Connectivity of the Space of Pointed Hyperbolic Surfaces

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Outline

1. The set $\mathcal{H}^2_\bullet$ of pointed hyperbolic surfaces
   - The space $\mathcal{H}^2_\bullet$
   - Hyperbolic surfaces
   - Understanding the geometry of a surface
2. The Topology on $\mathcal{H}^2_\bullet$
   - The pointed Gromov-Hausdorff topology
   - Relationship with the Chabauty topology on $\mathrm{Sub}(\mathrm{PSL}_2(\mathbb{R}))$
3. Sketch of Results
   - Continuous paths in $\mathcal{H}^2_\bullet$
   - Global path-connectivity
   - Local path-connectivity
     - At (X, p), where X is of the first kind
     - At $(\mathbb{H}^2, z_0)$

Definition

$$\begin{aligned}\mathcal{H}^2_\bullet := &\ \text{the set of (isometry classes) of hyperbolic surfaces with a basepoint} \\ = &\ \{(X,p) : X \text{ a hyperbolic surface}, p \in X\}/\sim\end{aligned}$$

where $(X, p) \sim (X', p')$ if there is a basepoint-preserving isometry between them. We equip $\mathcal{H}^2_\bullet$ with the *pointed Gromov-Hausdorff topology*.

In this talk, a hyperbolic surface is assumed to be **connected, oriented, and metrically complete without boundary**, unless specified otherwise.

Results

Theorem (W., 2021)

The space $\mathcal{H}^2_\bullet$ is path-connected.

Theorem (W., 2021)

$\mathcal{H}^2_\bullet$ is weakly locally path-connected at the following points:

- (X, p), *where X is of the first kind*
- $(\mathbb{H}^2, z_0)$.

Hyperbolic surfaces

A *hyperbolic surface* is a 2-dimensional Riemannian manifold locally modeled by a neighborhood of the Poincaré disk, which is the unit disk

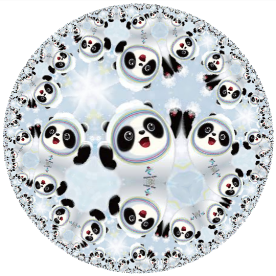
$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

with the metric

$$|ds|^2 = \frac{4|dz|^2}{(1-|z|^2)^2}.$$

The Poincaré disk is the unique (up to isometry) complete simply connected Riemann surface with constant sectional curvature -1.

The Poincaré disk



A tiling of the Poincaré disk by 冰墩墩 generated using Marlin Christersson’s tool

Beltrami’s original model (1869)
Source: M. Cornalba, Attualità di Eugenio Beltrami

Hyperbolic surfaces

Equivalently, we can also take the upper-half plane to be a model of hyperbolic geometry. This is

$$\mathbb{H}^2 = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

with the metric $|ds|^2 = \frac{1}{|\mathrm{Im}(z)|^2}|dz|^2$.

A hyperbolic surface X can be viewed as the quotient manifold

$$X \cong \mathbb{H}^2/\Gamma$$

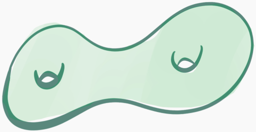
where Γ is a discrete torsion-free subgroup of $\mathrm{Isom}^+(\mathbb{H})^2 \cong \mathrm{PSL}_2(\mathbb{R})$ acting on $\mathbb{H}^2$.

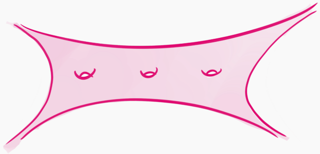
Examples of hyperbolic surfaces

Here, all hyperbolic surfaces are assumed to be **connected, oriented, and metrically complete without boundary**, unless specified otherwise.

Examples

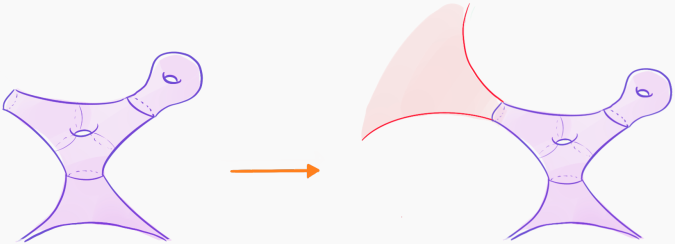
- The Poincaré disk $\mathbb{D}$
- Finite-type surfaces of genus g with p punctures, where $2 - 2g - p < 0$





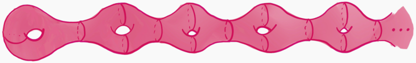
Examples of hyperbolic surfaces

- Nielsen extensions of finite-type hyperbolic surfaces with geodesic boundary

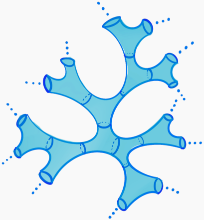


Examples of hyperbolic surfaces

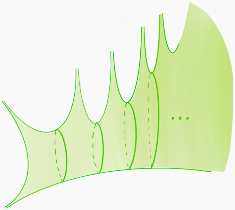
- Infinite-type hyperbolic surfaces



a Loch Ness monster



a Cantor tree



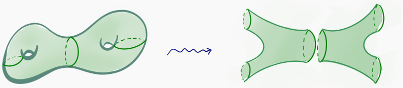
a (tight) flute surface

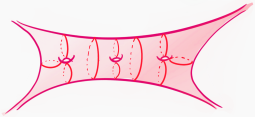
Examples of hyperbolic surfaces

- More generally, any topological surface that is homeomorphic to $\mathbb{S}^2 - K$, where K is a closed subset of a Cantor set, can be equipped with a complete hyperbolic metric.
  - This follows from the classification of non-compact surfaces by Kerékjártó (1923) and Richards (1962) and a decomposition of such surfaces by Bavard-Walker (2018).

Definition

A *geodesic pants decomposition* of a hyperbolic surface X is a collection of pairwise disjoint, mutually homotopically distinct simple closed geodesics $\{\gamma_i\}_{i\in\mathcal{I}}$ on X so that the closure of each component of $X - \bigcup \gamma_i$ is a geodesic pair of pants.





Theorem (Álvarez-Rodriguez, 2004; Basmajian-Šarić, 2019)

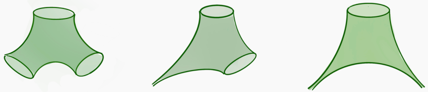
Let X be a complete hyperbolic surface without boundary with a nonabelian fundamental group that is not diffemorphic to a sphere with three points removed.
Then, the convex core $CC(X)$ admits a geodesic pants decomposition, and each component of $X - \overline{CC(X)}$ is either a funnel or a half-plane.

Definition

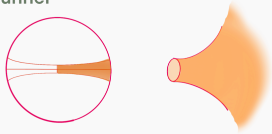
A hyperbolic surface X is **of the first kind** if $CC(X) = X$. Otherwise, it is **of the second kind.**

Types of building blocks of a hyperbolic surface

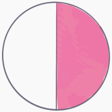
Geodesic pairs of pants



Hyperbolic funnel



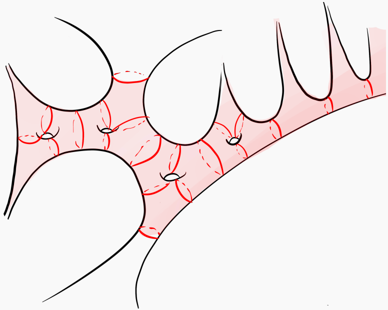
Half-plane



Understanding the geometry of a surface

Fenchel-Nielsen Coordinates

A more typical surface in $\mathcal{H}^2_\bullet$ might look like this...

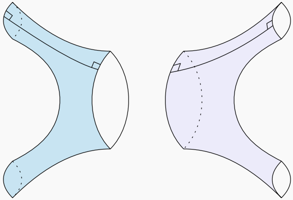


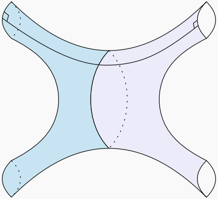
Fix a pants decomposition $\mathcal{P} = \{\gamma_i\}_{i\in\mathcal{I}}$ of X. Its hyperbolic structure is determined by the **Fenchel-Nielsen coordinates** of X with respect to $\mathcal{P}$:

$$\mathcal{FN}(X) = ((\text{length}[\gamma_i], \text{twist}[\gamma_i]))_{i\in\mathcal{I}} .$$

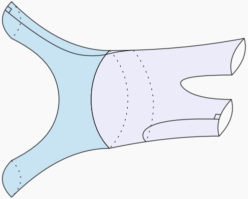
Understanding the geometry of a surface

On twist parameters



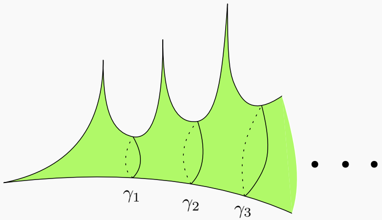


No twist



$3\pi/2$ twist

A tight flute surface

$X =$ 

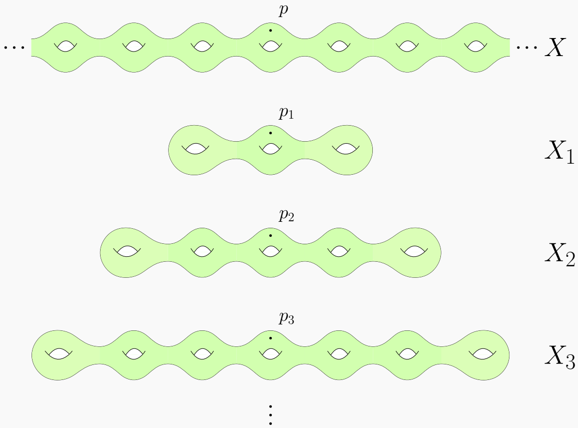
$$d_n = d_X(\gamma_n, \gamma_{n+1})$$

Theorem (Basmajian, 1993)

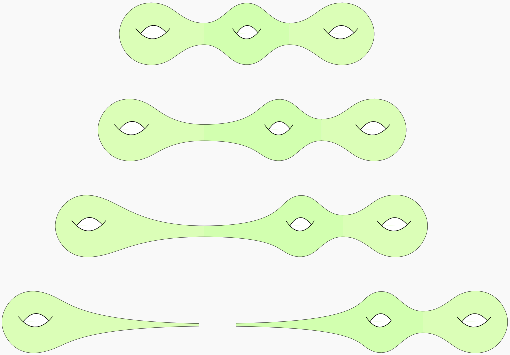
If $\sum_n d_n < \infty$ and $\sum_n |\text{twist}[\gamma_n]| < \infty$, then the metric completion of X contains a half-plane.

There are several useful notions to compare two closed surfaces that are diffeomorphic.

Motivation: A surface can be approximated by a sequence of larger and larger compact subsurfaces.



Caveat: It's important to use a basepoint keep track of the local geometry if we hope to get a unique limit.



Definition

For $K > 1$ and $r > 0$, a (K, r)-*quasi-isometry* between (X, p) and (Y, q) in $\mathcal{H}^2_\bullet$ is a diffeomorphism between two subsurfaces $(X_1, p) \subset (X, p)$ and $(Y_1, q) \subset (Y, q)$

$$f : (X_1, p) \to (Y_1, q)$$

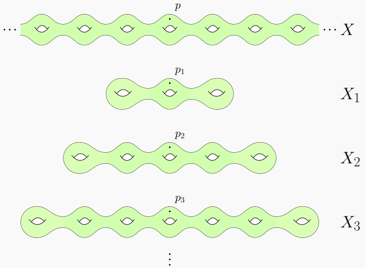
such that

1. $B_X(p, r) \subset (X_1, p)$ and $B_Y(q, r) \subset (Y_1, q)$;
2. $f(p) = q$;
3. For all $x, x' \in X_1$,

$$\frac{1}{K} d(x, x') \leq d(f(x), f(x')) \leq K d(x, x').$$

Convergence criterion

In $\mathcal{H}^2_\bullet$, $(X_n, p_n) \to (X, p)$ if for all $K > 1$, $r > 0$, there exists $n \in \mathbb{N}$ sufficiently large so that there is an (K, r)-quasi-isometry between (X_n, p_n) and (X, p).



Upshot: Two pointed surfaces are close in the pointed GH topology if large compact subsurfaces around their respective basepoints are almost isometric.

The pointed Gromov-Hausdorff topology

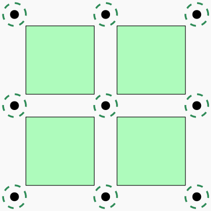
Some remarks

- The topology is Hausdorff. In fact, it's metrizable.
- Generalizes to $\mathcal{H}^n_\bullet$, the space of pointed n-dimensional hyperbolic manifolds
- Has rich applications in dimension 3. It's a crucial ingredient in the works of Jørgensen-Thurston in determining the volume spectrum.
- In the setting of hyperbolic manifolds, this version of the definition is equivalent to the notion of (ϵ, R)-relations introduced by Edwards (1975) and generalized by Gromov (1981).
- This is related to the Chabauty topology on the space of closed subgroups of $\mathrm{Isom}^+(\mathbb{H}^n)$.

Definition

Given a Lie group G, let $\mathrm{Sub}(G)$ be the set of closed subgroups of G. The **Chabauty topology** on $\mathrm{Sub}(G)$ is generated by all open sets of the form

1. $O_1(K) = \{H \leq G : H \cap K = \emptyset\}$, $K \subset G$ is compact;
2. $O_2(U) = \{H \leq G : H \cap U \neq \emptyset\}$, $U \subset G$ is open.



Chabauty convergence

In $\mathrm{Sub}(G)$, $H_n \to H$ if

1. $\forall h \in H, \exists h_n \in H_n$ such that $h_n \to h$ and
2. H contains the limits of all convergent sequences $h_n \in H_n$.

- $\mathrm{Sub}(G)$ is compact and Hausdorff.

- Let $\mathrm{Sub}_{DT}(\mathrm{PSL}_2\,\mathbb{R}) \subset \mathrm{Sub}(\mathrm{PSL}_2\,\mathbb{R})$ be the subspace of discrete torsion-free subgroups of $\mathrm{PSL}_2\,\mathbb{R}$ with the Chabauty topology.
- Let $\mathcal{H}^2_f$ be the space of hyperbolic surfaces with baseframe

  $$\mathcal{H}^2_f = \{(X, p, \mathbf{w}) : (X, p) \in \mathcal{H}^2_\bullet, \mathbf{w} \text{ an orthonormal basis of } T_p(X)\}/\sim$$

  where $\sim$ is by baseframe-preserving isometry.
- We equip $\mathcal{H}^2_f$ with the framed version of the pointed GH topology.

Relationship with the Chabauty topology

Framed surfaces

- Fix $z_0 \in \mathbb{H}^2$ and an oriented orthonormal basis $\mathbf{v}_0$ of $T_{z_0}(\mathbb{H}^2)$.
- Let $\pi : \mathbb{H}^2 \to \mathbb{H}^2/\Gamma$ be the projection. Then, the map

$$\begin{aligned} \mathrm{Sub}_{DT}(\mathrm{PSL}_2 \mathbb{R}) &\longrightarrow \mathcal{H}_f^2 \\ \Gamma &\longmapsto (\mathbb{H}^2/\Gamma, \pi(z_0), d\pi_{z_0}(\mathbf{v}_0)) \end{aligned}$$

gives a homeomorphism! See Canary-Marden-Epstein (2006).

Relationship with the Chabauty topology

Prior Results

- The Chabauty closure of the subspace of one-generator subgroups of $\mathrm{PSL}_2\,\mathbb{R}$ is simply connected. (Baik-Clavier, 2013)
- The subspace of closed elementary subgroups of $\mathrm{Sub}\,\mathrm{PSL}_2(\mathbb{R})$ is simply connected. (Biringer-Lazarovich-Leitner, 2021)

Relationship with the Chabauty topology

Prior Results

If S is a finite-type hyperbolic 2-orbifold, we write

$$\mathrm{Sub}(\mathrm{PSL}_2\,\mathbb{R}; S) := \{\Gamma \in \mathrm{Sub}(\mathrm{PSL}_2\,\mathbb{R}) : \mathbb{H}^2/\Gamma \cong S\}.$$

Let $\mathcal{M}(S)$ be the moduli space of S.

Theorem (Biringer-Lazarovich-Leitner, 2021)

For such an S, the map

$$\pi_{\mathrm{Sub}} : \mathrm{Sub}(\mathrm{PSL}_2\,\mathbb{R}; S) \to \mathcal{M}(S)$$

is a fiber orbibundle with fiber T^1S and $\mathrm{Sub}(\mathrm{PSL}_2\,\mathbb{R}; S)$ is a $6g + 2(k + l) - 3$ dimensional manifold, where g is the genus of S, and k is the number of cusps, and l is the number of cone points.

Theorem (Biringer-Lazarovich-Leitner, 2021)

Suppose that either a four-punctured sphere or a once-punctured torus embeds in S as the interior of a surface with geodesic boundary. Then $\mathrm{Sub}(\mathrm{PSL}_2(\mathbb{R}); S)$ *is simply connected.*

Results

Theorem (W., 2021)

The space $\mathcal{H}^2_\bullet$ is path-connected.

Theorem (W., 2021)

$\mathcal{H}^2_\bullet$ is weakly locally path-connected at the following points:

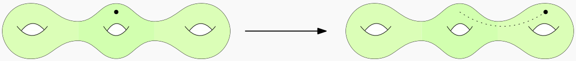
- (X, p), *where X is of the first kind*
- $(\mathbb{H}^2, z_0)$.

Continuous paths in $\mathcal{H}^2_\bullet$

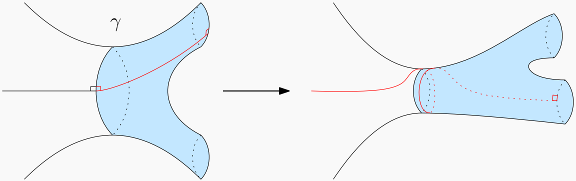
Examples

We can create a continuous path in $\mathcal{H}^2_\bullet$ by

- Moving a basepoint along a path on a fixed surface



- Adjusting finitely many length and twist parameters in the Fenchel-Nielsen coordinates

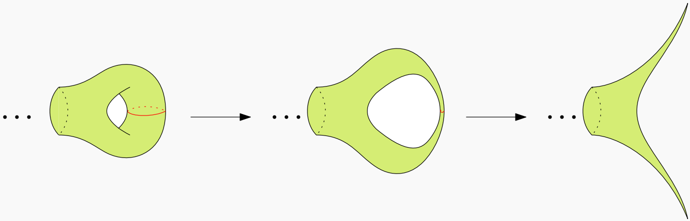


Continuous paths in $\mathcal{H}^2_\bullet$

Examples (cont.)

We can create a continuous path in $\mathcal{H}^2_\bullet$ by

- Pinching a simple closed geodesic to a cusp



Continuous paths in $\mathcal{H}^2_\bullet$

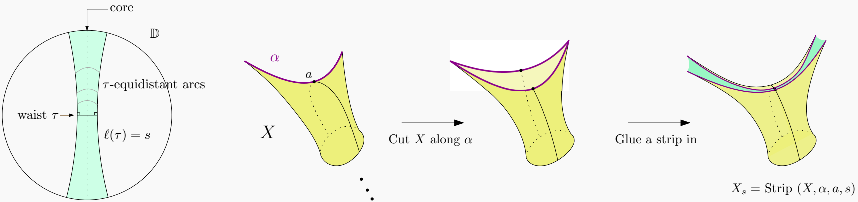
Examples (cont.)

We can create a continuous path in $\mathcal{H}^2_\bullet$ by

- Inserting and growing a strip along a properly embedded infinite geodesic

Setup

- a hyperbolic surface X with a proper geodesic α and $a \in \alpha$
- an s-strip, $s > 0$



Continuous paths in $\mathcal{H}^2_\bullet$

Strip insertion is continuous

Proposition (W., 2021)

Fixing $(X, p) \in \mathcal{H}^2_\bullet$ and α and a as above, the map

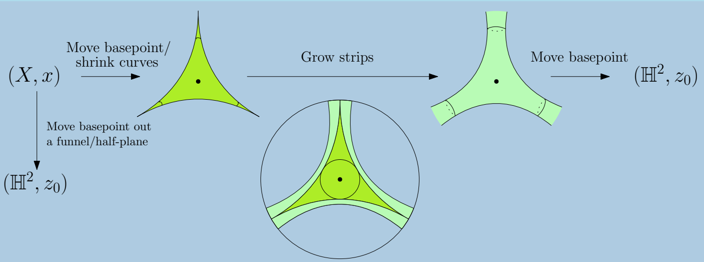
$$\begin{aligned} \mathbb{R}_+ &\to \mathcal{H}^2_\bullet \\ s &\mapsto (X_s, p) \end{aligned}$$

is continuous. That is, inserting and growing a strip is a continuous construction in $\mathcal{H}^2_\bullet$.

Theorem

The space $\mathcal{H}^2_\bullet$ is path-connected.

Proof.



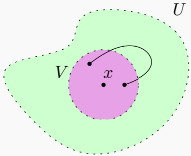
□

Local path-connectivity of $\mathcal{H}^2_\bullet$

Actually weakly

Definition

A space $\mathcal{X}$ is **weakly locally path-connected at** $x \in \mathcal{X}$ if every open neighborhood U of x contains an open neighborhood V of x such that any two points in V lie in some path-connected subset of U.



Upshot

If $\mathcal{X}$ is weakly locally path-connected at each point, then $\mathcal{X}$ is locally path-connected.

Local path-connectivity of $\mathcal{H}^2_\bullet$

Now and next

Theorem

$\mathcal{H}^2_\bullet$ is weakly locally path-connected at the following points:

- *(X, p), where X is of the first kind*
- *$(\mathbb{H}^2, z_0)$.*

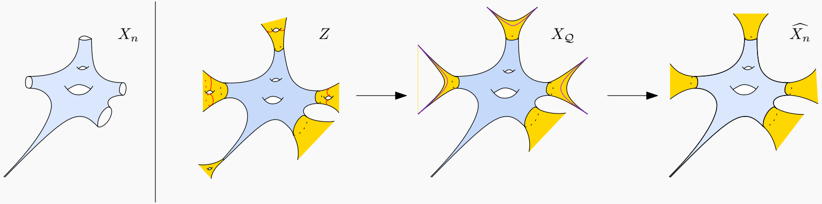
Goal

The space $\mathcal{H}^2_\bullet$ is locally path-connected? Contractible?

- Exhaust X by finite-area subsurfaces with geodesic boundary

$$X_1 \subset X_2 \cdots \subset X$$

- Define a neighborhood basis based on the X_n

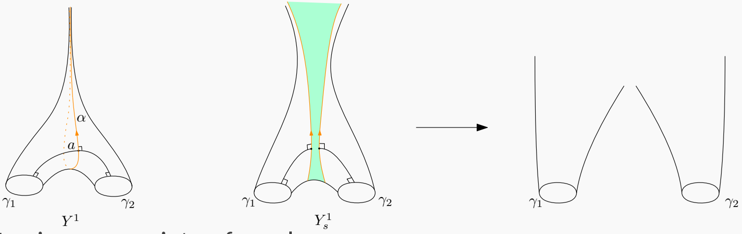


Continuous paths in $\mathcal{H}^2_\bullet$

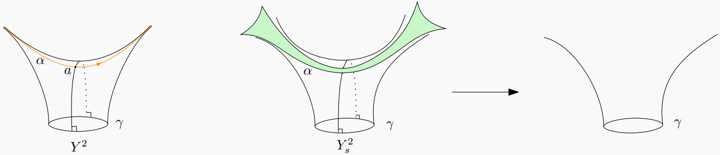
Two particular strip insertions

We make use of two special strip insertions.

- Turning 1 cusp into 2 funnels



- Turning 2 cusps into 1 funnel



Neighborhoods

For $r > 0$, let $U(r) = \{(X, p) : \text{injrad}_X(p) > r\}$.

Local path-connectivity of $\mathcal{H}^2_\bullet$

At $(\mathbb{H}^2, z_0)$

- When X is non-compact,

Theorem (Bavard-Walker, 2018)

There is a collection $\mathcal{A}$ of disjoint properly embedded geodesics in X whose complement is a union of simply connected regions.

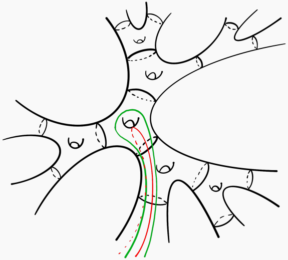
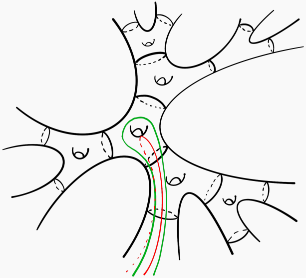


Figure: Fixing an end of X, we choose such geodesic arcs exiting that end for every genus. The picture is modified from their paper.

Local path-connectivity of $\mathcal{H}^2_\bullet$

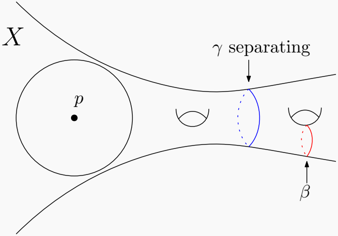
At $(\mathbb{H}^2, z_0)$



- When X is non-compact,
  - add strips, one at a time, along the geodesics in $\mathcal{A}$,
  - take the limit as the strip widths go to ∞, and
  - we get a path from (X, p) to $(\mathbb{H}^2, z_0)$.

Local path-connectivity of $\mathcal{H}^2_\bullet$

At $(\mathbb{H}^2, z_0)$



- When X is compact,
  - find a separating simple closed geodesic γ far from the basepoint p,
  - pinch β on the other side of γ from p, and
  - we return to the non-compact case!

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