$2\chi(M) - 3|\sigma(M)| \geq \frac{3}{2\pi^2} Vol_{Ric}(M)$$

where the minimal Ricci volume $Vol_{Ric}(M)$ is defined as

$$Vol_{Ric}(M) := inf_g\{Vol_g(M) \quad Ric_g \geq -3g\}.$$

Moreover, equality occurs if and only if g is **half-conformally flat** *and it realizes the minimal Ricci volume (up to scaling). Finally, if $\sigma(M) = 0$ and the equality is achieved, then M is real-hyperbolic.*

Lemma

Let M *be an extended graph* 4*-manifold* **without** *pure pieces. We have* $\chi(M) = \sigma(M) = 0$*. If* M *has* $k \geq 1$ *pure real-hyperbolic pieces say* $(V_i := \Gamma_i \backslash \mathbb{H}^4_{\mathbb{R}}, g_{-1})_{i=1}^{k}$*, we then have*

$$\chi(M) = \sum_{i=1}^{k} \chi(V_i) > 0, \quad \sigma(M) = 0.$$

Lemma

Let M be an extended graph 4-manifold **without** *pure pieces. We have $\chi(M) = \sigma(M) = 0$. If M has $k \geq 1$ pure real-hyperbolic pieces say $(V_i := \Gamma_i \backslash \mathbb{H}^4_{\mathbb{R}}, g_{-1})_{i=1}^{k}$, we then have*

$$\chi(M) = \sum_{i=1}^{k} \chi(V_i) > 0, \quad \sigma(M) = 0.$$

and finally

Lemma

Let M *be an extended graph* 4*-manifold* **without** *pure pieces. We have* $\chi(M) = \sigma(M) = 0$. *If* M *has* $k \geq 1$ *pure real-hyperbolic pieces say* $(V_i := \Gamma_i \backslash \mathbb{H}^4_{\mathbb{R}}, g_{-1})_{i=1}^{k}$, *we then have*

$$\chi(M) = \sum_{i=1}^{k} \chi(V_i) > 0, \quad \sigma(M) = 0.$$

and finally

Lemma (Connell-Suaréz-Serrato, 2019)

Let M *be an extended graph* 4*-manifold* **with** $k \geq 1$ *pure real-hyperbolic pieces say* $(V_i, g_{-1})_{i=1}^{k}$, *we then have*

$$Vol_{Ric}(M) = \sum_{i=1}^{k} Vol_{g_{-1}}(V_i) = \frac{4\pi^2}{3} \sum_{i=1}^{k} \chi(V_i) = \frac{4\pi^2}{3} \chi(M).$$

Concluding, we have two **distinct** cases:

Concluding, we have two **distinct** cases:

- $Vol_{Ric}(M) = 0$.

Concluding, we have two **distinct** cases:

- $Vol_{Ric}(M) = 0$. In this case, M has no pure real-hyperbolic pieces and M saturates the **Hitchin-Thorpe** Inequality.

Concluding, we have two **distinct** cases:

- $Vol_{Ric}(M) = 0$. In this case, M has no pure real-hyperbolic pieces and M saturates the **Hitchin-Thorpe** Inequality. The growth of $\pi_1(M)$ now provides an obstruction!

Concluding, we have two **distinct** cases:

- $Vol_{Ric}(M) = 0$. In this case, M has no pure real-hyperbolic pieces and M saturates the **Hitchin-Thorpe** Inequality. The growth of $\pi_1(M)$ now provides an obstruction!

- $Vol_{Ric}(M) > 0$.

Concluding, we have two **distinct** cases:

- $Vol_{Ric}(M) = 0$. In this case, M has no pure real-hyperbolic pieces and M saturates the **Hitchin-Thorpe** Inequality. The growth of $\pi_1(M)$ now provides an obstruction!

- $Vol_{Ric}(M) > 0$. In this case, M has at least one pure real-hyperbolic piece and M saturates the improved **Hitchin-Thorpe** Inequality.

Concluding, we have two **distinct** cases:

- $Vol_{Ric}(M) = 0$. In this case, M has no pure real-hyperbolic pieces and M saturates the **Hitchin-Thorpe** Inequality. The growth of $\pi_1(M)$ now provides an obstruction!

- $Vol_{Ric}(M) > 0$. In this case, M has at least one pure real-hyperbolic piece and M saturates the improved **Hitchin-Thorpe** Inequality. Thus, the Einstein metric on M has to be real-hyperbolic!

Concluding, we have two **distinct** cases:

- $Vol_{Ric}(M) = 0$. In this case, M has no pure real-hyperbolic pieces and M saturates the **Hitchin-Thorpe** Inequality. The growth of $\pi_1(M)$ now provides an obstruction!

- $Vol_{Ric}(M) > 0$. In this case, M has at least one pure real-hyperbolic piece and M saturates the improved **Hitchin-Thorpe** Inequality. Thus, the Einstein metric on M has to be real-hyperbolic! $\pi_1(M)$ now provides an obstruction as it contains at least a subgroup isomorphic to $\mathbb{Z}^3$.

Here are the relevant papers for some of the lemmas:

Here are the relevant papers for some of the lemmas:

C. LeBrun, *Four-dimensional Einstein Manifolds, and Beyond*, **Surveys in differential geometry: essays on Einstein manifolds** 247-285, Surv. Differ. Geom., 6, *Int. Press, Boston, MA* , (1999).

Here are the relevant papers for some of the lemmas:

C. LeBrun, *Four-dimensional Einstein Manifolds, and Beyond*, **Surveys in differential geometry: essays on Einstein manifolds** 247-285, Surv. Differ. Geom., 6, *Int. Press, Boston, MA* , (1999).

C. Connell, P. Suaréz-Serrato, *On higher graph manifolds*, **Int. Math. Res. Not.** (2019), no. 5, 1281-1311.

G. Besson, G. Coutois, S. Gallor, *Entropies and rigidités des espaces localment symétriques de curbure strictment négative*, **Geom. Func. Anal. 5** (1995), 731-799.

Thanks for listening!

Thanks for listening!

It was a great pleasure to speak here, and

Thanks for listening!

It was a great pleasure to speak here, and

I hope to meet you all one day **in person**!