Deformation space of circle patterns on surfaces with complex projective structures

Wai Yeung Lam

BIMSA

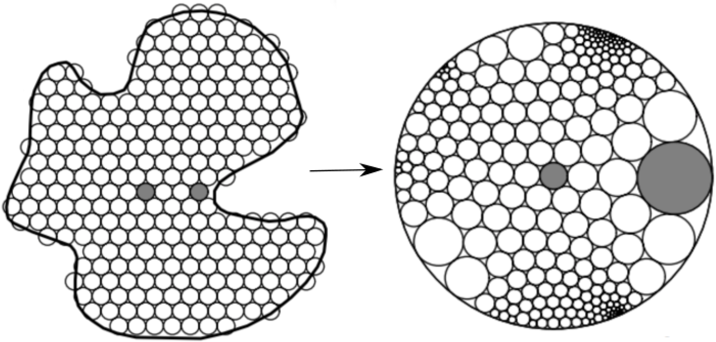
29 March 2022

Outline

- **Motivation: *Discrete holomorphic functions***
- Classical theory: *Complex projective structures*
- Main results: *Deformation space of circle patterns*
- Key ingredients: *Graph Laplacian (Cotangent weights), Harmonic conjugates*
- Open questions

Discrete holomorphic functions

Classical theory: A holomorphic function maps infinitesimal small circles to infinitesimal small circles.

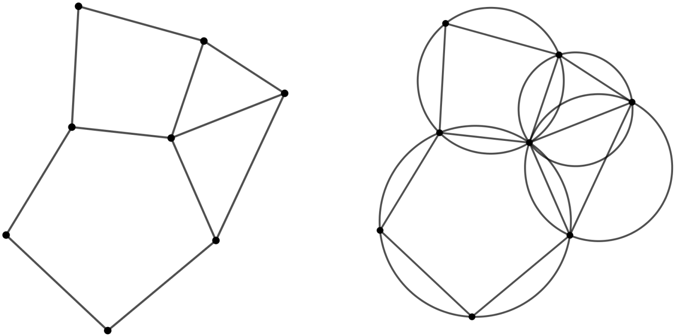


(Figure from Ken Stephenson)

- Circle Packings - Discrete holomorphic functions (Thurston 1985)
- Hexagonal packings $\mapsto$ Riemann mapping (Rodin and Sullivan 1987)

Circle patterns

- Circle pattern is a **realization of a planar graph in $\mathbb{C} \cup \infty$ such that the vertices of each face lie on a circle**



- Special case: Circle packing $+$ dual packing
  $\rightarrow$ Circle pattern with prescribed intersection angles $\Theta_{ij} \in \{0, \pi/2\}$

Outline

- Motivation: *Discrete holomorphic functions*
- **Classical theory: *Complex projective structures***
- Main results: *Deformation space of circle patterns*
- Key ingredients: *Graph Laplacian (Cotangent weights), Harmonic conjugates*
- Open questions

Complex projective structures

M_g closed surface of genus g

Definition

A conformal structure is a maximal atlas of charts to $\hat{\mathbb{C}}$ such that the transition functions are holomorphic.

$\mathcal{T}(M)$ Teichmüller space = space of marked conformal structures \constant-curvature metrics

Ω_{WP} Weil-Petersson symplectic form

Definition

A complex projective structure structure is a maximal atlas of charts to $\hat{\mathbb{C}}$ such that the transition functions are restrictions of *CP* transformations (Möbius transformations).

$P(M)$ space of all marked complex projective structures

Ω_G Goldman's complex symplectic form induced from $\text{Hom}(\pi_1(M), SL(2,\mathbb{C}))$

$\pi : P(M) \to \mathcal{T}(M)$ uniformization map

Why CP^1 structures?

Möbius transformations map circles to circles

$\rightarrow$ Circles are well defined on surfaces with complex projective structures

Examples for $P(M_g)$ $(g = 1)$

- Euclidean structures $\mathcal{T}(M)$
- Complex affine structures (transition function $z \mapsto az + b$)

Examples for $P(M_g)$ $(g > 1)$

- Hyperbolic structures $\mathcal{T}(M)$
- Quasi-Fuchsian
- More...

Facts about CP^1-structures

Teichmüller space

$$\mathcal{T}(M_g) \cong \begin{cases} \mathbb{R}^0 & \text{for } g = 0 \\ \mathbb{R}^2 & \text{for } g = 1 \\ \mathbb{R}^{6g-6} & \text{for } g > 1 \end{cases}$$

Marked CP^1-structures

$$P(M_g) \cong \begin{cases} \mathbb{C}^0 & \text{for } g = 0 \\ \mathbb{C}^2 & \text{for } g = 1 \\ \mathbb{C}^{6g-6} & \text{for } g > 1 \end{cases}$$

$\pi : P(M_g) \to \mathcal{T}(M_g)$ is a fiber bundle

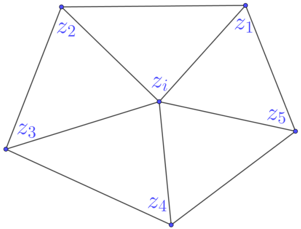
OUTLINE

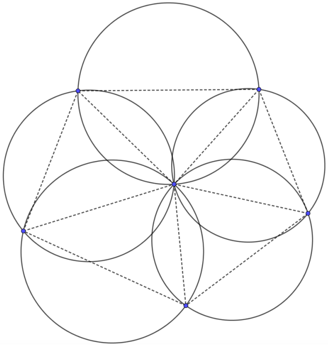
- Motivation: *Discrete holomorphic functions*
- Classical theory: *Complex projective structures*
- **Main results: *Deformation space of circle patterns***
- Key ingredients: *Graph Laplacian (Cotangent weights), Harmonic conjugates*
- Open questions

Cross ratios of 4 points $z_1, z_2, z_3, z_4 \in \mathbb{C}$:

$$X(z_1, z_2, z_3, z_4) := -\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)} \in \mathbb{C}$$

$\implies$ Cross ratio for every interior edge $X : E \to \mathbb{C}$ (Note: $X_{ij} = X_{ji}$)





Around each interior vertex i

$$1 = \Pi_{j=1}^{n} X_{ij} \tag{1}$$

$$0 = (X_{i1}) + (X_{i1}X_{i2}) + \cdots + (X_{i1}X_{i2}\ldots X_{in}) \tag{2}$$

Delaunay cross ratio system

Definition

Given $M = (V, E, F)$ a triangulation of a closed surface, a cross ratio system is a map $X : E \to \mathbb{C}$ such that for every vertex i

$$\begin{aligned} 1 &= \Pi_{j=1}^{n} X_{ij} \\ 0 &= (X_{i1}) + (X_{i1}X_{i2}) + \cdots + (X_{i1}X_{i2}\ldots X_{in}) \end{aligned}$$

Definition

A Delaunay angle structure is an assignment $\Theta : E \to [0, \pi)$ satisfying

1. For every vertex i, $\sum_j \Theta_{ij} = 2\pi$.
2. $\sum_{i=1}^{n} \Theta_{ij} > 2\pi$ for any closed loop on the dual graph bounding more than one face.

$P(\Theta)$ the space of all cross ratio systems X with $\mathrm{Arg}\, X \equiv \Theta$.

i.e. space of circle patterns with prescribed intersection angles

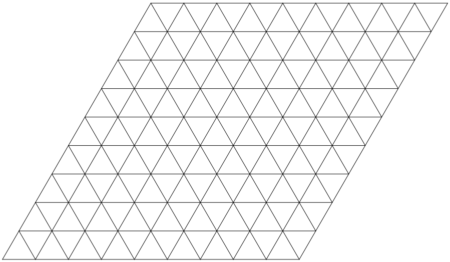
- Each Delaunay cross ratio system induces a complex projective structure on M together with a circle pattern by gluing circumdisks.
- It yields

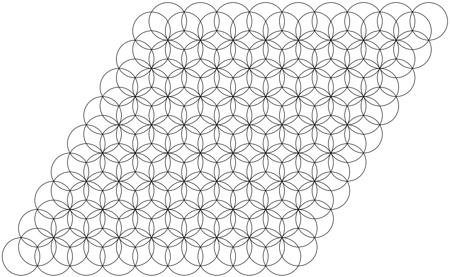
$$P(\Theta) \xrightarrow{f} P(M) \xrightarrow{\pi} \mathcal{T}(M)$$

- How does $P(\Theta)$ look like? Manifold? Dimension? $\pi \circ f : P(\Theta) \to \mathcal{T}(M)$?

ELEMENTS OF $P(\Theta)$

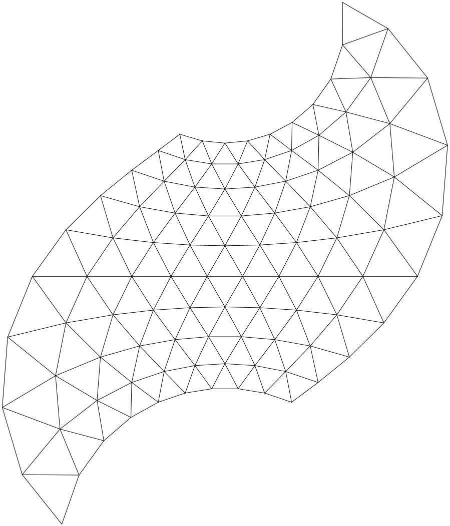
$\Theta \equiv \pi/3$ on a triangulated torus.

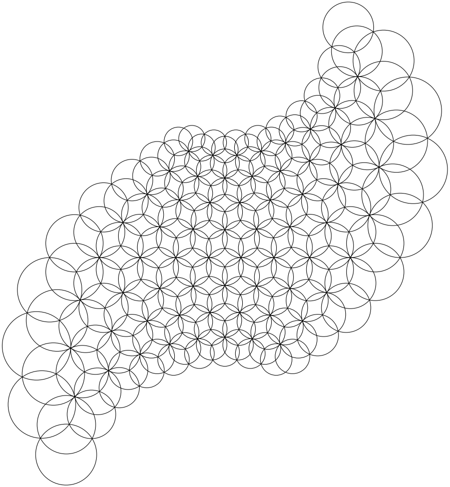




ELEMENTS OF $P(\Theta)$

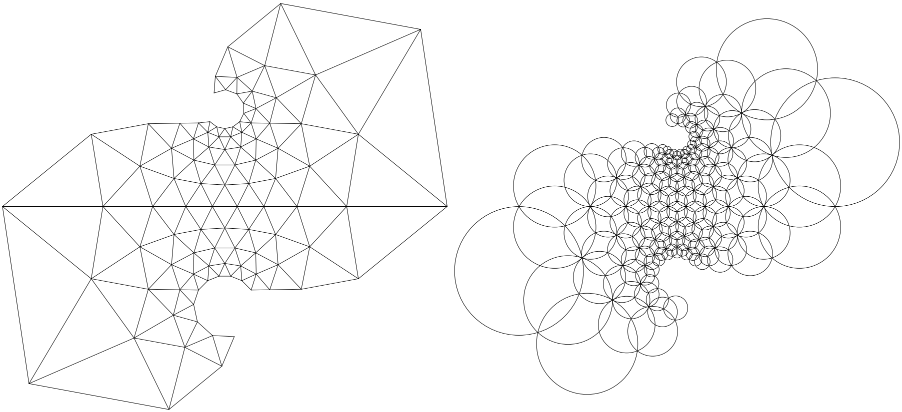
$\Theta \equiv \pi/3$ on a triangulated torus.





ELEMENTS OF $P(\Theta)$

$\Theta \equiv \pi/3$ on a triangulated torus.



How does $P(\Theta)$ look like? Manifold? Dimension? $\pi \circ f : P(\Theta) \to \mathcal{T}(M)$?

Conjecture (Kojima-Mizushima-Tan (2003))

The projection $\pi \circ f : P(\Theta) \to \mathcal{T}(M)$ is a homeomorphism.

- (Mizushima 2000) One-vertex triangulation on torus: $P(\Theta)$ homeomorphic to $\mathbb{R}^2$.
- (Kojima, Mizushima, and Tan 2003) General triangulation: **Neighbourhood** around the Euclidean Torus in $P(\Theta)$ is homeomorphic to $\mathbb{R}^2$. ($g = 1$, similar for $g > 1$)
- (Schlenker,Yarmola 2018) $\pi \circ f$ is proper ($g > 1$)

Analogous to Thurston's grafting construction via measured laminations

MAIN RESULTS (FOR TORUS $g = 1$)

(L. 2019)

THEOREM (A)

Fixing any triangulation and Delaunay angle structure Θ on a torus,

1. $P(\Theta)$ *is a real analytic surface homeomorphic to* $\mathbb{R}^2$.
2. $f : P(\Theta) \to P(M)$ *is embedding*
3. *The holonomy map is embedding*

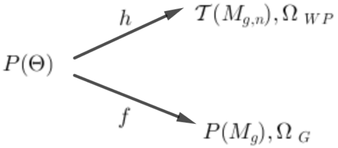
$$\text{hol} : P(\Theta) \to \text{Hom}(\pi_1(M), PSL(2,\mathbb{C})) /\!/ PSL(2,\mathbb{C})$$

THEOREM (B)

The projection $\pi \circ f : P(\Theta) \to \mathcal{T}(M)$ *is a homeomorphism.*

MAIN RESULTS (FOR TORUS $g = 1$)

(L. 2022) Symplectic structure on $P(\Theta)$.



$M_{g,n}$ denotes a genus-g surface with n punctures, where $n = |V|$.

THEOREM (C)

The pullback of the symplectic forms $h^*\Omega_{WP} = f^*\Omega_G$ coincides and are non-degenerate.

There is an induced real symplectic form on $P(\Theta)$.

Compare with Thurston's grafting

Theorem (Thurston)

$Gr : \mathcal{T}(M_g) \times ML(M_g) \to P(M_g)$ is a homeomorphism.

Theorem (Scannell-Wolf (2002))

Fix $\lambda \in ML(M_g)$, the map $gr_\lambda : \mathcal{T}(M_g) \to \mathcal{T}(M_g)$ is a homeomorphism.

Outline

- Motivation: *Discrete holomorphic functions*
- Classical theory: *Complex projective structures*
- Main results: *Deformation space of circle patterns*
- **Key ingredients: *Graph Laplacian (Cotangent weights), Harmonic conjugates***
- Open questions

Graph Laplacian (cotangent weights)

$G = (V, E, F)$ cell decomposition of a surface, $c : E \to \mathbb{R}_{\geq 0}$, with $c_{ij} = c_{ji}$.

Definition

$u : V \to \mathbb{R}$ is a discrete harmonic function on G if around each interior vertex $i \in V$

$$\sum_j c_{ij}(u_j - u_i) = 0$$

Proposition

$u : V \to \mathbb{R}$ is discrete harmonic if and only if there exists $v : F \to \mathbb{R}$ such that

$$v_{left(\vec{ij})} - v_{right(\vec{ij})} = c_{ij}(u_j - u_i)$$

where $left(\vec{ij})$ is the left face of the oriented edge $\vec{ij}$.

Check: The function v is a discrete harmonic function on the dual cell decomposition G^* with weights $c^* := \frac{1}{c}$.

Graph Laplacian (cotangent weights)

Circle patterns $\Longrightarrow$ radii of circles $R : F \to \mathbb{R}$

1-parameter family of circle patterns $\Longrightarrow$ $R_t : F \to \mathbb{R}$

Proposition

$v := \frac{d}{dt} \log R_t$ is a discrete harmonic function on G^* where $c_{ij} = \cot \angle jki + \cot \angle ilj$.

Note: No non-constant harmonic functions on Tori.

We consider harmonic 1-forms.

A **discrete 1-form** is a function $\omega : \vec{E} \to \mathbb{R}$ such that $\omega_{ji} = -\omega_{ij}$.
It is **closed** on G if $\forall \phi \in F, \sum_{ij \in \partial\phi} \omega_{ij} = 0$

Definition

A closed discrete 1-form ω is **harmonic** if around each vertex $i \in V$

$$\sum_j c_{ij}\omega_{ij} = 0$$

Proposition

A closed discrete 1-form ω on G is harmonic if and only if there exists a closed discrete 1-form η on G^* such that

$$\eta_{ij} = c_{ij}\omega_{ij}.$$

We call $*\omega := \eta$ harmonic conjugate of ω.

Recall: Harmonic 1-forms on Riemann surfaces are parameterized by periods

$$(A, B) = (\sum_{\gamma_1} \omega, \sum_{\gamma_2} \omega)$$

Harmonic conjugate on periods

For each triangulated affine tori, we define an action of harmonic conjugate on periods

$$*_G : \mathbb{R}^2 \to \mathbb{R}^2.$$

1. Given any $(A, B) \in \mathbb{R}^2$, find discrete harmonic 1-form ω such that

$$(A, B) = (\sum_{\gamma_1} \omega, \sum_{\gamma_2} \omega)$$

2. Compute periods of the harmonic conjugate

$$(\tilde{A}, \tilde{B}) = (\sum_{\gamma_1} *\omega, \sum_{\gamma_2} *\omega)$$

3. $*_G(A, B) := (\tilde{A}, \tilde{B})$. Known: $*_G$ is an isomorphism.

The period space is equipped with an inner product where $|(A, B)|^2$ is the Dirichlet energy of the corresponding smooth harmonic 1-form.

Note: smooth harmonic conjugate $*$ is an isometry, i.e. $|| * || = 1$.

PROPOSITION

For non-Euclidean affine torus, $|| *_G^{-1} || < 1$.

Outline

- Motivation: *Discrete holomorphic functions*
- Classical theory: *Complex projective structures*
- Main results: *Deformation space of circle patterns*
- Key ingredients: *Graph Laplacian (Cotangent weights), Harmonic conjugates*
- **Open questions**

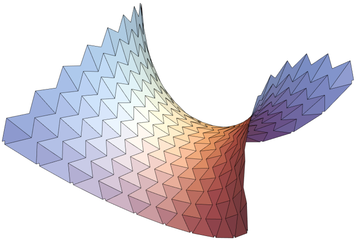
Open questions for Tori

1. Algorithm for Thm (B): Fixing a triangulation and intersection angle Θ, how to find the complex projective structure and circle pattern for any marked conformal structure?
2. Deformation space of circle patterns diffeomorphic to the Teichmuller space near Euclidean circle packing?

Further connections

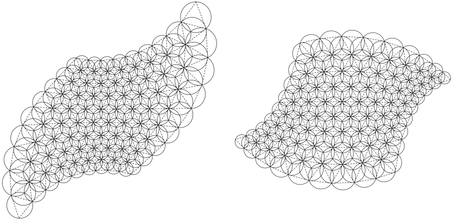
Why deformation space of circle patterns?

1. Discrete holomorphic functions
2. Classical Teichmuller theory
3. Discrete surface theory
4. Dimers and Circle patterns

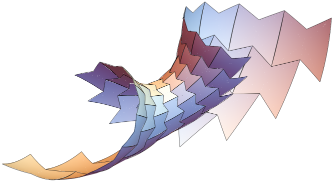


Weierstrass representation

Osculating Möbius transformation $A_h : F \to SL(2, \mathbb{C})/\{\pm I\}$



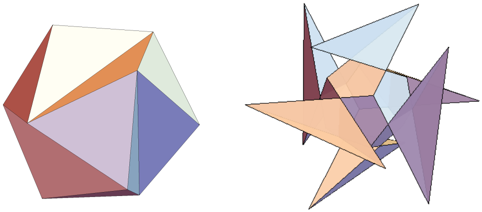
$\mathbf{i} : SL(2, \mathbb{C})/\{\pm I\} \to SL(2, \mathbb{C})/SU(2, \mathbb{C}) \cong \mathbb{H}^3$



CMC-1 surfaces in $\mathbb{H}^3$ (L.2020)

Good discretization = Rich in mathematical structures 🙂

Thank you!



W.Y. Lam. Quadratic differentials and circle patterns on complex projective tori. *Geom. Topol.* (2019)